Theory of matroids and applications II

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Curso : Introducción a la Teoría de Matroides,
Valladolid, Spain, March, 2025

Duality

Let $M = (\mathcal{B}, E)$ be a matroid. Then,

$$\mathcal{B}^* = \{E \backslash B \mid B \in \mathcal{B}\}$$

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is the set of bases of a matroid on E [Exercise].

The matroid on E having $\mathcal{B}^*$ as set of bases, denoted by M^*, is called the dual of M.

A base of M^* is also called cobase of M.

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Properties [Exercise]

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- The rank function of M^* is given by

$$r_{M^*}(X) = |X| + r_M(E \backslash X) - r(M),$$

for $X \subset E$.

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The matroid obtained on this way is called bond matroid, denoted by $B(G)$.

Planarity

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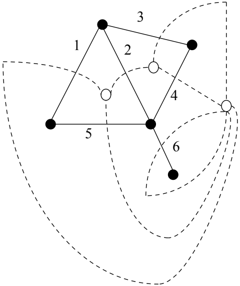

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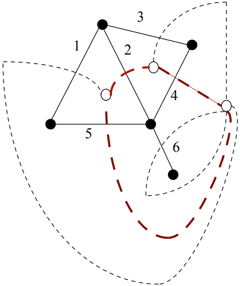

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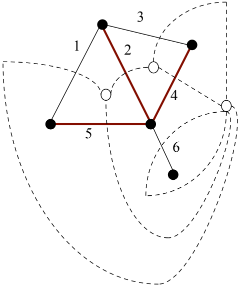

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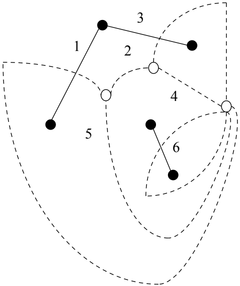

Remark The dual of a graphic matroid is not necessarily graphic ($M^*(K_5)$ is not graphic [Exercise]).

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M^* can be obtained from the set of columns of the matrix

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Remark If the space V is generated by the columns of $(I \mid A)$ then the orthogonal space $V^{\perp}$ is generated by the columns of $(-{}^tA \mid I_{n-r})$.

Operation : deletion

Let M be a matroid on the set E and let $A \subset E$. Then,

$$\{X \subset E \backslash A \mid X \in \mathcal{I}(M)\}$$

is a set of independent of a matroid on $E \backslash A$.

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Proposition [Exercise]

- The circuits of $M \setminus A$ are the circuits of M contained in $E \setminus A$.
- For $X \subset E \setminus A$ we have $r_{M \setminus A}(X) = r_M(X)$.

Operation : contraction

Let M be a matroid on the set E and for $A \subset E$ let $M|_A = \{X \subseteq A | X \in \mathcal{I}(M)\}$. Then,

$\{X \subseteq E \setminus A |$there exists a base B of $M|_A$ such that $X \cup B \in \mathcal{I}(M)\}$

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Proposition [Exercise]

- The circuits of M/A are the non-empty minimal (by inclusion) sets of the form $C \setminus A$ where C is a circuit of M.
- For $X \subset E \setminus A$ we have $r_{M/A}(X) = r_M(X \cup A) - r_M(A)$.

Deletion and contraction

Properties [Exercise]

- $(M\backslash A)\backslash A' = M\backslash(A \cup A')$
- $(M/A)/A' = M/(A \cup A')$
- $(M\backslash A)/A' = (M/A')\backslash A$
- $M/A = (M^* \setminus A)^*$.

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Question : Is it true that any family of matroids is closed under deletions/contractions operations ?

Uniform matroids

Proposition Any minor of a uniform matroid is uniform.

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Proof Deletion : let $T \subseteq E$ with $|T| = t$. Then,

$$U_{n,r} \backslash T = \begin{cases} U_{n-t,n-t} & \text{if } n \geq t \geq n-r \\ U_{n-t,r} & \text{if } t < n-r. \end{cases}$$

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Contraction : it follows by using duality.

