Theory of matroids and applications IV

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Notation

A signed set X is a set $\underline{X}$ partitioned in two parts (X^+, X^-), where X^+ is the set of positive elements of X and X^- is the set of negatives elements.

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We say that X is a restriction of Y if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. If A is a not signed set and X a signed set then $X \cap A$ design the signed set Y with $Y^+ = X^+ \cap A$ et $Y^- = X^- \cap A$.

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Generally, given a signed set X and a set A we denote by ${}_{-A}X$ the signed set defined by $({}_{-A}X)^+ = (X^+ \setminus A) \cup (X^- \cap A)$ and $({}_{-A}X)^- = (X^- \setminus A) \cup (X^+ \cap A)$. We say that the signed set ${}_{-A}X$ is obtained by an reorientation of A.

Circuits

A collection $\mathcal{C}$ of signed sets of a finite set E is the set of circuits of a oriented matroid on E if and only if the following axioms are verified :

$(C0)$ $\emptyset \notin \mathcal{C}$,

$(C1)$ $\mathcal{C} = -\mathcal{C}$,

$(C2)$ for any $X, Y \in \mathcal{C}$, if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$,

$(C3)$ for any $X, Y \in \mathcal{C}, X \neq -Y$, and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

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(c) Let M be an oriented matroid E and $\mathcal{C}$ the collection of circuits. We clearly have that $_{-A}\mathcal{C}$ is the set of circuits of an oriented matroid, denoted by $_{-A}M$ and obtained from M by a reorientation of A.

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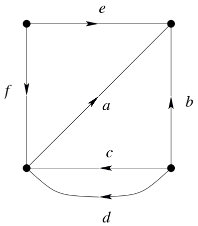
Notation. We may write $X = a\overline{bc}de$ the signed circuit X defined by $X^+ = \{a, d, e\}$ and $X^- = \{b, c\}$.

Oriented graph

Let G be an oriented graph. We obtain the signed circuits from the cycles of G.



Then,

$$\begin{aligned}\mathcal{C} &= \{(a\bar{b}c), (a\bar{b}d), (a\bar{e}f), (c\bar{d}), (b\bar{c}\bar{e}f), (b\bar{d}\bar{e}f),\\ &\quad (\bar{a}b\bar{c}), (\bar{a}b\bar{d}), (\bar{a}e\bar{f}), (\bar{c}d), (\bar{b}ce\bar{f}), (\bar{b}de\bar{f})\}.\end{aligned}$$

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We obtain an oriented matroid on E by considering the signed sets $X = (X^+, X^-)$ where

$$X^+ = \{i : \lambda_i > 0\} \text{ et } X^- = \{i : \lambda_i < 0\}$$

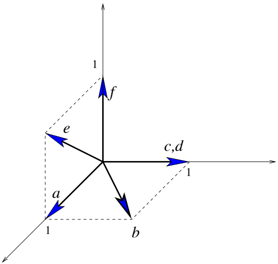
for all minimal dependencies among the v_i.

Example

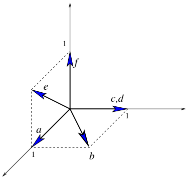
Let

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

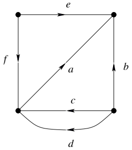
The columns of A correspond to the following vectors



We can check that the circuits of



are the same as those arising from



For exemple, $(a\bar{b}c)$ correspond to the linear combination $a - b + c = 0$ or the circuit $(b\overline{de}f)$ correspond to the linear combination $b - d - e + f = 0$.

