A new method to compute the determinant of a knot

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A challenging problem

Question : have you ever tried to untangle a rope, headphone cables, necklace or any other strand ?

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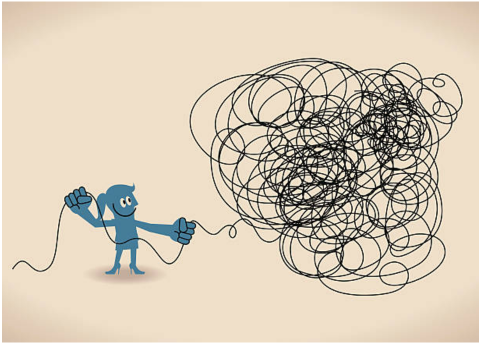

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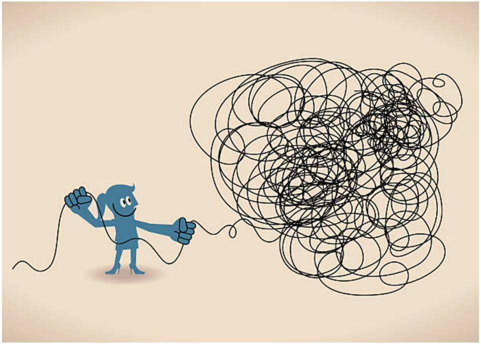

The longer a strand, the more likely it is to tangle.

Knot theory

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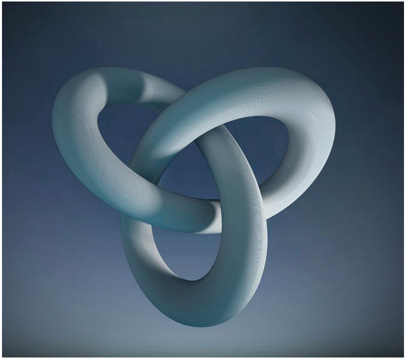

Knot theory : diagrams

Trivial knot
0_1

Trefoil knot
3_1

Figure-eight knot
4_1

Pentafoil knot
5_1

Trivial link
0^2_1

Hopf link
2^2_1

Solomon link
4^2_1

Borromean link
6^3_2

Unknotting problem

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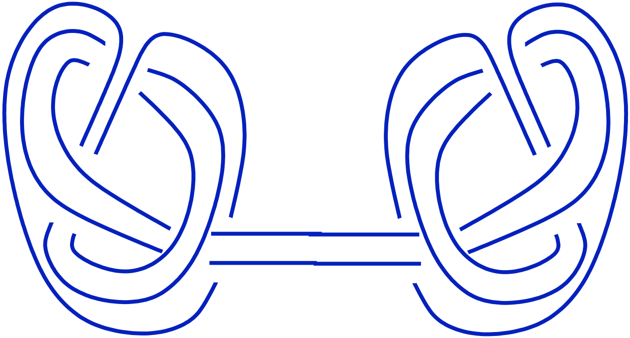

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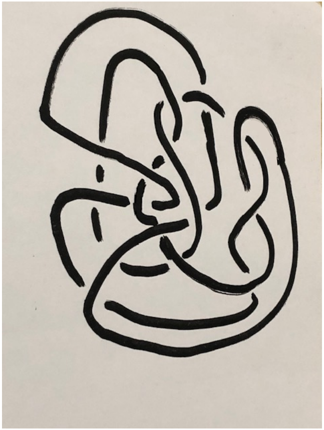
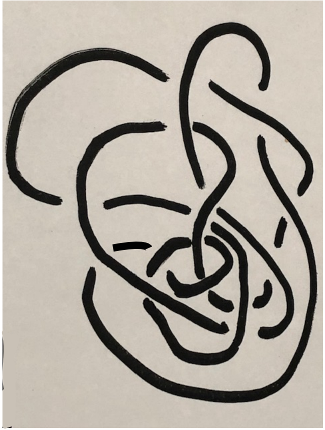
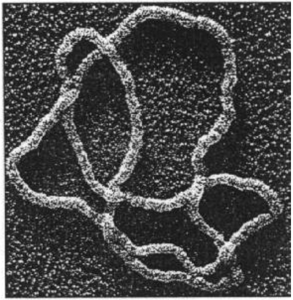




