Knots and the Cyclic Polytope

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Spatial graphs

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Spatial representation of K_5.

Let $m(L)$ be the smallest integer such that any spatial representation of K_n with $n \geq m(L)$ contains cycles isotopic to L.

Question (Bothe 1973) Is it true that $m(2_1^2) = 6$?

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- For any spatial representation of K_6, it holds

$$\sum_{(\lambda_1,\lambda_2)} lk(\lambda_1,\lambda_2) \equiv 1 \bmod 2$$

where (λ_1,λ_2) is a 2-component link contained in K_6 and lk denotes the linking number.

- For any spatial representation of K_7, it holds

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where λ is a 7-cycle of K_7 and Arf denotes the Arf invariant.

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A spatial representation is linear if the curves are line segments.

Let $\bar{m}(L)$ be the smallest integer such that any spatial linear representation of K_n with $n \geq \bar{m}(L)$ contains cycles isotopic to L.

Let $s(L)$ be the smallest number of segments needed to represent link L.

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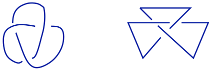
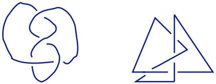
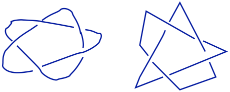
Trefoil
(*T*)



Figure–eight
F_8

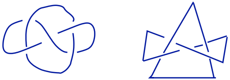


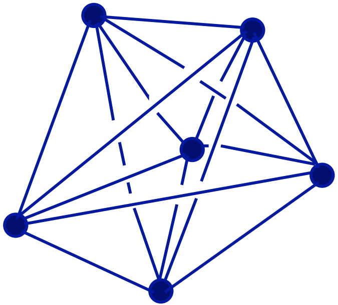
$T(5,2)$

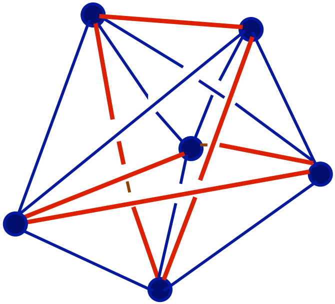


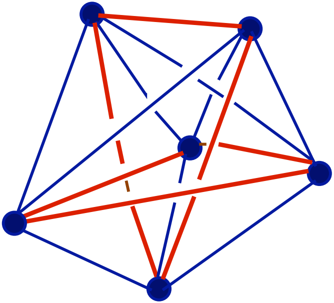
Hopf link
2^2_1

4^2_1









Theorem (Negami 1991) $\bar{m}(L)$ exists and it is finite for any link L.

Oriented matroids

Let E a finite set. An oriented matroid is a family $\mathcal{C}$ of signed subsets of E verifying certain axioms (the family $\mathcal{C}$ is called the circuits of the oriented matroid).

There is a natural way to obtain an oriented matroid from a configuration of points in $\mathbb{R}^d$.

If $C \in \mathcal{C}$ $conv(\text{pos. elements } C) \cap conv(\text{neg. elements } C) \neq \emptyset$.

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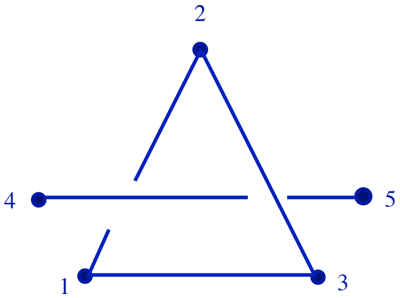
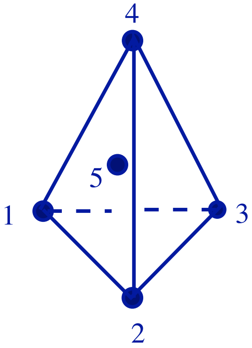
Example : $d = 3$.