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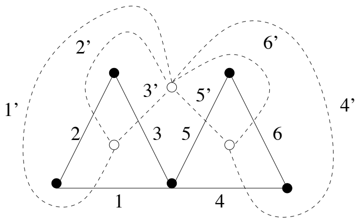
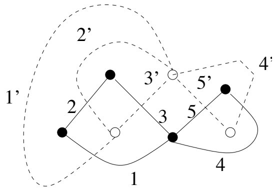


(a)



(b)

Contracting element 6

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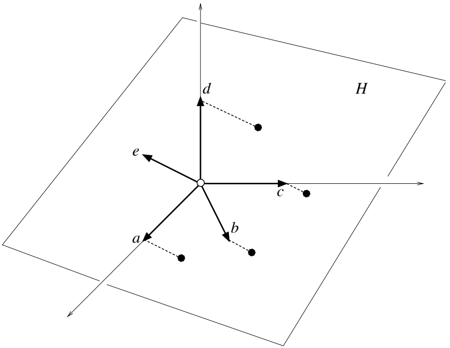
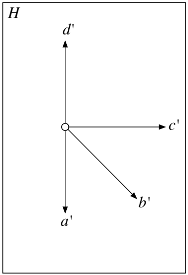
Contracting : M/a is the matroid obtained from the vectors $(v'_e)_{e\in E\setminus a}$ where v'_e is the vector obtained from v_e by deleting the non zero entry of v_a.

Geometric interpretation of contraction

M/e is the matroid induced by the vectors a', b', c', d' obtained from the projection of a, b, c, d to the plane orthogonal to e passing through 0.





Transversal matroids

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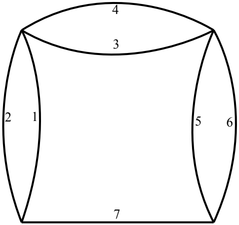
The matroid $M(G)$ is transversal (with $A_1 = \{1, 2, 7\}$, $A_2 = \{3, 4, 7\}, A_3 = \{5, 6, 7\}$). However, $M(G/7)$ is not transversal [Exercise].



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For any field $\mathbb{F}$, there exists a list of excluded minors, that is, nonrepresentable matroids over $\mathbb{F}$ but any of its proper minors is representable over $\mathbb{F}$.

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• For $\mathbb{F} = GF(2) = \mathbb{Z}_2$ (Binary matroids) the list has only one matroid $U_{2,4}$ (3 pages proof)

Rota's conjecture

• For $\mathbb{F} = GF(3) = \mathbb{Z}_3$ (Ternary matroids) the list has four matroids F_7, F_7^*, $U_{2,5}$, $U_{3,5}$ (10 pages proof)

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Conjecture (Rota, 1970) Representability over any finite field is characterized by a finite list of excluded minors.

Theorem (Geelen, Gerards, Whittle, 2014) For each finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.

Regular Matroids

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Theorem Graphic matroids are regular.

Proof (idea) Let $G = (V, E)$ be a graph. We orient the edges of G and let $A = (a_{i,j})$ be the matrix

$$a_{ie} = \begin{cases} 1 & \text{if } i \text{ is the initial vertex of } e \\ -1 & \text{if } i \text{ is the end vertex of } e \\ 0 & \text{if } i \text{ is not incident to } e \text{ or if } e \text{ is a loop} \end{cases}$$

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Problem (H. Poincaré, beginning of the 20th century) How the unimodular matrices be constructed ?

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Theorem (Seymour) A matroid M is regular if and only if it can be built with graphic, cographic and R_{10} matroids where R_{10} is the matroid of the linear dependencies over $\mathbb{Z}_2$ of the 10 vectors of $\mathbb{Z}_2^5$ having 3 components equal to one and 2 equal to zero.

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• M is built with bricks (graphic, cographic and R_{10}) via 3 operations :

1-sum : direct sum of two matroids

2-sum : patching two matroids on one common element

3-sum : patching two binary matroids on 3 common elements forming a 3-circuit in each matroid.

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Remark Most of the combinatorial optimization problems can be realized as a unimodular linear programming.