Configurations of points

Any configuration of points induce an oriented matroid in the affine space where the signed set of circuits are are the coefficients of minimal affine dependencies of the form

$$\sum_i \lambda_i \mathsf{v}_i = 0 \quad \text{with} \quad \sum_i \lambda_i = 0,\ \lambda_i \in \mathbb{R}$$

$$\begin{array}{c} \\ A' = \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{pmatrix} \end{array}$$

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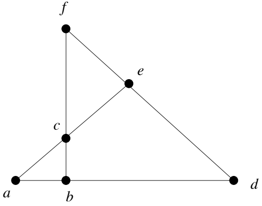

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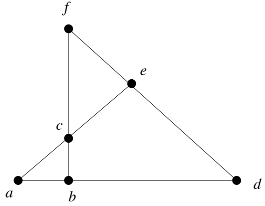

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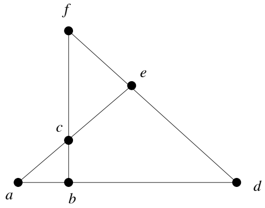

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For instance, circuit $(a\bar{b}d)$ correspond to the affine dependency $3(-1,0)^t - 4(0,0)^t + 1(3,0)^t = (0,0)^t$ with $3-4+1=0$.

Radon Partitions

There is a natural way to obtain an oriented matroid from a configuration of points in $\mathbb{R}^d$
If $C \in \mathcal{C}$ then $conv(\text{pos. elements } C) \cap conv(\text{neg. elements } C) \neq \emptyset$

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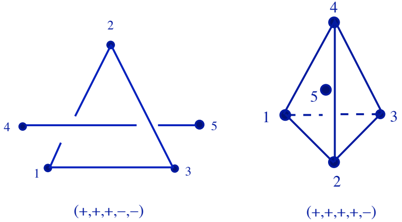
Example : $d = 3$.



These are called minimal Radon partitions

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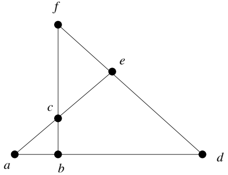
- ${}_{-d}M(A')$ is graphic. Moreover, it correspond to the oriented matroid



under the permutation
$\sigma(a) = b, \sigma(b) = a, \sigma(c) = c, \sigma(d) = d, \sigma(e) = f, \sigma(f) = e.$

Duality

Two signed sets X and Y are called orthogonal, denoted by $X \perp Y$, if either $\underline{X} \cap \underline{Y} = \emptyset$ or $X|_{X \cap \underline{Y}}$ and $Y|_{Y \cap \underline{X}}$ are neither the same or opposite.

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Proposition Let $M = (E, \mathcal{C})$ be an oriented matroid. Then,

1) there is a unique signature of the cocircuits $\mathcal{C}^*$ of $\underline{M}$ such that $X \perp Y$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}^*$

2) $\mathcal{C}^*$ is the set of signed circuits of a matroid, denoted by M^*

3) $M^{**} = M$.

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H is a hyperplane of a matroid $M = (E, \mathcal{C})$ of rank r if $r(H) = r - 1$ and $cl(H) = H$.

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If M is realizable with points in the space then H is a geometric hyperplane (generated by the corresponding points). in this case, the cocircuit $D = (D^+, D^-)$ is given by

$$D^+ = \{e \notin H | h(e) > 0\} \text{ and } D^- = \{e \notin H | h(e) < 0\}$$

where h is the linear function with $Ker(h) = H$.

Bases and Chirotope

$\mathcal{B}$ is the set of bases of an oriented matroid if and only if there is an application, called chirotope, $\chi : E^r \to \{+,-,0\}$ such that

(i) $\mathcal{B} \neq \emptyset$;

(ii) for any B and B' in $\mathcal{B}$ and $e \in B \setminus B'$ there existes $f \in B' \setminus B$ such that $B \setminus e \cup f \in \mathcal{B}$;

(iii) $\{b_1, \ldots, b_r\} \in \mathcal{B}$ if and only if $\chi(b_1, \ldots, b_r) \neq 0$

(iv) χ is alternating, i.e. $\chi(b_{\sigma(1)}, \ldots, b_{\sigma(r)}) = sign(\sigma)\chi(b_1, \ldots, b_r)$ for any $b_1, \ldots, b_r \in E$ for any permutation σ

