THE SWITCHING GAME ON UNIONS OF ORIENTED MATROIDS

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Abstract. In 1986, Hamidoune and Las Vergnas [3] introduced an oriented matroid version of the so-called Shannon's switching game. They conjectured that the classification of the directed switching game on oriented matroids is identical to the classification of the non-oriented version. In this note, we support this conjecture by showing its validity for an infinity class of oriented matroids obtained as unions of rank-1 and/or rank-2 uniform oriented matroids.

Keywords: Oriented Matroid, Switching Game

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1. Introduction

In 1960, C.E. Shannon introduced the following game.

> **Switching game for graphs.** Let G be a graph and e an edge of G. Two players, *Maker* and *Breaker*, play alternatively. A move of Maker consists of making an unplayed edge invulnerable to deletion (the objective of Maker is to construct a path between the end points of the unplayable edge e). A move of Breaker consists of deleting an unplayed edge (the objective of Breaker is to prevent Maker to succeed). The game proceed until one of the players reaches its objective.

The above game has been generalized and elegantly solved for matroids by Lehman [8]. In 1986, Hamidoune and Las Vergnas [3] introduced the directed switching game on graphs (see [4] for a *arborescence rooted* variant). They naturally considered the following oriented matroid version.

> **Directed switching game for oriented matroids.** Let $\mathcal{M}$ be an oriented matroid and e one of its elements. Maker and Breaker alternatively play by choosing an unplayed element of $\mathcal{M}$ different from e, Maker signs it and Breaker deletes it. Maker wins the game

if the final orientation of $\mathcal{M}$ contains a positive circuit containing e. Breaker wins otherwise.

Hamidoune and Las Vergnas [3] presented a complete solution for graphic and cographic oriented matroids and conjectured that the classification of the oriented game is identical to the classification of the non-oriented version.

Conjecture 1. [3, Conjecture 8.1] *Let $\mathcal{M}$ be an oriented matroid on E. If E is the union of two disjoint bases, then the directed switching game on $\mathcal{M}$ is winning for Maker playing first.*

Recently, Forge and Vieilleribière [2] proved the above conjecture for the class of *Lawrence* oriented matroids.

In this note, we show that Conjecture 1 holds for oriented matroids obtained as union of rank-1 and/or rank-2 uniform oriented matroids (Theorem 1). The latter contains Forge and Vieilleribière's result in the particular case when the union consists of only rank-1 uniform oriented matroids.

2. Main result

We assume that the reader is familiar with basic oriented matroid theory [1]. Let $\mathcal{M}$ be an oriented matroid on the ground set $E = \{1, \ldots, n\}$. Let $\mathcal{C}(\mathcal{M}), \mathcal{B}(\mathcal{M})$ and $r(\mathcal{M})$ denote the set of circuits, the set of bases and the rank of $\mathcal{M}$ respectively. The *union* operation for oriented matroids is the oriented analogue to the union of (ordinary) matroids. The oriented version has first appeared as a particular case of a more general result due to Las Vergnas [5] who proved that principal extensions of oriented matroids can be oriented. Las Vergnas [6] used this to show that images of orientable matroids are also orientable (and thus unions of orientable matroids are orientables).

Lawrence and Weinberg [7] have shown that the union operation can be described in terms of bases orientation (or chirotopes). Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented matroids on the (totally ordered) set E with n elements of rank r_1 and r_2 respectively, and let $\chi_{\mathcal{M}_1}$ and $\chi_{\mathcal{M}_2}$ denote their corresponding chirotopes. We assume that $r_1+r_2 \leq n-1$. Let

$$\chi_{\mathcal{M}}(j_1, \ldots, j_{r_1+r_2}) := \chi_{\mathcal{M}_1}(j_1, \ldots, j_{r_1}) \cdot \chi_{\mathcal{M}_2}(j_{r_1+1}, \ldots, j_{r_1+r_2}) \qquad (1)$$

for all (r_1+r_2)-tuple $j_1 < \cdots < j_{r_1+r_2}$ of E. Lawrence and Weinberg [7, Theorem 3] proved that $\chi_{\mathcal{M}}$ is the base orientation of an oriented matroid of rank $r_1 + r_2$ on

E, called *union* of $\mathcal{M}_1$ and $\mathcal{M}_2$ (and denoted by $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$). If $\mathcal{M}_1$ and $\mathcal{M}_2$ are uniform[1] then $\mathcal{M}_1 \cup \mathcal{M}_2$ is also uniform.