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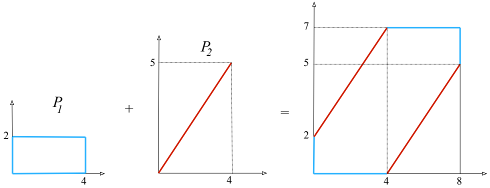

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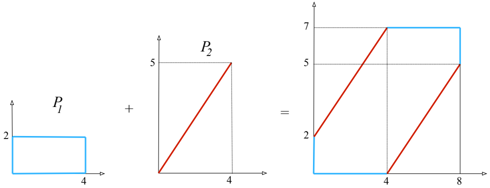
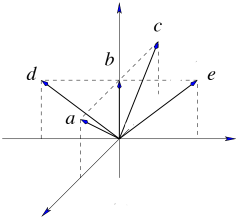


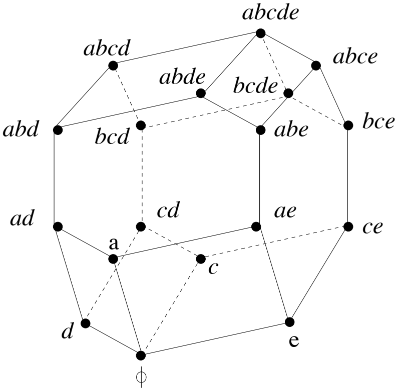
Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.

A zonotope, generated by A and denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

Permutahedron





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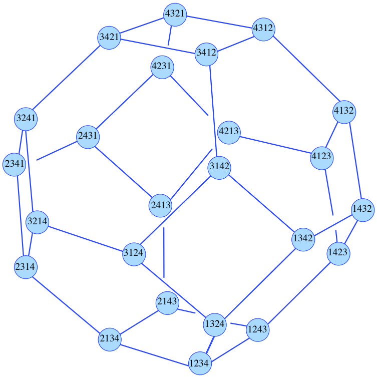
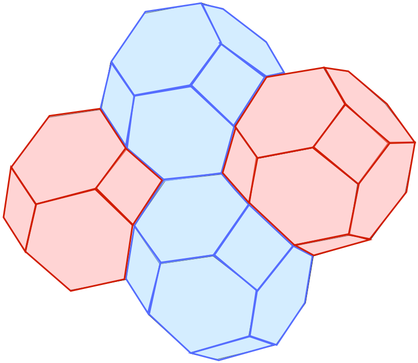


Permutahedron tiling the space



Application 6 : regular matroids

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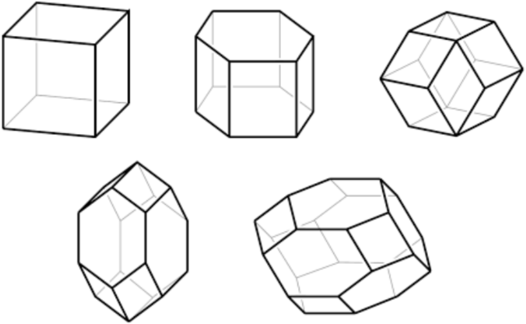
Theorem A zonotope tiles the space by translations if and only if the associated matroid is regular.

The five Fedorov's solid



Consequence : there exist exactly 5 regular matroids of rank 3.

Non Representable Matroids

There exists matroids that are not representable in ANY field.

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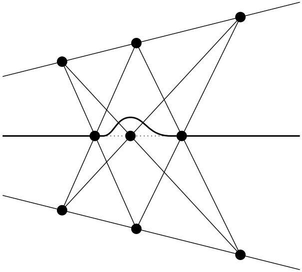
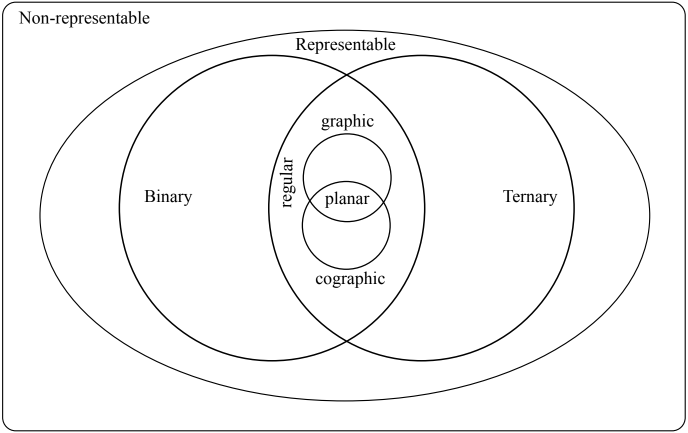
Example (classic) : the rank 3 matroid on 9 elements obtained from the Non-Pappus configuration



Matroid representability : Venn diagram



Toric ideal associated to a matroid

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We consider the homomorphism of k-algebras $\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

$$y_B \mapsto \prod_{i \in B} x_i.$$

The image of φ is a standard graded k-algebra, which is called the bases monomial ring of the matroid M and it is denoted by S_M.

