Packing of Rigid Spanning Subgraphs and Spanning Trees

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Abstract

We prove that every $(6k+2\ell, 2k)$-connected simple graph contains k rigid and ℓ connected edge-disjoint spanning subgraphs. This implies a theorem of Jackson and Jordán [4] and a theorem of Jordán [6] on packing of rigid spanning subgraphs. Both these results are generalizations of the classical result of Lovász and Yemini [9] saying that every 6-connected graph is rigid for which our approach provides a transparent proof. Our result also gives two improved upper bounds on the connectivity of graphs that have interesting properties: (1) every 8-connected graph packs a spanning tree and a 2-connected spanning subgraph; (2) every 14-connected graph has a 2-connected orientation.

1 Definitions

Let $G = (V, E)$ be a graph. We will use the following connectivity concepts. G is called **connected** if for every pair u, v of vertices there is a path from u to v in G. G is called **k-edge-connected** if $G - F$ is connected for all $F \subseteq E$ with $|F| \leq k-1$. G is called **k-connected** if $|V| > k$ and $G - X$ is connected for all $X \subset V$ with $|X| \leq k-1$. For a pair of positive integers (p, q), G is called **(p, q)-connected** if $G - X$ is $(p - q|X|)$-edge-connected for all $X \subset V$. By Menger theorem, G is (p, q)-connected if and only if for every pair of disjoint subsets X, Y of V such that $Y \neq \emptyset, X \cup Y \neq V$,

$$d_{G-X}(Y) \geq p - q|X|. \tag{1}$$

For a better understanding we mention that G is $(6, 2)$-connected if G is 6-edge-connected, $G - v$ is 4-edge-connected for all $v \in V$ and $G - u - v$ is 2-edge-connected for all $u, v \in V$. It follows from the definitions that k-edge-connectivity is equivalent to (k, k)-connectivity. Moreover, since loops and parallel edges do not play any role in vertex connectivity, every k-connected graph contains a $(k, 1)$-connected simple spanning subgraph. Note also that $(k, 1)$-connectivity implies (k, q)-connectivity for all $q \geq 1$. (Remark that this connectivity concept

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is (very slightly) different from the one introduced by Kaneko and Ota [7] since p is not required to be a multiple of q.)

Let $D = (V, A)$ be a directed graph. D is called **strongly connected** if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed path from u to v in D. D is called k**-arc-connected** if $G - F$ is strongly connected for all $F \subseteq A$ with $|F| \leq k - 1$. D is called k**-connected** if $|V| > k$ and $G - X$ is strongly connected for all $X \subset V$ with $|X| \leq k - 1$.

For a set X of vertices and a set F of edges, denote $\boldsymbol{G_F}$ the subgraph of G on vertex set V and edge set F, that is $G_F = (V, F)$ and $\boldsymbol{E(X)}$ the set of edges of G induced by X. Denote $\boldsymbol{\mathcal{R}(G)}$ the **rigidity matroid** of G on ground-set E with rank function $\boldsymbol{r_{\mathcal{R}}}$ (for a definition we refer the reader to [9]). For $F \subseteq E$, by a theorem of Lovász and Yemini [9],

$$r_{\mathcal{R}}(F) = \min \sum_{X \in \mathcal{H}} (2|X| - 3), \tag{2}$$

where the minimum is taken over all collections $\mathcal{H}$ of subsets of V such that $\{E(X) \cap F, X \in \mathcal{H}\}$ partitions F.

Remark 1. *If $\mathcal{H}$ achieves the minimum in (2), then each $X \in \mathcal{H}$ induces a connected subgraph of G_F.*

We will say that G is **rigid** if $r_{\mathcal{R}}(E) = 2|V| - 3$.

2 Results

Lovász and Yemini [9] proved the following sufficient condition for a graph to be rigid.

Theorem 1 (Lovász and Yemini [9])**.** *Every 6-connected graph is rigid.*

Jackson and Jordán [4] proved a sharpenning of Theorem 1.