Proposition 1. *Let $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ where $\mathcal{M}_1$ and $\mathcal{M}_2$ are uniform oriented matroids of rank r_1 and r_2 respectively. Let $C = (i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2+1})$ be a circuit of $\mathcal{M}$ such that $C_1 = (i_1, \dots, i_{r_1}, j_1)$ (resp. $C_2 = (j_1, \dots, j_{r_2+1})$) is a circuit of $\mathcal{M}_1$ (resp. a circuit of $\mathcal{M}_2$) with $C_1(j_1) = C_2(j_1) = C(j_1)$. Then,*

$$C_1(i_k) = C(i_k) \text{ for each } 1 \leq k \leq r_1 \text{ and } C_2(j_k) = C(j_k) \text{ for each } 1 \leq k \leq r_2 + 1$$

where $C(f)$ denotes the sign of element f in C.

Proof. Since $\mathcal{M}$ is uniform then,

$$\chi_{\mathcal{M}}(B_j) \cdot \chi_{\mathcal{M}}(B_{j+1}) = -C(i_j) \cdot C(i_{j+1}), \ 1 \leq j \leq r_1 + r_2 \tag{2}$$

where $B_j = C \setminus \{i_j\}$ and $B_{j+1} = C \setminus \{i_{j+1}\}$ are bases of $\mathcal{M}$. Therefore,

$$\chi_{\mathcal{M}}(i_1, \dots, i_{r_1-1}, j_1, \dots, j_{r_2+1}) \cdot \chi_{\mathcal{M}}(i_1, \dots, i_{r_1}, j_2, \dots, j_{r_2+1}) = -C(i_{r_1}) \cdot C(j_1). \tag{3}$$

Now, by equation (1), we have

$$\chi_{\mathcal{M}}(i_1, \dots, i_{r_1-1}, j_1, \dots, j_{r_2+1}) = \chi_{\mathcal{M}_1}(i_1, \dots, i_{r_1-1}, j_1) \cdot \chi_{\mathcal{M}_2}(j_2, \dots, j_{r_2+1}) \tag{4}$$

and

$$\chi_{\mathcal{M}}(i_1, \dots, i_{r_1}, j_2, \dots, j_{r_2+1}) = \chi_{\mathcal{M}_1}(i_1, \dots, i_{r_1-1}, i_{r_1}) \cdot \chi_{\mathcal{M}_2}(j_2, \dots, j_{r_2+1}). \tag{5}$$

Combining equations (2), (3), (4) and (5) we obtain

$$C(i_{r_1}) \cdot C(j_1) = -\chi_{\mathcal{M}_1}(i_1, \dots, i_{r_1-1}, j_1) \cdot \chi_{\mathcal{M}_1}(i_1, \dots, i_{r_1-1}, i_{r_1}) = C_1(i_{r_1}) \cdot C_1(j_1).$$

Since $C_1(j_1) = C(j_1)$ then $C(i_{r_1}) = C_1(i_{r_1})$. The result follows by recursively carrying on the above argument. □

Lemma 1. *The directed switching game on $\mathcal{U}_{2,4}$ is winning for Maker playing first. Moreover, there are two winning choices for the first move of Maker.*

[1]An *uniform matroid*, denoted by $\mathcal{U}_{r,n}$, is a matroid with set of bases all r-subsets of a set of n elements.

Proof. Without loss of generality, we suppose that the given element is 1 and so Maker's goal is to create a positive circuit containing 1. We claim that Maker's goal can be achieved if there exists an element $f \in \{2,3,4\}$ such that either $C(1) \cdot C(f) = +$ or $C(1) \cdot C(f) = -$ for any circuit C containing 1 and f. Indeed, suppose that $C(1) \cdot C(f) = +$ and let C be a circuit containing 1 and f. In this case, Maker keeps the sign $C(f)$ and for every choice of Breaker the remaining element g belongs to a circuit C' containing 1 and f. Maker then signs g such that $C'(1) = C'(f) = C'(g)$ and thus circuit C' is positive. In the case when $C(1) \cdot C(f) = -$ we may proceed as above except that at the beginning Maker changes the sign of $C(f)$.