Application : DNA (Deoxyribonucleic Acid)

A fundamental application of this problem can be found in the study of DNA



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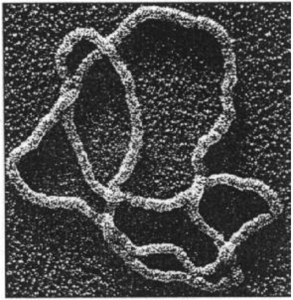

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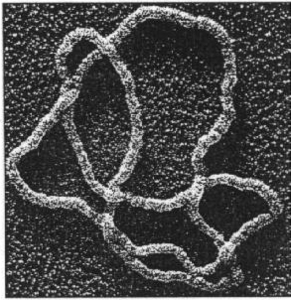
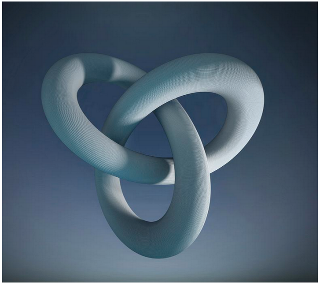


when an enzyme acts on the initial form of DNA, it gives new forms of DNA. We would like to know if they are still unraveled. Difficulty : DNA is very tangled inside the cell (equivalent to approximately 200 km of fishing line inside a football).

First step ...

Is the trefoil trivial ?



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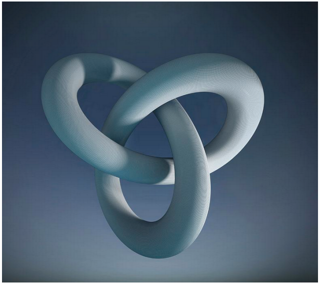
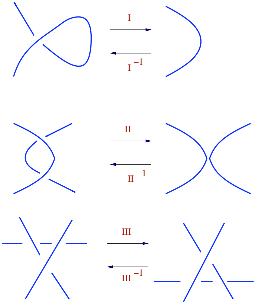


Theorem (Papakyrikopoulos, 1957) Un knot K is trivial if and only if the fundamental group of the complementary space of K is abelian.

Reidemeister moves



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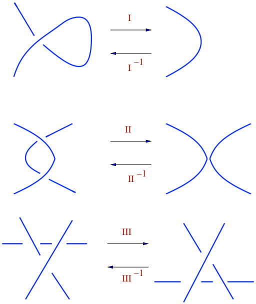

Theorem (Reidemeister 1927) Two knots (or links) are isotopic if and only if their diagrams are related by a sequence of Reidemeister moves.

3-coloring

(R. Fox) A knot diagram K is 3-colorable if one can color each arc of the diagram with **red**, **blue** and **green**, such that

- at least 2 colors are used,

- at each crossing we have either 3 different colors or only one color.

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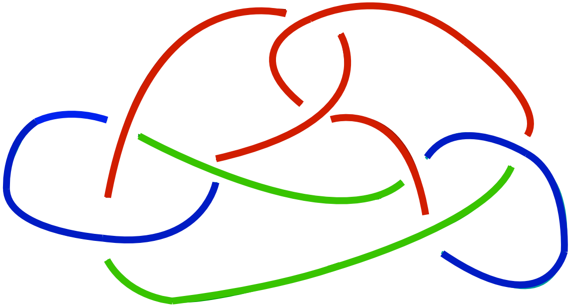

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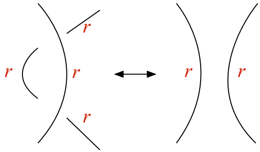
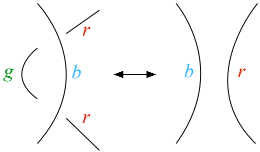
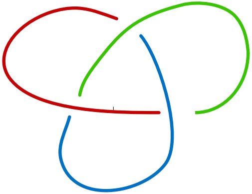
For instance if a move II is performed we have essentially two cases





