Theory of matroids and applications III

J.L. Ramírez Alfonsín

Institut Montpelliérain Alexander Grothendieck,
Université de Montpellier, France

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Tutte Polynomial - generating function

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$$\begin{aligned} t(U_{2,3};x,y) &= \sum_{X\subseteq E,\ |X|=0} (x-1)^{2-0}(y-1)^{0-0} + \sum_{X\subseteq E,\ |X|=1} (x-1)^{2-1}(y-1)^{1-1} \\ &+ \sum_{X\subseteq E,\ |X|=2} (x-1)^{2-2}(y-1)^{2-2} + \sum_{X\subseteq E,\ |X|=3} (x-1)^{2-2}(y-1)^{3-2} \\ &= (x-1)^2 + 3(x-1) + 3(1) + y - 1 \\ &= x^2 - 2x + 1 + 3x - 3 + 3 + y - 1 = x^2 + x + y. \end{aligned}$$

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The Tutte polynomial can be expressed recursively as follows

$$t(M;x,y) = \begin{cases} t(M \setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M \setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

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$$\begin{aligned} t(U_{2,2}; x, y) &= t(U_{2,2}(2); x, y) t(U_{2,2} \setminus 2; x, y) \\ &= t(I; x, y) t(U_{1,1}; x, y) \\ &= x t(U_{1,1}; x, y) \\ &= x t(I; x, y) = x^2 \end{aligned}$$

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Therefore, $t(U_{2,3}; x, y) = x^2 + x + y$.

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$$t(M(K_n); x, y) = \sum_{k=1}^{n} \binom{(n-1)}{(k-1)} \left(x + \sum_{i=1}^{k-1} y^i \right) t(K_{k-1}; 1, y) \cdot t(K_{n-k}; x, y)$$

(due to Gessel and Pak)

Application 7 : acyclic orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

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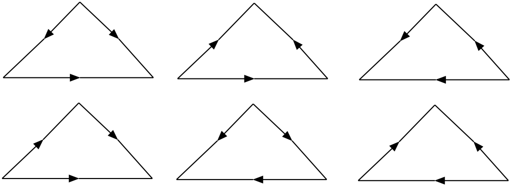
Theorem The number of acyclic orientations of G is equals to

$$t(M(G); 2, 0).$$

Example : acyclic orientations

There are 6 acyclic orientations of C_3



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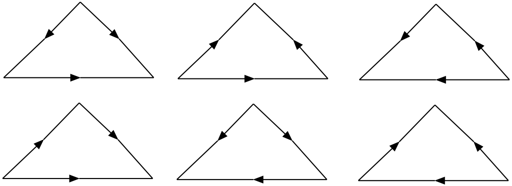

Notice that $M(C_3)$ is isomorphic to $U_{2,3}$.

Since $t(U_{2,3}; x, y) = x^2 + x + y$ then the number of acyclic orientations of C_3 is $t(U_{2,3}; 2, 0) = 2^2 + 2 + 0 = 6$.

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Conjectures (Conde, Merino, Welsh)

- $\max\{t(M;2,0),t(M;0,2)\} \geq t(M;1,1)$
- $t(M;2,0)+t(M;0,2) \geq 2t(M;1,1)$ (additive version)
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Theorem (Knauer, Martínez-Sandoval, R.A., 2018) Let M be a lattice path matroid. Then, $t(M;2,0)\cdot t(M;0,2) \geq \frac{4}{3}t^2(M;1,1)$

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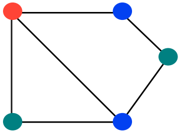

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Theorem $\chi(G, \lambda)$ is a polynomial on λ. Moreover

$$\chi(G, \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\omega(G[X])}$$

where $\omega(G[X])$ denote the number of connected components of the subgraph generated by X.

Proof (idea) By using the inclusion-exclusion formula [Exercise]

Application 9 : chromatic polynomial

The chromatic polynomial $\chi(G, \lambda)$ has been introduced by Birkhoff as a tool to attack the 4-color problem.