Theorem 2 (Jackson and Jordán [4])**.** *Every $(6, 2)$-connected simple graph is rigid.*

Jordán [6] generalized Theorem 1 and gave a sufficient condition for the existence of a packing of rigid spanning subgraphs.

Theorem 3 (Jordán [6])**.** *Let $k \geq 1$ be an integer. Every $6k$-connected graph contains k edge-disjoint rigid spanning subgraphs.*

The main result of this paper contains a common generalization of Theorems 2 and 3. It provides a sufficient condition to have a packing of rigid spanning subgraphs and spanning trees.

Theorem 4. *Let $k \geq 1$ and $\ell \geq 0$ be integers. Every $(6k + 2\ell, 2k)$-connected simple graph contains k rigid spanning subgraphs and ℓ spanning trees pairwise edge-disjoint.*

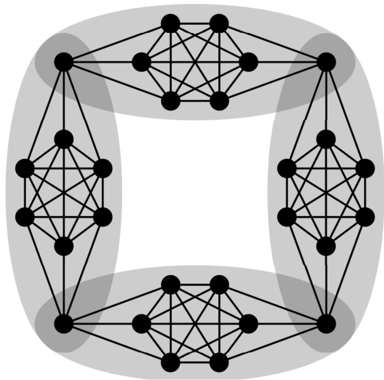

Figure 1: A $(6,3)$-connected non-rigid graph $G=(V,E)$. The collection $\mathcal{H}$ of the four grey vertex-sets partitions E. Hence, by (2), $\mathcal{R}_G(E) \leq \sum_{X\in\mathcal{H}}(2|X|-3) = 4(2\times 8-3) = 52 < 53 = 2\times 28-3 = 2|V|-3$. Thus G is not rigid. The reader can easily check that G is $(6,3)$-connected.

Note that in Theorem 2, the connectivity condition is the best possible since there exist non-rigid $(5,2)$-connected graphs (see [9]) and non-rigid $(6,3)$-connected graphs, for an example see Figure 1.

Let us see some corollaries of the previous results. Theorem 4 applied for $k=1$ and $\ell=0$ provides Theorem 2. Since $6k$-connectivity implies $(6k,2k)$-connectivity of a simple spanning subgraph, Theorem 4 implies Theorem 3.

One can easily derive from the rank function of $\mathcal{R}(G)$ that rigid graphs with at least 3 vertices are 2-connected (see Lemma 2.6 in [5]). Thus, Theorem 4 gives the following corollary.

Corollary 1. *Let $k \geq 1$ and $\ell \geq 0$ be integers. Every $(6k+2\ell, 2k)$-connected simple graph contains k 2-connected and ℓ connected edge-disjoint spanning subgraphs.*

Corollary 1 allows us to improve two results of Jordán. The first one deals with the following conjecture of Kriesell, see in [6].

Conjecture 1 (Kriesell)**.** *For every positive integer λ there exists a (smallest) $f(\lambda)$ such that every $f(\lambda)$-connected graph G contains a spanning tree T for which $G-E(T)$ is λ-connected.*

As Jordán pointed out in [6], Theorem 3 answers this conjecture for $\lambda=2$ by showing that $f(2)\leq 12$. Corollary 1 applied for $k=1$ and $\ell=1$ directly implies that $f(2)\leq 8$.

Corollary 2. *Every 8-connected graph G contains a spanning tree T such that $G-E(T)$ is 2-connected.*

The other improvement deals with the following conjecture of Thomassen [10].

Conjecture 2 (Thomassen [10])**.** *For every positive integer λ there exists a (smallest) $g(\lambda)$ such that every $g(\lambda)$-connected graph G has a λ-connected orientation.*

By applying Theorem 3 and an orientation result of Berg and Jordán [1], Jordán proved in [6] the conjecture for $\lambda = 2$ by showing that $g(2) \leq 18$. Corollary 1 allows us to prove a general result that implies $g(2) \leq 14$. For this purpose, we use a result of Király and Szigeti [8].

Theorem 5 (Király and Szigeti [8])**.** *An Eulerian graph $G = (V, E)$ has an Eulerian orientation D such that $D - v$ is k-arc-connected for all $v \in V$ if and only if $G - v$ is $2k$-edge-connected for all $v \in V$.*

Corollary 1 and Theorem 5 imply the following corollary which gives the claimed bound for $k = 1$.