We now show that there are two elements verifying the above condition. Let us suppose that there are two different circuits $C_1 = \{1,2,3\}$ and $C_2 = \{1,2,4\}$ (containing elements 1 and 2) such that $C_1(1) = C_2(1) = +$. Then, $C_1(1) \cdot C_1(2) \neq C_2(1) \cdot C_2(2)$ and $C_1(2) \neq C_2(2)$. Without loss of generality we suppose that $C_1(2) = +$. Since $C_1 \neq C_2$ and $2 \in C_1^+ \cap C_2^-$ then there exists a circuit C_3 such that $C_3^+ \subset (C_1^+ \cup C_2^+) \setminus \{2\}$ and $C_3^- \subset (C_1^- \cup C_2^-) \setminus \{2\}$. So, $C_3 = \{1,3,4\}$ with $C_3(1) = C_1(1) = C_2(1)$, $C_3(3) = C_1(3)$ and $C_3(4) = C_2(4)$. Therefore, both elements 3 and 4 verify the above conditions. □

Theorem 1. *Let $\mathcal{M} = \cup_{i=1}^{p} \mathcal{M}_i$ where $\mathcal{M}_i$ is a uniform oriented matroid on E of rank $r_i = 1$ or 2 for each $1 \leq i \leq p$. Then, the directed switching game on $\mathcal{M}$ is winning for Maker playing first if and only if $|E| \geq 2r$ where $r = \sum_{i=1}^{p} r_i$ is the rank of $\mathcal{M}$.*

Proof. Let e be an element in $\mathcal{M}$. If $|E| < 2r$ then there are not enough elements for Maker to create a positive circuit containing e and so Maker loses. Suppose that $|E| = 2r$. Maker's strategy would be to construct a positive circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$ for each $1 \leq i \leq p$ such that

(a) $C_i \cap C_j = \emptyset$ for all $1 \leq i < j \leq p$ and $j \neq i+1$,

(b) the last element of C_i is the first element of C_{i+1} for each $1 \leq i \leq p-1$ (and this is the only element in common) and

(c) $e \in C_i$ for some $1 \leq i \leq p$.

The above strategy is winning for Maker since, by Proposition 1, it yields to the positive circuit $C = \cup_{i=1}^{p} C_i$ in $\mathcal{C}(\mathcal{M})$ containing e.

We partition E into p intervals I_i where

$$I_i = \begin{cases} (n_i - 3, n_i - 2, n_i - 1, n_i) & \text{if } r_i = 2, \\ (n_i - 1, n_i) & \text{if } r_i = 1 \end{cases}$$

with $n_i = 2\sum_{j=1}^{i} r_j$ for each $1 \leq i \leq p$ (notice that $n_p = 2r$ and that I_i contains $2r_i$ elements). Let us suppose that $e \in I_i$ for some $1 \leq i \leq p$. We have two cases.

Case A) If $r_i = 1$ then either $e = n_i$ or $n_i - 1$. Maker chooses either $e_M = n_i - 1$ or n_i. This permits Maker to sign n_i or $n_i - 1$ such that $\chi_{\mathcal{M}_i}(n_i) \cdot \chi_{\mathcal{M}_i}(n_i - 1) = -1$. Of course, by equation (2), the latter is done in order to have at the end $C_i(n_i) = C_i(n_i - 1)$ and thus constructing a positive circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$.

Now, if Breaker plays an element $e_B > n_i$ (resp. $e_B < n_i - 1$) then we set $E' = E \setminus \{e_M, e_B\}$ and $e' = n_i$ (resp. $e' = n_i - 1$) and restart our strategy with E' and e'. The latter is done so that in the next step, Maker will construct a positive circuit $C_{i+1} \in \mathcal{C}(\mathcal{M}_{i+1})$ whose first element will be the last element of C_i (resp. a positive circuit $C_{i-1} \in \mathcal{C}(\mathcal{M}_{i-1})$ whose last element will be the first element of C_i).

Case B) If $r_i = 2$ then, we have four subcases.

Subcase 1) If $e = n_i$ then Maker chooses either $n_i - 2$ or $n_i - 1$. By Lemma 1, there are two winning choices among $n_i - 1, n_i - 2, n_i - 3$ and thus either $n_i - 2$ or $n_i - 1$ is a winning choice for making a positive circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$. This winning element, say $e_M^1 \in \{n_i - 1, n_i - 2\}$ is chosen by Maker.

Now, the next move of Maker depends on the element that will be played by Breaker.