(+,+,+,−,−)



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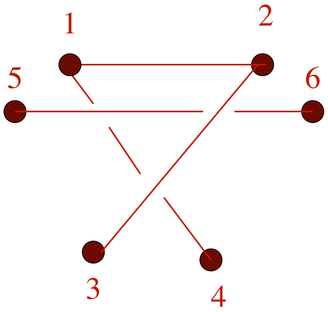

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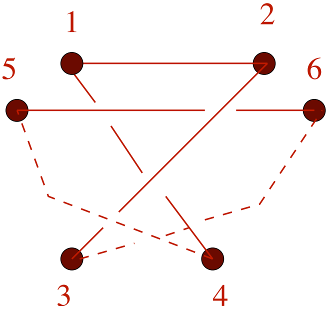
Proof (idea) : Give conditions on circuits in order to have the desired knot.



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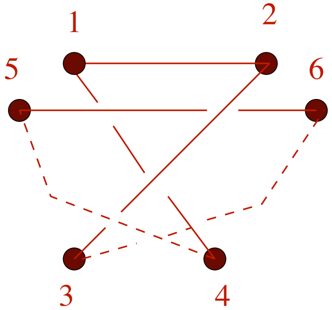

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Cyclic Polytope

Let $t_1, \ldots, t_n \in \mathbb{R}$. The cyclic polytope of dimension d with n vertices is defined as

$$C_d(t_1, \ldots, t_n) := conv(x(t_1), \ldots, x(t_n))$$

where $x(t_i) = (t_i, t_i^2, \ldots, t_i^d)$ are points of the moment curve

$$C_d(t_1, \ldots, t_n) \rightarrow C_d(n)$$

Upper bound theorem (McMullen 1970) The number of j-faces of a d-dimensional polytope with n vertices is maximal for $C_d(n)$.

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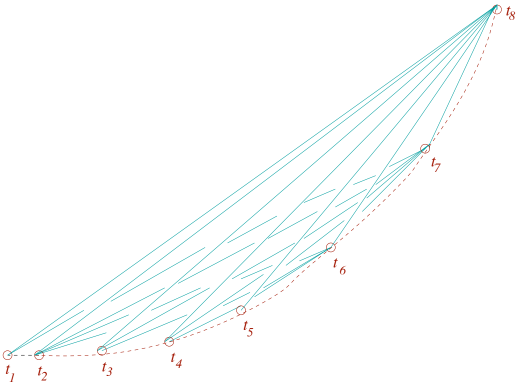
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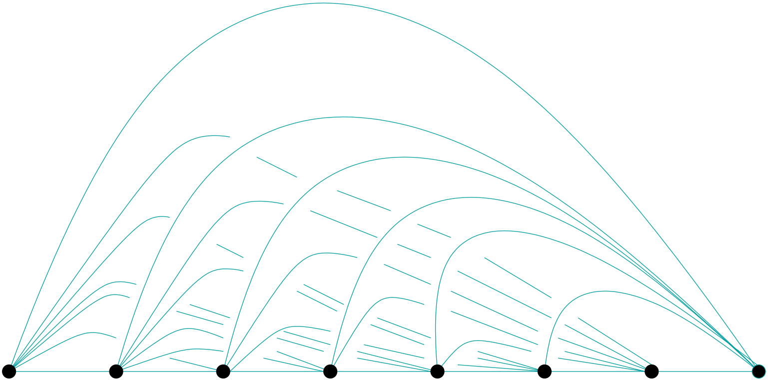

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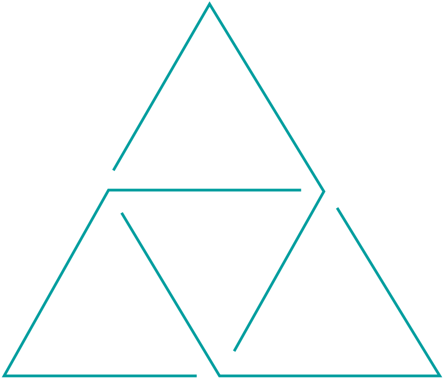