Corollary 3. *Every simple $(12k + 2, 2k)$-connected graph G has an orientation D such that $D - v$ is k-arc-connected for all $v \in V$.*

Proof. Let $G = (V, E)$ be a simple $(12k + 2, 2k)$-connected graph. By Theorem 5 it suffices to prove that G contains an Eulerian spanning subgraph H such that $H - v$ is $2k$-edge-connected for all $v \in V$. By Corollary 1, G contains $2k$ 2-connected spanning subgraphs $H_i = (V, E_i), i = 1, \dots, 2k$ and a spanning tree F pairwise edge-disjoint. Define $H' = (V, \cup_{i=1}^{2k} E_i)$. For all $i = 1, \dots, 2k$, since H_i is 2-connected, $H_i - v$ is connected; hence $H' - v$ is $2k$-edge-connected for all $v \in V$. Denote T the set of vertices of odd degree in H'. We say that F' is a ***T*-join** if the set of odd degree vertices of $G_{F'}$ coincides with T. It is well-known that the connected graph F contains a T-join. Thus adding the edges of this T-join to H' provides the required spanning subgraph of G. ■

Finally we mention that the following conjecture of Frank, that would give a necessary and sufficient condition for a graph to have a 2-connected orientation, would imply that $g(2) \leq 4$.

Conjecture 3 (Frank [3])**.** *A graph has a 2-connected orientation if and only if it is $(4, 2)$-connected.*

3 Proofs

To prove Theorem 4 we need to introduce two other matroids on the edge set E of G. Denote $\boldsymbol{\mathcal{C}(G)}$ the **circuit matroid** of G on ground-set E with rank function $\boldsymbol{r_{\mathcal{C}}}$ given by (3). Let n be the number of vertices in G, that is $\boldsymbol{n = |V|}$. For $F \subseteq E$, denote $\boldsymbol{c(G_F)}$ the number of connected components of G_F, it is well known that,

$$r_{\mathcal{C}}(F) = n - c(G_F). \tag{3}$$

To have k rigid spanning subgraphs and ℓ spanning trees pairwise edge-disjoint in G, we must find k basis in $\mathcal{R}(G)$ and ℓ basis in $\mathcal{C}(G)$ pairwise disjoint. To do that we will need the following matroid. For $k \geq 1$ and $\ell \geq 0$, define $\boldsymbol{\mathcal{M}_{k,\ell}(G)}$ as the matroid on ground-set E, obtained by taking the matroid union of k copies of the rigidity matroid $\mathcal{R}(G)$ and ℓ copies of the circuit matroid $\mathcal{C}(G)$. Let $\boldsymbol{r_{\mathcal{M}_{k,\ell}}}$ be the rank function of $\mathcal{M}_{k,\ell}(G)$. By a theorem of Edmonds [2], for the rank of matroid unions,

$$r_{\mathcal{M}_{k,\ell}}(E) = \min_{F \subseteq E} k r_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus F|. \tag{4}$$

In [6], Jordán used the matroid $\mathcal{M}_{k,0}(G)$ to prove Theorem 3 and pointed out that using $\mathcal{M}_{k,\ell}(G)$ one could prove a theorem on packing of rigid spanning subgraphs and spanning trees. We tried to fulfill this gap by following the proof of [6] but we failed. To achieve this aim we had to find a new proof technique. Let us first demonstrate this technique by giving a transparent proof for Theorems 1 and 2.

Proof of Theorem 1. By (2), there exists a collection $\mathcal{G}$ of subsets of V such that $\{E(X), X \in \mathcal{G}\}$ partitions E and $r_{\mathcal{R}}(E) = \sum_{X\in\mathcal{G}}(2|X|-3)$. If $V \in \mathcal{G}$ then $r_{\mathcal{R}}(E) \geq 2|V| - 3$ hence G is rigid. So in the following we may assume that $V \notin \mathcal{G}$.