(v) (Three-terms Grassmann-Plücker relation)

For any $b_1, \ldots, b_r, x, y \in E$, if

$$\chi(x, b_2, \ldots, b_r)\chi(b_1, y, b_3, \ldots, b_r) \geq 0$$

and

$$\chi(y, b_2, \ldots, b_r)\chi(x, b_1, b_3, \ldots, b_r) \geq 0$$

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Remark In the realizable case, axiom (v) is directly verified with the Grassmann-Plücker's relation, it is thus a combinatorial reformulation :

$det(b_1, \ldots, b_r) \cdot det(b'_1, \ldots, b'_r) =$
$\sum_{1 \leq i \leq r} det(b'_i, b_2, \ldots, b_r) \cdot det(b'_1, \ldots, b'_{i-1}, b_1, b'_{i+1}, \ldots, b'_r).$

Relation between bases and circuits

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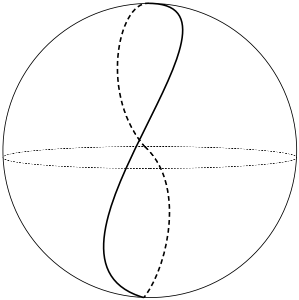
For any two ordered bases of M of the form $(e, x_2, \ldots, x_r)$ and $(f, x_2, \ldots, x_r)$, $e \neq f$, we have

$$\chi(f, x_2, \ldots, x_r) = -C(e)C(f)\chi(e, x_2, \ldots, x_r)$$

where C is one of the two opposite signed circuits of M in the set $(e, f, x_2, \ldots, x_r)$ and $C(e)$ and $C(f)$ correspond to the sign of elements e and f in C respectively.

Arrangement of pseudospheres

A sphere s of $\mathbb{S}^{d-1}$ is a pseudo-sphere if s is homeomorphic to $\mathbb{S}^{d-2}$ in an homomorphism of $\mathbb{S}^{d-1}$.



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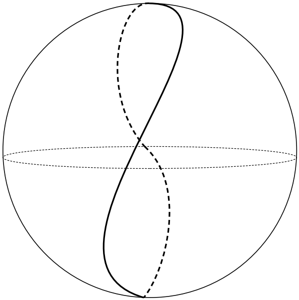

We have two connected components in $\mathbb{S}^{d-1} \setminus s$, each homeomorphic to the $d-1$ dimensional ball (called sides of s).

Arrangement of pseudo-spheres

A finite collection $\{s_1, \ldots, s_n\}$ of pseudo-spheres in $\mathbb{S}^{d-1}$ is an arrangement of pseudo-spheres if

$(PS1)$ for all $A \subseteq E = \{1, \ldots, n\}$ the set $S_A = \cap_{e \in A} s_e$ is a (topological) sphere

$(PS2)$ If $S_A \not\subseteq s_e$ for $A \subseteq E, e \in E$ and s_e^+, s_e^- denotes the two sides of s_e then $S_A \cap s_e$ is a pseudo-sphere of S_A having as sides $S_A \cap s_e^+$ and $S_A \cap s_e^-$.

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We say that the arrangement is signed if for each pseudosphere S_e, $e \in E$ it is chosen a positive and a negative side.

Topological representation

Topological Representation (Folkman+Lawrence) Any loop-free oriented matroid of rank $d+1$ (up to isomorphism) are in one-to-one correspondence with arrangements of pseudo-spheres in $\mathbb{S}^d$ (up to topological equivalence).