(**i**) If Breaker plays an element $e_B^1 < n_i$ then Maker will choose $e_M^2 = \max\{j \mid j < n_i\}$ creating the desired positive circuit $C_i = \{e_M^1, e_M^2, e\} \in \mathcal{C}(\mathcal{M}_i)$. Moreover, if Breaker plays next $e_B^2 < e_M^2 < n_i$ (resp. $e_B^2 > n_i$) then we set $E' = E \setminus \{e_M^1, n_i, e_B^1, e_B^2\}$ and $e' = e_M^2$ (resp. $E' = E \setminus \{e_M^1, e_M^2, e_B^1, e_B^2\}$ and $e' = n_i$) and restart our strategy with E' and e'. The latter is done so that in the next step, Maker will construct a positive circuit $C_{i-1} \in \mathcal{C}(\mathcal{M}_{i-1})$ whose last element will be the first element of C_i (resp. $C_{i+1} \in \mathcal{C}(\mathcal{M}_{i+1})$ whose first element will be the last element of C_i)

(**ii**) If Breaker plays an element $e_B^1 > n_i$ then we will form the circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$ consisting of elements $\{e_M^1, e_M^2, n_i\}$ where $e_M^2 \in \{n_i - 1, n_i - 2, n_i - 3\} \setminus \{e_M^1\}$. This will be done as soon as Breaker plays an element $e_B^2 < n_i$ (the final choice of e_M^2 will depend on the element played by Breaker since $e_M^2 \neq e_B^2$). In this case, we shall carry on our strategy by setting $E' = E \setminus \{e_M^1, e_B^1\}$, $e' = n_i$, and as soon as Breaker plays an element $e_B^2 < n_i$ (at any stage) circuit C_i will be completed right away (that is, Maker plays e_M^2 as above), elements e_B^2, e_M^2 will be deleted and Breaker restarts with a new move.

Remark 1. *(a) While Breaker keeps playing elements strictly bigger than n_i then our strategy will construct circuits C_j with $i+1 \leq j \leq p$.*

(b) We might have the case in which several circuits $C_{i_1}, \dots, C_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq i$ might be awaiting to be completed. If Breaker plays an element $e'_B < i_k$ then the circuit to be completed will be C_{i_j} where i_j is the smallest index such that $e'_B < i_j$.

The following subcase is completely symmetric, with respect to the interval $[n_i - 3, n_i - 2, n_i - 1, n_i]$, to subcase 1) and thus we may use analogous arguments.

Subcase 2) If $e = n_i - 3$ then Maker chooses either $n_i - 2$ or $n_i - 1$. By Lemma 1, there are two winning choices among $n_i - 1, n_i - 2, n_i - 3$ and thus either $n_i - 2$ or $n_i - 1$ is a winning choice for making a positive circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$. This winning element, say $e_M^1 \in \{n_i - 1, n_i - 2\}$ is chosen by Maker.

Now, the next move of Maker depends on the element that will be played by Breaker.

(**i**) If Breaker plays an element $e_B^1 > n_i - 3$ then Maker will choose $e_M^2 = \min\{j \mid j > n_i - 3\}$ creating the desired positive circuit $C_i = \{e_M^1, e_M^2, e\} \in \mathcal{C}(\mathcal{M}_i)$. Moreover, if Breaker plays next $e_B^2 > e_M^2 > n_i - 3$ (resp. $e_B^2 < n_i - 3$) then we set $E' = E \setminus \{e_M^1, n_i - 3, e_B^1, e_B^2\}$ and $e' = e_M^2$ (resp. $E' = E \setminus \{e_M^1, e_M^2, e_B^1, e_B^2\}$ and $e' = n_i - 3$) and restart our strategy with E' and e'. The latter is done so that in the next step, Maker will construct a positive circuit $C_{i+1} \in \mathcal{C}(\mathcal{M}_{i+1})$ whose first element will be the last element of C_i (resp. $C_{i-1} \in \mathcal{C}(\mathcal{M}_{i-1})$ whose last element will be the first element of C_i)

(**ii**) If Breaker plays an element $e_B^1 < n_i - 3$ then we will form the circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$ consisting of elements $\{e_M^1, e_M^2, n_i\}$ where $e_M^2 \in \{n_i - 1, n_i - 2, n_i\} \setminus \{e_M^1\}$. This will be done as soon as Breaker plays an element $e_B^2 > n_i - 3$ (the final choice of e_M^2 will depend on the element played by Breaker since $e_M^2 \neq e_B^2$). In this case, we shall carry on our strategy by setting $E' = E \setminus \{e_M^1, e_B^1\}$, $e' = n_i - 3$, and as soon as Breaker plays an element $e_B^2 > n_i - 3$ (at any stage) circuit C_i will be completed right away (that is, Maker plays e_M^2 as above), elements e_B^2, e_M^2 will be deleted and Breaker restarts with a new move.

Remark 2. *(a) While Breaker keeps playing elements strictly smaller than* $n_i - 3$ *then our strategy will construct circuits* C_j *with* $1 \leq j \leq i-1$.