Let $\mathcal{H} = \{X \in \mathcal{G} : |X| \geq 3\}$ and $F = \bigcup_{X\in\mathcal{H}} E(X)$. We define, for $X \in \mathcal{H}$, the border of X as $X_B = X \cap (\cup_{Y\in\mathcal{H}-X} Y)$ and the proper part of X as $X_I = X \setminus X_B$ and $\mathcal{H}' = \{X \in \mathcal{H} : X_I \neq \emptyset\}$.

Since every edge of F is induced by an element of $\mathcal{H}$, for $X \in \mathcal{H}'$, by definition of X_I, no edge of F contributes to $d_{G-X_B}(X_I)$; and for a vertex $v \in V - V(\mathcal{H})$, no edge of F contributes to $d_G(v)$. Thus, since for $X \in \mathcal{H}'$, $X_I \neq \emptyset$ and $X_I \cup X_B = X \neq V$, by 6-connectivity of G, we have $|E \setminus F| \geq \frac{1}{2}(\sum_{X\in\mathcal{H}'} d_{G-X_B}(X_I) + \sum_{v\in V-V(\mathcal{H})} d_G(v)) \geq \frac{1}{2}(\sum_{X\in\mathcal{H}'}(6 - |X_B|) + \sum_{v\in V-V(\mathcal{H})} 6) \geq 3|\mathcal{H}'| - \sum_{X\in\mathcal{H}'} |X_B| + 3(|V| - |V(\mathcal{H})|)$.

Since for $X \in \mathcal{H} \setminus \mathcal{H}'$, $|X_B| = |X| \geq 3$, we have $\sum_{X\in\mathcal{H}}(2|X| - 3) = \sum_{X\in\mathcal{H}} 2|X| - 3|\mathcal{H}| + 3|\mathcal{H}'| - 3|\mathcal{H}'| \geq \sum_{X\in\mathcal{H}} 2|X| - \sum_{X\in\mathcal{H}\setminus\mathcal{H}'} |X_B| - 3|\mathcal{H}'|$.

Since G is simple, by Remark 1 every $X \in \mathcal{G}$ of size 2 induces exactly one edge. Hence, by the above inequalities, we have $\sum_{X\in\mathcal{G}}(2|X|-3) = \sum_{X\in\mathcal{H}}(2|X|-3) + |E \setminus F| \geq \sum_{X\in\mathcal{H}} 2|X| - \sum_{X\in\mathcal{H}} |X_B| + 3(|V| - |V(\mathcal{H})|) = (\sum_{X\in\mathcal{H}} 2|X_I| + \sum_{X\in\mathcal{H}} |X_B| - 2|V(\mathcal{H})|) + (|V| - |V(\mathcal{H})|) + 2|V| \geq 2|V|$.

To see the last inequality, let $v \in V(\mathcal{H})$. Then $v \in V$ and hence $n - |V(\mathcal{H})| \geq 0$. If v belongs to exactly one $X' \in \mathcal{H}$, then $v \in X'_I$; so v contributes 2 in $\sum_{X\in\mathcal{H}} 2|X_I|$. If v belongs to at least two $X', X'' \in \mathcal{H}$, then $v \in X'_B$ and $v \in X''_B$; so v contributes at least 2 in $\sum_{X\in\mathcal{H}} |X_B|$ and hence $\sum_{X\in\mathcal{H}} 2|X_I| + \sum_{X\in\mathcal{H}} |X_B| - 2|V(\mathcal{H})| \geq 0$.

Hence $2|V| - 3 \geq r_{\mathcal{R}}(E) \geq 2|V|$, a contradiction. ■

Proof of Theorem 2. Note that in the lower bound on $|E \setminus F|$, $d_{G-X_B}(X_I) \geq 6 - |X_B|$ can be replaced by $d_{G-X_B}(X_I) \geq 6 - 2|X_B|$, and the same proof works. This means that instead of 6-connectivity, we used in fact $(6, 2)$-connectivity. ■