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Theorem If G is a graph with $\omega(G)$ connected components. Then,

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Example :

$$\begin{aligned}\chi(C_3,3) &= 3^1(-1)^{3-1}t(M(C_3);1-3,0)\\ &= 3\cdot 1\cdot t(U_{2,3};-2,0)\\ &= 3((-2)^2-2+0)\\ &= 6.\end{aligned}$$

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Ehrhart studied the function i_P that counts the number of integer points in the integer polytope P dilated by a factor of t

$$i_P: \quad \mathbb{N} \longrightarrow \mathbb{N}^* \\ t \mapsto |tP \cap \mathbb{Z}^d|$$

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Theorem (Ehrhart 1962) i_P is a polynomial on t of degree d,

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All others coefficients remain a mystery ! !

Application 10 : Ehrhart Polynomial

Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.

Let $Z(A)$ be the zonotope formed by the following Minkowski's sum of line segments

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$$i_{int(Z(A))}(q) = (-1)^d i_{Z(A)}(-q) = (-1)^d q^{r(M)} t\left(M(A); 1 - \frac{1}{q}, 1\right)$$

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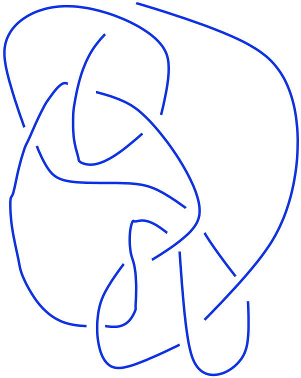
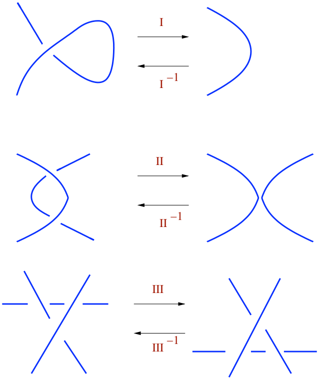
and $(-1)^{r(M)} t(M(A); 0, 1)$ counts the number of integer points in the interior of $Z(A)$.

Knots



Knot diagram

Reidemeister moves



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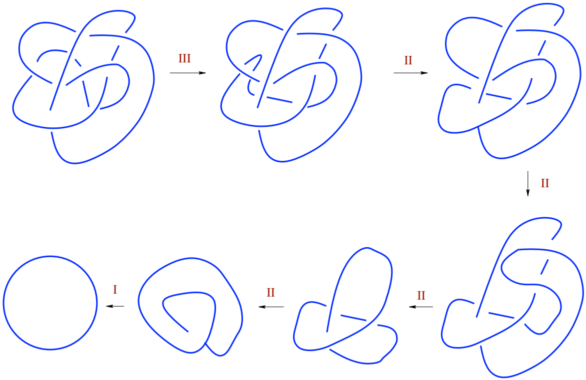

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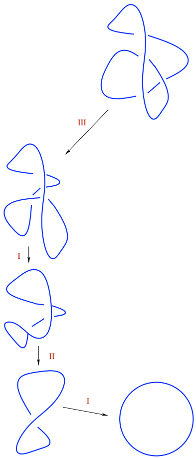

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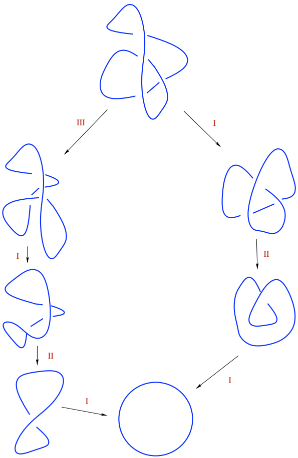

Bracket polynomial

For any link diagram D define a Laurent polynomial $\langle D \rangle$ in one variable A which obeys the following three rules where U denotes the unknot :

i) $\langle U \rangle = 1$

ii) $\langle U + D \rangle = -(A^{2} + A^{-2}) \langle D \rangle$

iii) 

Bracket polynomial

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i) - (iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves II and III but it is NOT invariant under Reidemeister move I!!

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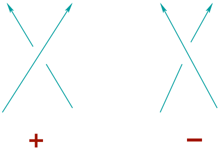
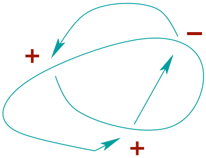
The writhe of an oriented link diagram D is the sum of the signs at the crossings of D (denoted by $\omega(D)$).

Writhe





$\omega(D)=1$

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$$f_D(A) = (-A^3)^{\omega(D)} \langle L \rangle$$

Then, $f_D(A)$ is an invariant of ambient isotopy.