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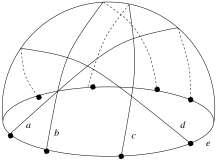

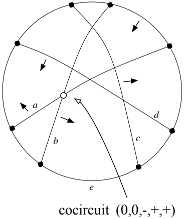

Arrangement of pseudolines

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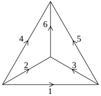
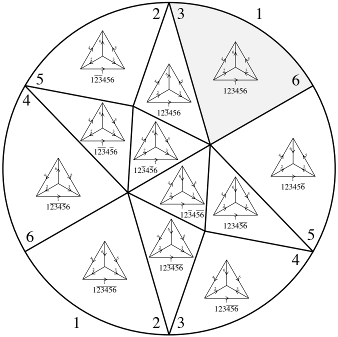
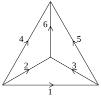
Theorem (Las Vergnas 1975, Zaslavsky 1975) The set of acyclic orientations of M are in bijection with the set of cells of the corresponding arrangement of pseudospheres.

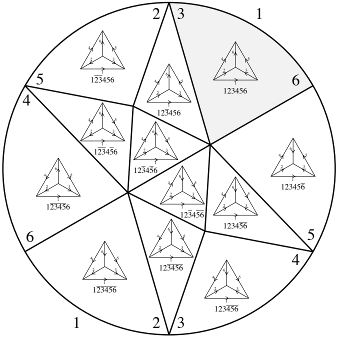
Example





Example





Remark A cell that is bounded by hyperplanes $\{h_{i_1}, \ldots, h_{i_k}\}$ correspond to an acyclic reorientation having $[n] \setminus \{i_1, \ldots, i_k\}$ as interior points.

Application 14 : McMullen problem

A projective transformation $P : \mathbb{R}^d \to \mathbb{R}^d$ is such that $p(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$ where A is a linear transformation of $\mathbb{R}^d$, $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$ such that at least one of $c \neq 0$ or $\delta \neq 0$.

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Oriented matroid version (Cordovil, Silva 1985) Determine the largest integer $g(d)$ such that given any uniform oriented matroid M of rank r on g elements there is an acyclic orientation of M having no interior points.

Application 13 : McMullen problem

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Theorem (R.A. 2001) $f(d) \leq 2d + \lceil \frac{d}{2} \rceil$ for any $d \geq 2$.

Lawrence oriented matroid

A Lawrence oriented matroid M of rank r on the $E = \{1, \ldots, n\}$, $r \leq n$, is a uniform oriented matroid obtained as the union of r uniform oriented matroids $M_1, \ldots, M_r$ of rank 1 on $(E, <)$.

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The chirotope χ corresponds to some Lawrence oriented matroid M_A if and only if there exists a matrix $A = (a_{ij})_{1 \leq i \leq r, 1 \leq j \leq n}$ with entries from $\{+1, -1\}$ where the i^{th}-row is given by $\chi(M_i)$, and thus,

$$\chi(B) = \prod_{i=1}^{r} a_{ij_i}$$

where B is an ordered r-tuple $j_1 \leq \ldots \leq j_r$ elements of E.

Lawrence oriented matroid

elements

rank	1	2	3	4	5	6	7	
1	+	−	−	+	+	+	+	$\longleftarrow \chi(M_1)$
2	+	−	+	+	+	+	+	$\longleftarrow \chi(M_2)$
3	+	+	+	+	+	+	+	
4	+	−	+	+	+	+	+	

Matrix A arising a Lawrence oriented matroid $M = \bigcup_{i=1}^{n} M_i$.

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Reorientation

elements

rank	1	2	3	4	5	**6**	7	
1	+	−	−	+	+	**−**	+	$\longleftarrow \chi(M_1)$
2	+	−	+	+	+	**−**	+	$\longleftarrow \chi(M_2)$
3	+	+	+	+	+	**−**	+	
4	+	−	+	+	+	**−**	+	

Reorientation of element 6 arising a Lawrence oriented matroid $-_6 M$.

Top and Bottom Travels

	1	2	3	4	5	6	7
1	+	−	−	+	+	+	+
2	+	−	+	+	+	+	+
3	+	+	+	+	+	+	+
4	+	−	+	+	+	+	+

We define Top Travel $[TT]$ and the Bottom Travel $[BT]$ on the entries of A, both formed by horizontal and vertical movements.