Proof of Theorem 4. Suppose that there exist integers k, ℓ and a graph $G = (V, E)$ contradicting the theorem. We use the matroid $\mathcal{M}_{k,\ell}$ defined above. Choose F a smallest-size set of edges that minimizes the right hand side of (4). By (2), we can define $\mathcal{H}$ a collection of subsets of V such that $\{E(X) \cap F, X \in \mathcal{H}\}$ partitions F and $r_{\mathcal{R}}(F) = \sum_{X\in\mathcal{H}}(2|X| - 3)$. Since G is a counterexample and by (2) and (3),

$$k(2n-3)+\ell(n-1) > r_{\mathcal{M}_{k,\ell}}(E) = k \sum_{X\in\mathcal{H}} (2|X|-3)+\ell(n-c(G_F))+|E\setminus F|. \quad (5)$$

By $k \geq 1$, G is connected, thus, by (5), $V \notin \mathcal{H}$. Recall the notations, for $X \in \mathcal{H}$, $X_B = X \cap (\cup_{Y\in\mathcal{H}-X} Y)$ and $X_I = X \setminus X_B$ and the definition $\mathcal{H}' = \{X \in$

$\mathcal{H} : X_I \neq \emptyset\}$. Denote $\mathcal{K}$ the set of connected components of G_F intersecting no set of $\mathcal{H}'$. By Remark 1, for $X \in \mathcal{H}'$, X induces a connected subgraph of G_F, thus a connected component of G_F intersecting $X \in \mathcal{H}'$ contains X and is the only connected component of G_F containing X. So by definition of $\mathcal{K}$,

$$|\mathcal{H}'| \geq c(G_F) - |\mathcal{K}|. \tag{6}$$

Let us first show a lower bound on $|E \setminus F|$.

Claim 1. $|E \setminus F| \geq k\Big(3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|\mathcal{K}|\Big) + \ell c(G_F)$.

Proof. For $X \in \mathcal{H}'$, $X_I \neq \emptyset$ and $X_I \cup X_B = X \neq V$. Thus by $(6k + 2\ell, 2k)$-connectivity of G, for $X \in \mathcal{H}'$ and for $K \in \mathcal{K}$,

$$d_{G-X_B}(X_I) \geq (6k + 2\ell) - 2k|X_B|, \tag{7}$$
$$d_G(K) \geq 6k + 2\ell. \tag{8}$$

Since every edge of F is induced by an element of $\mathcal{H}$ and by definition of X_I, for $X \in \mathcal{H}'$, no edge of F contributes to $d_{G-X_B}(X_I)$. Each $K \in \mathcal{K}$ is a connected component of the graph G_F, thus no edge of F contributes to $d_G(K)$. Hence, by (7), (8), (6) and $\ell \geq 0$, we obtain the required lower bound on $|E \setminus F|$,

$$\begin{aligned}
|E \setminus F| &\geq \frac{1}{2}\Bigg(\sum_{X \in \mathcal{H}'} d_{G-X_B}(X_I) + \sum_{K \in \mathcal{K}} d_G(K)\Bigg) \\
&\geq \frac{1}{2}\Bigg((6k + 2\ell)|\mathcal{H}'| - 2k \sum_{X \in \mathcal{H}'} |X_B| + (6k + 2\ell)|\mathcal{K}|\Bigg) \\
&\geq k\Bigg(3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|\mathcal{K}|\Bigg) + \ell\Big(|\mathcal{H}'| + |\mathcal{K}|\Big) \\
&\geq k\Bigg(3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|\mathcal{K}|\Bigg) + \ell c(G_F).
\end{aligned}$$

■

Claim 2. $\sum_{X \in \mathcal{H} \setminus \mathcal{H}'} |X_B| \geq 3(|\mathcal{H}| - |\mathcal{H}'|)$.