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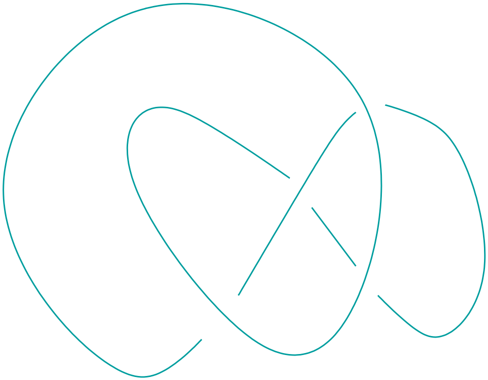
It is known that the so-called Jones' polynomial of an oriented link L is given by

$$V_L(z) = f_D(z^{-1/4})$$

where D is any diagram representing L.

Tait's graph



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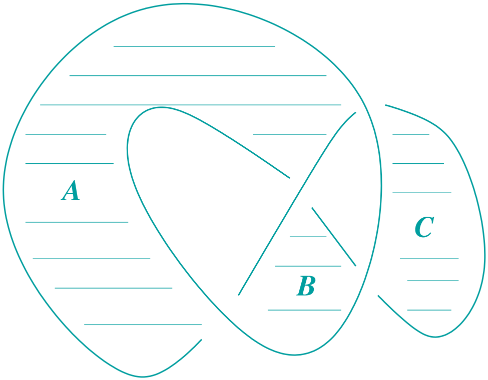

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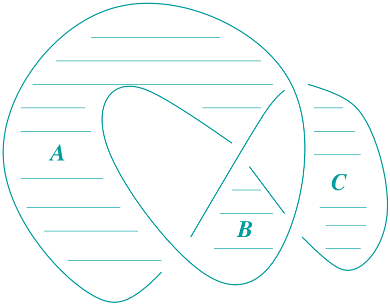

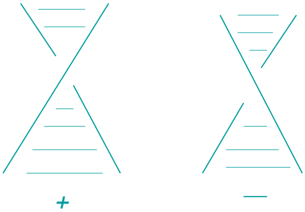

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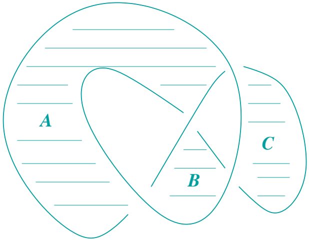

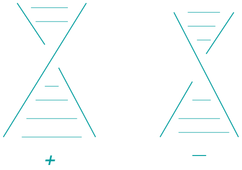

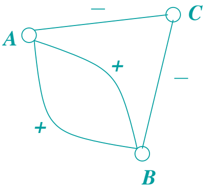

Application 11 : Jones' polynomial

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Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram then

$$V_L(z) = (z^{-1/4})^{3\omega(D)-2} t(M(G); -z, -z^{-1})$$

where G is the graph associated to the knot diagram.

Application 12 : Codes

Let C be the linear code over field $\mathbb{F}_q$ with $q = p^n$, p-prime generated by a matrix A.

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Useful to give a combinatorial proof of the following

Theorem (MacWilliams 1963)

$$W_{C^\perp}(x,y) = \frac{1}{|C|} W_C(x+(q-1)y, x-y).$$

Simplicial complex

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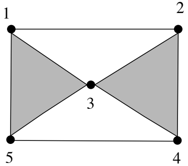
- Let $dim(\Delta) = d - 1$. The f-vector of Δ is $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta | dim(F) = i\}|$.

Example

Simplicial complexe Δ of dimension 2.



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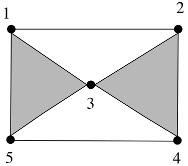

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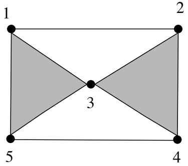

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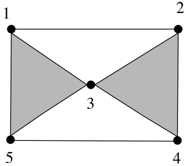

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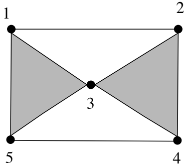

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- $f(\Delta) = (1, 5, 8, 2)$.
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- The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

Matroid complex

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on a set E are equivalent to $\mathcal{I}$ being an abstract simplicial complex on E.