Top and Bottom Travels

	1	2	3	4	5	6	7
1	$+$	$-$	$-$	$+$	$+$	$+$	$+$
2	$+$	$-$	$+$	$+$	$+$	$+$	$+$
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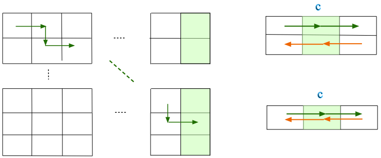
Acyclic and interior points



TT ends at last column

TT and BT parallel at column c

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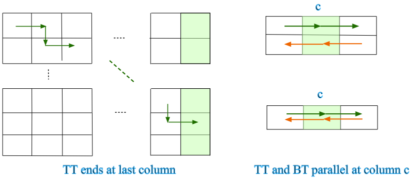

- M_A is acyclic iff TT arrives at the last column of A.

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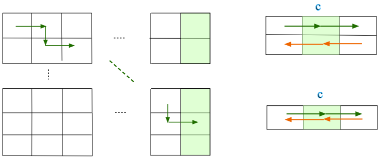

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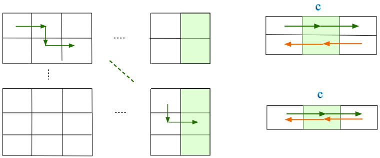

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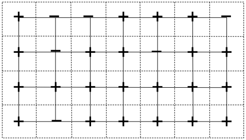
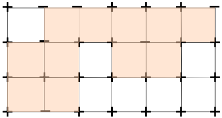
4	+	−	+	+	+	+	+

M_A is acyclic and 4, 5 and 6 are interior elements.

Chessboard

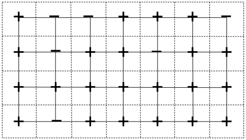
+	−	−	+	+	+	−
+	−	+	+	−	+	+
+	+	+	+	+	+	+
+	−	+	+	+	+	+

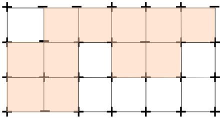




Chessboard of matrix A invariant under reorientations

+	−	−	+	+	+	−
+	−	+	+	−	+	+
+	+	+	+	+	+	+
+	−	+	+	+	+	+





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Chessboard

+	−	−	+	+	+	−
+	−	+	+	−	+	+
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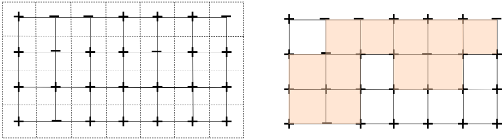

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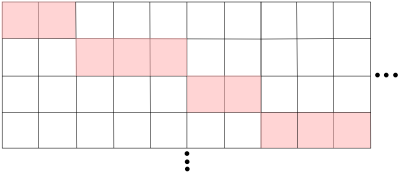

Spatial graphs

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Spatial representation of K_5

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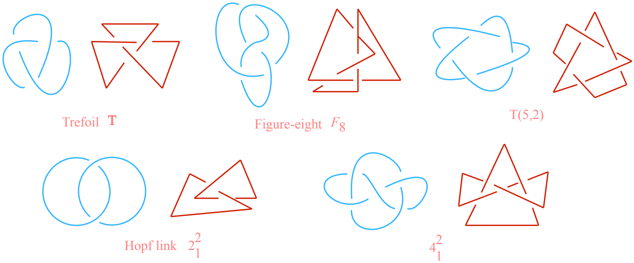

Some values of $m(L)$

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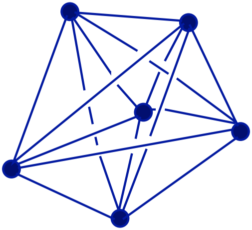
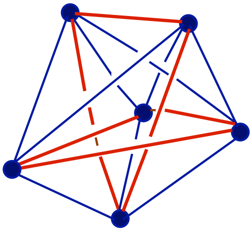