Existence of nontrivial knots



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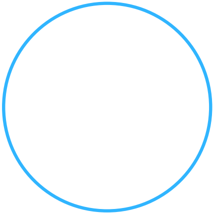

non 3-colorable

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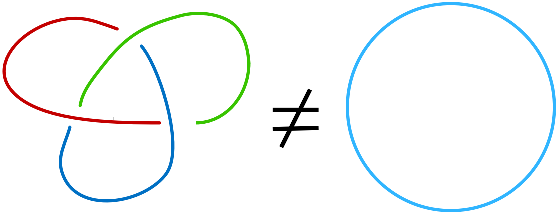

3-colorable $\neq$ non 3-colorable

Colorability $\pmod{p}$

A knot diagram K is colorable $\pmod{p}$ if each arc of the diagram can be labeled with an integer in $\{1, \ldots, p-1\}$ such that

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- at each crossing the relation $2x - y - z = 0 \pmod{p}$ is verified where x is the label on the over crossing and y and z the other two labels.

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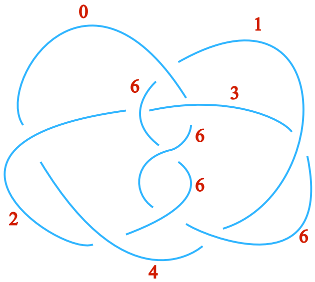

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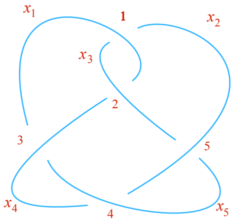

$$M = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 2 & -1 & 0 & -1 \end{pmatrix}$$

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Theorem If a link L is splittable then L is colorable $\pmod{n}$ for all $n \geq 3$.

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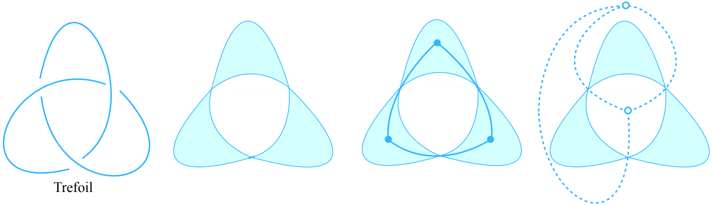
Theorem (Stoimenow 2005) Let n be an odd integer. Then, $det(L) = n$ for some achiral rational link L if and only if $n = p^2 + q^2$ with $pgc(p, q) = 1$.

Tait graphs



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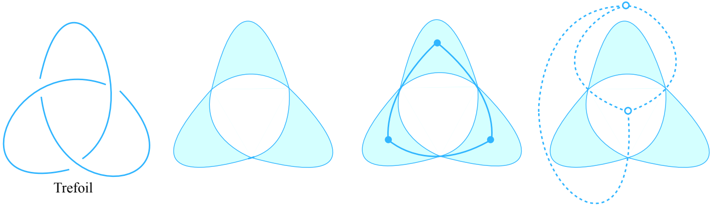

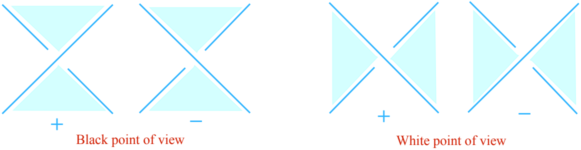

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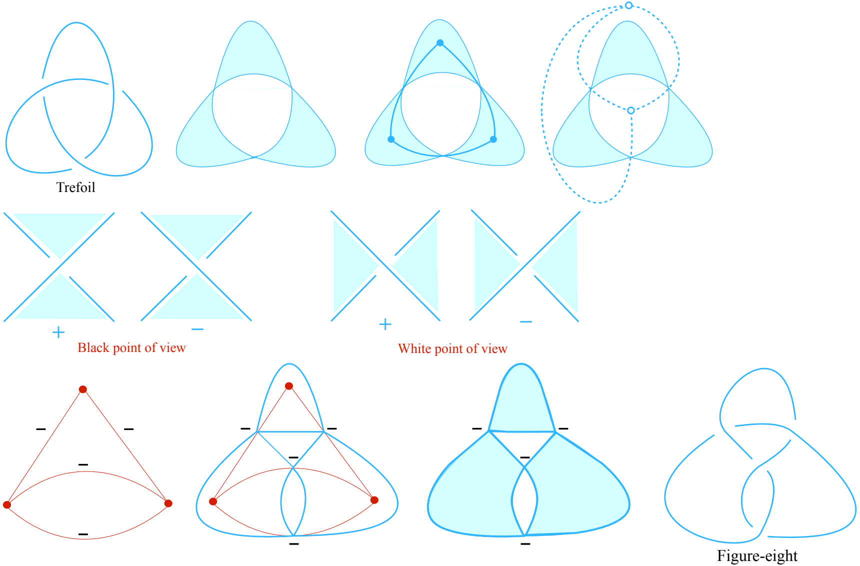