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I_M is a prime, binomial and homogeneous ideal.

Example

Observation Let b be the number of bases of a matroid M on n elements. Then, I_M is generated by the kernel of the integer $n \times b$ matrix whose columns are the zero-one incidence vectors of the bases of M.

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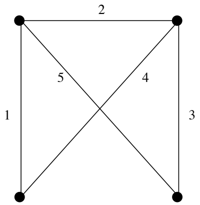
Matroid $M(G)$ associated to graph G. We have $r(M(G)) = 3$.



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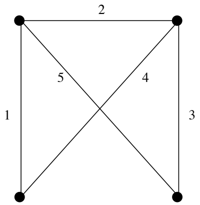

$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

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$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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By considering $\varphi : k[y_{B_1}, \dots, y_{B_8}] \longrightarrow k[x_1, \dots, x_5]$ we have that

$$y_{B_1} \mapsto x_1x_2x_3, \quad y_{B_2} \mapsto x_1x_2x_5, \quad y_{B_3} \mapsto x_1x_3x_4, \quad \dots$$

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An element of the kernel of φ (i.e., $I_{M(G)}$) is :

$y_{B_7}y_{B_4} - y_{B_2}y_{B_8} = 0.$

Symmetric exchange axiom

Let $\mathcal{B}$ denote the set of bases of M.

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Theorem (Brualdi) The exchange axiom :

> *for every* $B_1, B_2 \in \mathcal{B}$ *and for every* $e \in B_1 \setminus B_2$*, there exists* $f \in B_2 \setminus B_1$ *such that* $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$*.*

is equivalent to

the symmetric exchange axiom :

> *for every* $B_1, B_2 \in \mathcal{B}$ *and for every* $e \in B_1 \setminus B_2$*, there exists* $f \in B_2 \setminus B_1$ *such that both* $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$ *and* $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$*.*

White's conjecture

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is, $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

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Conjecture (White 1980) For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

White's conjecture

Remark for $B_1, \ldots, B_s, D_1, \ldots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ (as multisets).

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Since I_M is a homogeneous binomial ideal, it follows that

$$I_M = \left(\{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \mid B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \ (\textit{as multisets})\}\right)$$

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Observation White's conjecture does not depend on the field k.

Example (continued) and known results

Recall $\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$.
We had that $y_{B_7} y_{B_4} - y_{B_2} y_{B_8} \in I_{M(G)}$.
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- Blasiak (2008) has confirmed the conjecture for graphical matroids.
- Kashiwaba (2010) checked the case of matroids of rank ≤ 3.
- Schweig (2011) proved the case of lattice path matroids.
- Bonin (2013) confirmed the conjecture for sparse paving matroids
- Lasoń, Michałek (2014) proved for strongly base orderables matroids.

Detecting minors

We consider the following binary equivalence relation $\sim$ on the set of pairs of bases :

$$\{B_1, B_2\} \sim \{B_3, B_4\} \iff B_1 \cup B_2 = B_3 \cup B_4 \text{ (as multisets)},$$

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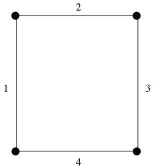

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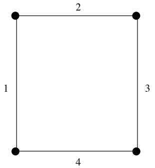

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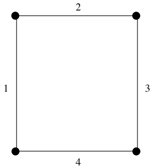

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It can be checked that $\Delta_{\{B_i,B_j\}} = 1$ for any pair $1 \leq i \neq j \leq 4$ [Exercise]

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Lemma For every $B_1, B_2 \in \mathcal{B}$ we have $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$ where $d := |B_1 \setminus B_2|$.

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Theorem (Garcia-Marco, R.A, 2014) If M has $U_{d,2d}$ as minor for some $d \geq 2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = \binom{2d-1}{d}$.

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Proposition Let $\{g_1, \ldots, g_s\}$ be a minimal set of binomial generators of I_M. Then,

$$\Delta_{\{B_1,B_2\}} = 1 + |\{g_i = y_{B_{i_1}} y_{B_{i_2}} - y_{B_1} y_{B_2} \mid B_{i_1} \cup B_{i_2} = B_1 \cup B_2\}|$$

for every $B_1, B_2 \in \mathcal{B}$.

Variants of White's conjecture

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Remark : Conjectures 2 and 3 together imply White's conjecture.