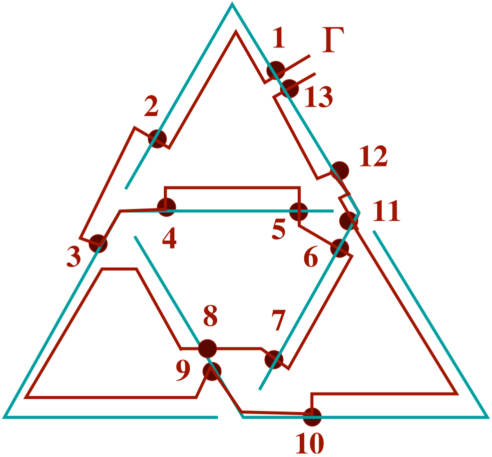
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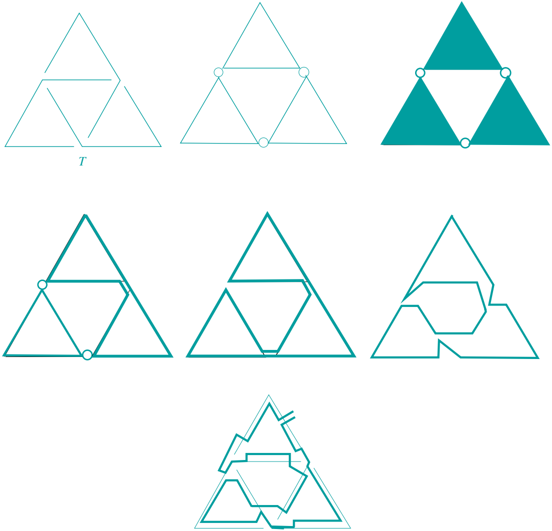
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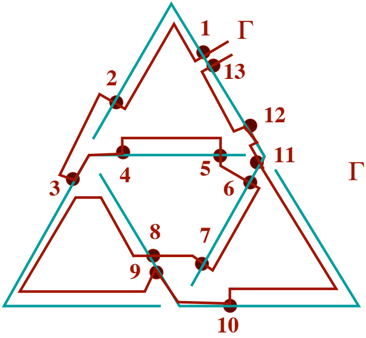
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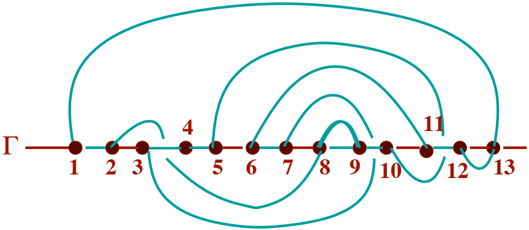
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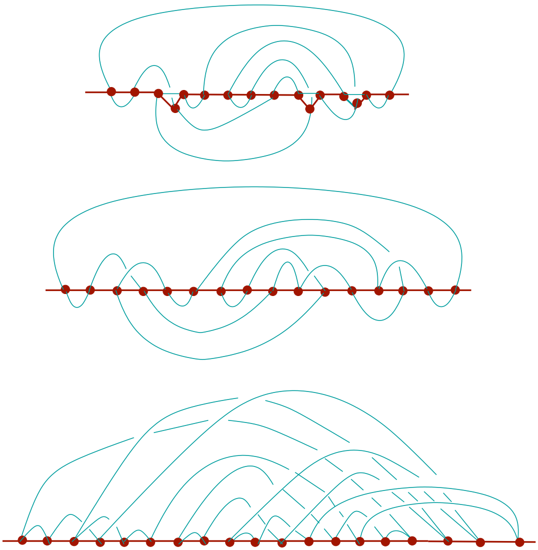

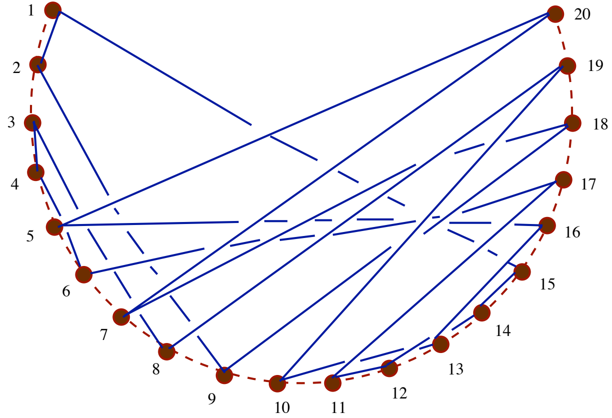
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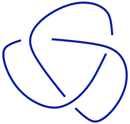

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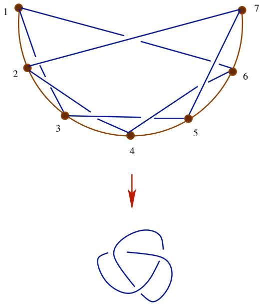

Knots physical models

For a given diameter, one needs certain minimum length of rope in order to tie a (nontrivial) knot.