Trefoil
+
−
Black point of view
+
−
White point of view
Figure-eight

Spanning trees

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Remark : If L_G is alternating then

$$det(L_G) = \# \text{ of spanning trees of } G$$

Fourier-Hadamard transforms

Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function.
Let $\mathrm{supp}(f) = \{\boldsymbol{x} \in \mathbb{F}_2^n \mid f(\boldsymbol{x}) \neq 0\}$ be its support.

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The Fourier-Hadamard transform of f is defined as

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f can be represented by elements in the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/(x_1^2 - x_1, \ldots, x_n^2 - x_n)$, called Numerical Normal Form (NNF). It can be written as

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{F}_2^n} \lambda_{\boldsymbol{y}} \boldsymbol{x}^{\boldsymbol{y}}$$

where $\boldsymbol{x}^{\boldsymbol{y}} = \prod_{i=1}^{n} x_i^{y_i}$.

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We define $FH_G(x_1, \ldots, x_n)$, to be the NNF of the Fourier-Hadamard transform of f_G, that is,

$$FH_G(x_1, \ldots, x_n) = \widehat{f}_G(x_1, \ldots, x_n).$$

Formula for the determinant

Theorem (Gros, Pastor-Díaz, R.A. 2024) Let (G, χ_E) be an edge-signed connected planar graph and let L_G be the link arising from G. Then,

$$\det(L_G) = \left| FH_G(\boldsymbol{v}) \right|$$

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Our method yields to a straightforward procedure to answer this question.

Computing FH_G

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$$FH_G(x_1, \ldots, x_n) = \sum_{T \in \mathcal{T}_G} \left(\prod_{i \in E(T)} (1 - 2x_i) \right).$$

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$$FH_G(x_1,\ldots,x_n)=\sum_{T\in\mathcal{T}_G}\left(\prod_{i\in E(T)}(1-2x_i)\right).$$

Theorem (Gros, Pastor-Díaz, R.A. 2024) Let $G=(V,E)$ be a graph.

- If G consist of one edge, say e, then

$$FH_G(x_e)=\begin{cases}1-2x_e & \text{if } e \text{ is an isthmus,}\\ 1 & \text{if } e \text{ is a loop.}\end{cases}$$

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$$FH_G(\mathrm{x})=FH_G(x_e)\ FH_{G\setminus e}(\mathrm{x}')$$

where x' is obtained from x from which the entry x_e, corresponding to edge e, is deleted.

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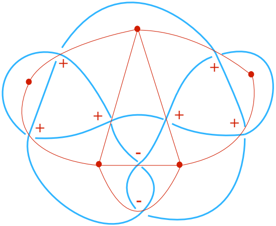
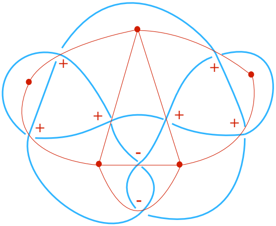
$$FH_G(\mathrm{x}) = FH_{G \setminus e}(\mathrm{x}') + (1 - 2x_e) FH_{G/e}(\mathrm{x}')$$



$\det(8_{21}) = \left|FH_G(1,1,0,0,0,0,0,0)\right| = 15$ (difference between the 24 negative-spanning trees and 9 positive-spanning trees.



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$det(L)$	# of edge-signatures
1	46
3	44
5	14
7	7
9	9
11	2
13	2
15	1
19	2
33	1

Determinants obtained from the half of the 2^8 possible edge-signatures (corresponding to vectors beginning with 1). The other half yields to the same set of determinants)

Thanks for your attention ! !