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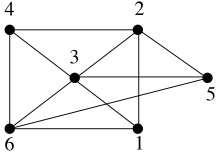
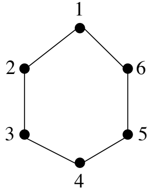
A simplicial complex Δ over the vertices E is called matroid complex if axiom $(I3)'$ is verified.

Examples

Two 1-dimensional simplicial complexes.



(a)



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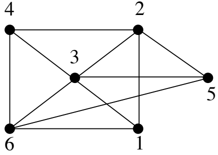

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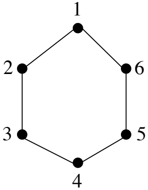

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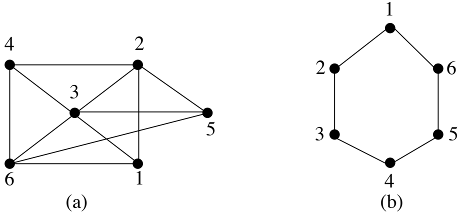

(a) Matroid complex (check that every $A \subseteq \{1, \ldots, 6\}, \Delta_A$ is pure) [Exercise].

(b) It is not a matroid complex (check that the restriction $\Delta_{\{1,3,4\}}$ is not pure) [Exercise].

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A matroid complex Δ_M is a cone if and only if M has a coloop (or an isthmus), which corresponds to the apex (defined above).

Stanley-Reisner ideal

Let k be a field. We can associate to a simplicial complex Δ, the following square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

$$I_\Delta = \left(x_F = \prod_{i \in F} x_i \mid F \notin \Delta \right) \subseteq S.$$

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The ideal I_Δ is called the Stanley-Reisner ideal of Δ and S/I_Δ the Stanley-Reisner ring of Δ.

Stanley-Reisner ring

- Hilbert function

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In particular, for any $j = 0, \dots, d$, we have

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-1} h_i$$

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-1} f_{i-1}.$$

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Remark v_j is externally passive in B if it is internally passive in $E \setminus B$ in M^*.

Application 13 : *h*-vector

Theorem $t(M; x, y) = \sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}$

where $i(B)$ (resp. $e(B)$) counts the number of internally (resp. externally) passive elements in B.

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Consequence The h-numbers of a matroid complex are nonnegative.

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&= (1-z)^2 + 3z(1-z) + 3z^2 \\
&= 1 - 2z + z^2 + 3z - 3z - 3z^2 + 3z^2 \\
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Obtaining that $h(\Delta_{U_{2,3}}) = (1, 1, 1)$.

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Let $\mathcal{O}_i$ denote the collection of monomials in $\mathcal{O}$ of degree i.
Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}), \ldots, F_r(\mathcal{O}))$.

Order ideal

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A vector $h = (h_0, \ldots, h_d)$ is a pure O-sequence if there is a pure ideal $\mathcal{O}$ such that $h = F(\mathcal{O})$.

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\boldsymbol{xy^3z}$ and $\boldsymbol{x^2z^3}$ is :

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Hence the h-vector of X is the pure O-sequence $h = (1, 3, 6, 7, 5, 2)$.

Stanley's conjecture

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True for several families of matroid complexes.

(Merino, Noble, Ramirez-Ibañez, Villarroel, 2010) Paving matroids

(Oh, 2010) Cotranversal matroids

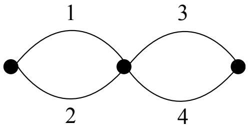
(Schweig, 2010) Lattice path matroids

(Stokes, 2009) Matroids of rank at most three

(De Loera, Kemper, Klee, 2012) for all matroids on at most nine elements all matroids of corank two.

Example

We consider the matroid complexe $\Delta_{M(G)}$ associated to the rank 2 matroid induced by the graph G



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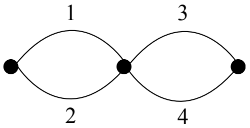

We have that $dim(\Delta) = 1$ and $f_{-1} = 1, f_0 = 3$ and $f_1 = 4$.

Example

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$$\sum_{i=0}^{2} h_i z^i = \sum_{B \in \mathcal{B}(M(G))} z^{i(B)} = 1 + z + z + z^2 = 1 + 2z + z^2.$$

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Obtaining the h-vector $h(1,2,1)$. Since $\mathcal{O} = (1, x_1, x_2, x_1x_2)$ is an order ideal then $h(1,2,1)$ is pure O-sequence.