Proof. By definition of $\mathcal{H}'$, $X_B = X$ for all $X \in \mathcal{H} \setminus \mathcal{H}'$. So to prove the claim it suffices to show that every $X \in \mathcal{H}$ satisfies $|X| \geq 3$. Suppose there exists $Y \in \mathcal{H}$ such that $|Y| = 2$. By Remark 1 and since G is simple, Y induces exactly one edge e. Define $F'' = F - e$ and $\mathcal{H}'' = \mathcal{H} - Y$. Note that $\{E(X) \cap F'', X \in \mathcal{H}''\}$ partitions F'', hence by (2) and the choice of $\mathcal{H}$,

$$r_{\mathcal{R}}(F'') \leq \sum_{X \in \mathcal{H}''} (2|X| - 3) = r_{\mathcal{R}}(F) - (2|Y| - 3) = r_{\mathcal{R}}(F) - 1. \tag{9}$$

Note also that $c(G_{F''}) \geq c(G_F)$, thus by (3) and $\ell \geq 0$,

$$\ell r_{\mathcal{C}}(F'') \leq \ell r_{\mathcal{C}}(F). \tag{10}$$

Since $|F''| < |F|$, the choice of F implies that F'' doesn't minimizes the right hand side of (4). Hence by (9), (10), the definition of F'', $|Y| = 2$, and $k \geq 1$, we have the following contradiction:

$$\begin{aligned}
0 &< \Big(kr_{\mathcal{R}}(F'') + \ell r_{\mathcal{C}}(F'') + |E \setminus F''|\Big) - \Big(kr_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus F|\Big) \\
&= k\Big(r_{\mathcal{R}}(F'') - r_{\mathcal{R}}(F)\Big) + \ell\Big(r_{\mathcal{C}}(F'') - r_{\mathcal{C}}(F)\Big) + \Big(|E \setminus F''| - |E \setminus F|)\Big) \\
&\leq -k + 0 + |\{e\}| \\
&\leq 0.
\end{aligned}$$

■

To finish the proof we show the following inequality with a simple counting argument.

Claim 3. $2|\mathcal{K}| + \sum_{X\in\mathcal{H}} 2|X_I| + \sum_{X\in\mathcal{H}} |X_B| \geq 2n$.

Proof. Let $v \in V$. If v belongs to no $X \in \mathcal{H}$, then $\{v\} \in \mathcal{K}$ and v contributes 2 in $2|\mathcal{K}|$. If v belongs to exactly one $X' \in \mathcal{H}$, then $v \in X'_I$ and v contributes 2 in $\sum_{X\in\mathcal{H}} 2|X_I|$. If v belongs to at least two $X', X'' \in \mathcal{H}$, then $v \in X'_B, v \in X''_B$ and v contributes at least 2 in $\sum_{X\in\mathcal{H}} |X_B|$. The claim follows. ■

Thus we get, by Claims 1, 2 and 3,

$$\begin{aligned}
&k \sum_{X\in\mathcal{H}} (2|X| - 3) + |E \setminus F| + \ell(n - c(G_F)) \\
&\geq k \sum_{X\in\mathcal{H}} 2|X| - 3k|\mathcal{H}| + k\Big(3|\mathcal{H}'| - \sum_{X\in\mathcal{H}'} |X_B| + 3|\mathcal{K}|\Big) + \ell c(G_F) + \ell(n - c(G_F)) \\
&\geq k\Big(\sum_{X\in\mathcal{H}} 2|X| - 3|\mathcal{H}| + 3|\mathcal{H}'| - \sum_{X\in\mathcal{H}'} |X_B| + 3|\mathcal{K}|\Big) + \ell n \\
&\geq k\Big(\sum_{X\in\mathcal{H}} 2|X| - \sum_{X\in\mathcal{H}} |X_B| + 2|\mathcal{K}|\Big) + \ell n \\
&\geq k\Big(2|\mathcal{K}| + \sum_{X\in\mathcal{H}} 2|X_I| + \sum_{X\in\mathcal{H}} |X_B|\Big) + \ell n \\
&\geq 2kn + \ell n.
\end{aligned}$$

By $k \geq 1$ and $\ell \geq 0$, this contradicts (5). ■

Remark that the proof actually shows that if G is simple and $(6k + 2\ell, 2k)$-connected and if $F \subseteq E$ is such that $|F| \leq 3k + \ell$, then $G' = (V, E \setminus F)$ contains k rigid spanning subgraphs and ℓ spanning trees pairwise edge disjoint.

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