(b) We might have the case in which several circuits $C_{i_1}, \ldots, C_{i_k}$ *with* $i+1 \leq i_1 < \cdots < i_k \leq p$ *are awaiting to be completed. If Breaker plays an element* $e'_B > i_k$ *then the circuit to be completed will be* C_{i_j} *where* i_j *is the largest index such that* $e'_B > i_j$.

Subcase 3) If $e = n_i - 1$ then Maker chooses either $n_i - 2$ or n_i. By Lemma 1, there are two winning choices among $n_i, n_i - 2, n - 3$ and thus either $n_i - 2$ or n_i is a winning choice for making a positive circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$. This winning element, say $e_M^1 \in \{n_i, n_i - 2\}$ is chosen by Maker.

If $e_M^1 = n_i$ then we are back to subcase 1). Let us suppose then that $e_M^1 = n_i - 2$. If Breaker plays element $e_B^1 < n_i - 2$ (resp. $e_B^1 > n_i - 1$) then Maker plays $e_M^2 = n_i$ (resp. $e_M^2 = n_i - 3$).

Moreover, if Breaker plays next $e_B^2 > n_i$ (resp. $e_B^2 < n_i - 3$) then we set $E' = E \setminus \{e_M^1, e_M^2, e_B^1, e_B^2\}$ and $e' = n_i$ (resp. $E' = E \setminus \{e_M^1, e_M^2, e_B^1, e_B^2\}$ and $e' = n_i - 3$) and restart our strategy with E' and e'. The latter is done so that in the next step, Maker will construct a positive circuit $C_{i+1} \in \mathcal{C}(\mathcal{M}_{i+1})$ whose first element will be the last element of C_i (resp. $C_{i-1} \in \mathcal{C}(\mathcal{M}_{i-1})$ whose last element will be the first element of C_i)

The following subcase is completely symmetric, with respect to the interval $[n_i - 3, n_i - 2, n_i - 1, n_i]$, to subcase 3) and thus we may use analogous arguments.

Subcase 4) If $e = n_i - 2$ then Maker chooses either $n_i - 3$ or $n_i - 1$. By Lemma 1, there are two winning choices among $n_i, n_i - 2, n - 3$ and thus either $n_i - 3$ or $n_i - 1$ is a winning choice for making a positive circuit $C_i \in \mathcal{C}(\mathcal{M}_i)$. This winning element, say $e_M^1 \in \{n_i - 1, n_i - 3\}$, is chosen by Maker.

If $e_M^1 = n_i - 3$ then we are back to subcase 2). Let us suppose then that $e_M^1 = n_i - 1$. If Breaker plays element $e_B^1 > n_i - 2$ (resp. $e_B^1 < n_i - 1$) then Maker plays $e_M^2 = n_i - 3$ (resp. $e_M^2 = n_i$).

Moreover, if Breaker plays next $e_B^2 < n_i - 3$ (resp. $e_B^2 > n_i$) then we set $E' = E \setminus \{e_M^1, e_M^2, e_B^1, e_B^2\}$ and $e' = n_i - 3$ (resp. $E' = E \setminus \{e_M^1, e_M^2, e_B^1, e_B^2\}$ and $e' = n_i$) and restart our strategy with E' and e'. The latter is done so that in the next step, Maker will construct a positive circuit $C_{i-1} \in \mathcal{C}(\mathcal{M}_{i-1})$ whose last element will be the first element of C_i (resp. $C_{i+1} \in \mathcal{C}(\mathcal{M}_{i+1})$ whose first element will be the last element of C_i).

Notice that after r moves of Maker, the desired positive circuit is formed. We finally notice that in the case when $|E| > 2r$ then Maker first select a subset A from $\{1,\ldots,|E|\}$ such that $|A| = 2r$ and $e \in A$ and then uses *fictitious* moves as done in [3]. That is, Maker applies the above strategy to A by choosing an element for Breaker in the case Breaker plays outside A. □

2.1. **Concluding remarks.** We have checked by computer that there is always a winning strategy for Maker (playing first) for any oriented matroid of rank 3 and rank 4 with 6 and 8 elements respectively (supporting further Conjecture 1). Moreover, we noticed that in each case there are always at least two winning choices for the first move of Maker (similar to Lemma 1, that was a key ingredient for the proof of our main result). The latter leads us to consider a possible extension of Theorem 1 by taking union of uniform oriented matroids of rank 1, 2, 3 or 4 (work in progress).

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