Moreover, the more complicated the knot you want to tie, the more rope you need.

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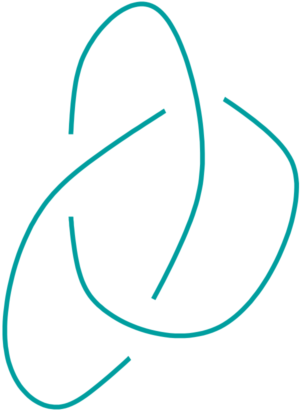

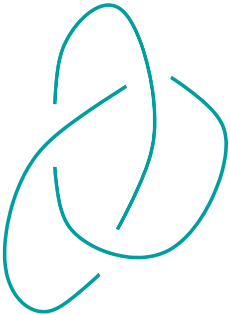

A number $r > 1$ is nice if for any distinct points x and y on K we have $D(x, r) \cap D(y, r) = \emptyset$. The disk thickness of K is defined to be $t(K) = sup\{r | r \text{ is nice}\}$.

A thick realization K_0 of K is a knot of unit thickness which is of the same type as K.

The rope length $L(K)$ of K is the infimum of the length of K_0 taken over all thick realizations of K.

Theorem (Cantarella, Kusner and Sullivan 2002) $L(K)$ exists.

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Theorem (Diao, Ernst and Yu 2004) There exists a constant c such that for any knot K

$$L(K) \leq c \cdot (cr(K))^{3/2}$$

where $cr(K)$ is the crossing number of K.

The cubic lattice consists of all points in $\mathbb{R}^3$ with integral coordinates and all unit line segments joining these points.

A cubic lattice knot is a polygonal knot represented in the cubic lattice.

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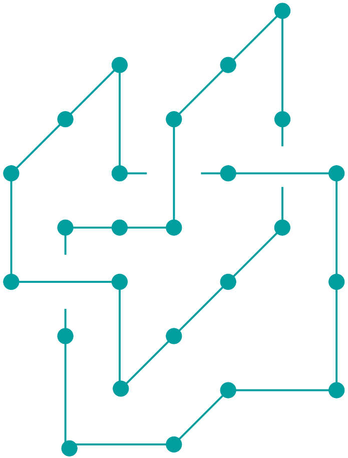
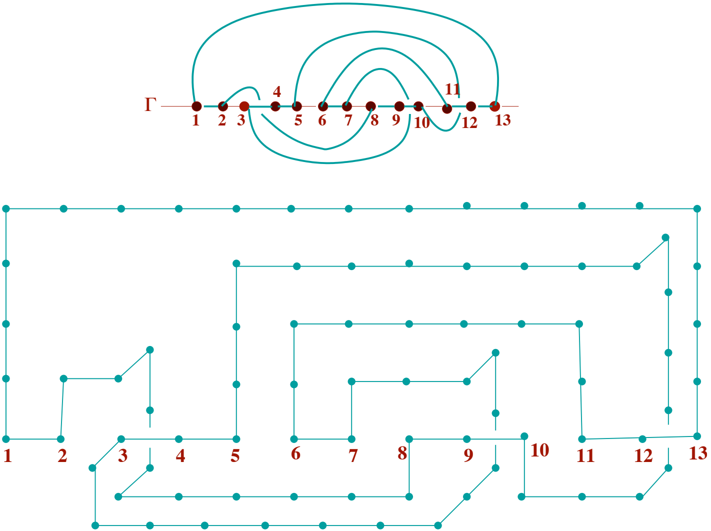
The trefoil represented in the cubic lattice.



Theorem (Diao, Ernst and Yu 2004) Let K be a knot. Then, K can be embedded into the cubic lattice with length at most

$$136\,(cr(K))^{3/2} + 84cr(K) + 22\sqrt{cr(K)} + 11$$

.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