Application 15 : values of $m(L)$

Theorem $m(2_1^2) = 6$



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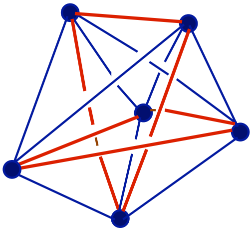

Theorem (R.A. 1998, 2000, 2009)
$m(T \,\text{or}\, T^*) = 7,\ m(4^2_1) > 7,\ m(F_8) > 8,\ m(T(5,2)) > 8$

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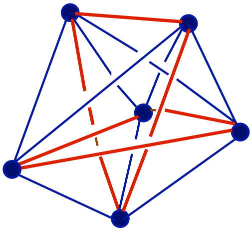

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(By using Radon partition arising from oriented matroids of rank 4 and some computer verification)

Isotopy Conjecture

Isotopy Conjecture for Oriented Matroid (Ringel 1956) The realization space over the real number field of an oriented matroid is path-connected. In other words, can one given realization of M be continuously deformed, through realizations, to another given one ?

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Theorem (Richter 1989) The realization spaces of all realizable uniform oriented matroids of rank 3 and at most 9 elements are contractible.

Universality Theorem

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Mnëv's Universality Theorem (1988) For every basic primary semialgebraic set V defined over $\mathbb{Z}$ there is a chirotope χ of rank 3 such that V and $\mathcal{R}(\chi)$ are stably equivalent.

Proof based in the algebra of throws

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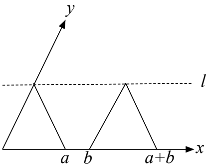
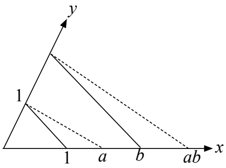


For any line l parallel to x (meeting at infinity)



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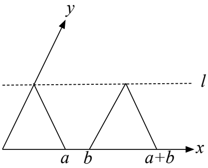

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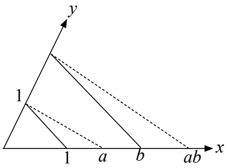

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By using this, polynomial algebraic relations can be translated into corresponding point-and-line configurations.

Some consequences of the universality theorem

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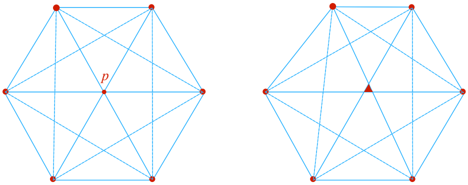
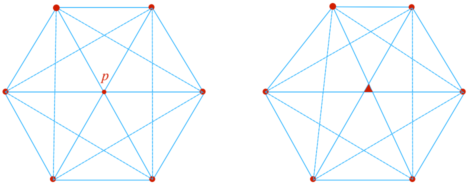
- (Bokowski, Sturmfels 1989) Realizability of rank 3 oriented matroids cannot be characterized by excluding a finite set of forbidden minors.
- For every finite simplicial complex Δ, there is an oriented matroid whose realization space is homotopy equivalent to Δ.

Strong geometry



Two configurations of points having the same oriented matroid

Strong geometry



Two configurations of points having the same oriented matroid

Gros, R.A. (2025) introduced a new oriented matroid $M_{\wedge}(X)$ arising from the set of lines spanned by X.

Strong geometry

Let X be a n-uple of points in the space.

We define the strong geometry associated to X, denoted by $SGeom(X)$, as the structure composed by $M(X)$ and $M_\wedge(X)$.

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Strong geometries encode nicely the combinatorics of the cells of the arrangement of the spanned lines.

Theorem (Gros, R.A. 2025) Any basic primary semialgebraic set V is stable equivalent to the realization space of a strong matroid.