ON THE NUMBER OF VERTICES OF PROJECTIVE POLYTOPES

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Abstract. Let X be a configuration of n points in $\mathbb{R}^d$.

What is the maximum number of vertices that $conv(T(X))$ can have among all the possible permissible projective transformations T?

In this paper, we investigate this and other related questions. After presenting several upper bounds, we study a closely related problem (via Gale transforms) concerning the number of minimal Radon partitions of a set of points. The latter led us to a result toward a question due to Pach and Szegedy. Related problem concerning the size of topes in arrangements of hyperplanes and some tolerance-type problem of finite sets are also discussed.

1. Introduction

Consider the following question

> Given a set of n points in general position $X \subset \mathbb{R}^d$, what is the maximum number of k-faces that $conv(T(X))$ can have among all the possible *permissible projective transformations* T?

More precisely, let $d \geq k \geq 0$ be integers and let $X \subset \mathbb{R}^d$ be a set of points in general position, we define the number of *projective k–faces* of X as

$$h_k(X,d) = \max_{T} \{f_k(conv(T(X)))\}, \tag{1}$$

where the maximum is taken over all possible permissible projective transformations T of X and $f_k(P)$ denotes the number of k-faces of a polytope P.

Now we can define the function $H_k(n,d)$ which determines the *maximum* number of *projective k-faces* that *any* X configuration of n points in $\mathbb{R}^d$ must have as,

$$H_k(n,d) = \min_{X \subset \mathbb{R}^d, |X|=n} \{h_k(X,d)\}.$$

In this paper, we focus our attention on the behavior of $H_0(n,d)$ (the number of projective vertices). It turns out that $H_0(n,d)$ is the source of several applications.

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1.1. **Scope/general interest.** The function $H_0(n,d)$ is closely connected with different notions/problems : McMullen's problem, bounds for $H_k(n,d)$, a connection of $H_{d-1}(n,d)$ with minimal Radon partitions and its relation with an open question due to Pach and Szegedy, tolerance-type problems of finite sets, topes in arrangements of hyperplanes.

1.1.1. *McMullen's problem.* $H_0(n,d)$ is a natural generalization of the following well-known problem of McMullen [9]:

What is the largest integer $\nu(d)$ such that any set of $\nu(d)$ points in general position, $X \subset \mathbb{R}^d$, can de mapped by a permissible projective transformation onto the vertices of a convex polytope?

The best known bounds for McMullen's problem are:

$$2d+1 \leq \nu(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil . \tag{2}$$

The lower bound was given by Larman [9] while the upper bound was provided by Ramirez Alfonsin [13]. In the same spirit, the following function has also been investigated:

> $\nu(d,k) :=$ the largest integer m such that any set of n points in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a k–neighborly polytope.

As a consequence of [6, Lemma 9] and [6, Equation (1)] it can be obtained that

$$\nu(d,k) \geq d + \left\lfloor \frac{d}{k} \right\rfloor + 1 \tag{3}$$

This inequality will be useful later for our proposes.

Let $t \geq 0$ be an integer. Let us define the following function.

> $n(t,d) :=$ the largest integer n such that any set of n points in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation onto the vertices of a convex polytope with at most t points in its interior.

The function $n(t,d)$ will allow us to study $H_0(n,d)$ in a more general setting, that of oriented matroids.

Remark 1. We have that $n(0,d) = \nu(d)$ and $H_0(n(d,t),d) = n(t,d) - t$.

Our first main contribution is the following

Theorem 1. *Let $d, l \geq 1$ and $n \geq 2$ be integers. Then,*

$$H_0(n,d) \begin{cases} = 2 & \text{if } d=1,\ n \geq 2, \\ = 5 & \text{if } d=2,\ n \geq 5, \\ \leq 7 & \text{if } d=3,\ n \geq 7, \\ = n & \text{if } d \geq 2,\ n \leq 2d+1, \\ \leq n-1 & \text{if } d \geq 4,\ n \geq 2d + \lceil \frac{d+1}{2} \rceil, \\ \leq n-2 & \text{if } d \geq 4,\ n \geq 2d + \lceil \frac{d+1}{2} \rceil + 1, \\ \leq n-(l+2) & \text{if } d \geq 4,\ 2d+3+l(d-2) \leq n < 2d+3+(l+1)(d-2). \end{cases}$$

A straightforward consequence of Theorem 1 is the following

Corollary 1. *Let $d, l \geq 1$, $n \geq 2$ and $1 \leq k \leq d-1$ be integers. Then,*

$$H_k(n,d) \begin{cases} = 5 & \textit{if } d = 2,\ n \geq 5, \\ \leq f_k(C_3(7)) & \textit{if } d = 3,\ n \geq 7, \\ \leq f_k(C_d(n)) & \textit{if } d \geq 2,\ n \leq 2d+1, \\ \leq f_k(C_d(n-1)) & \textit{if } d \geq 4,\ n \geq 2d + \lceil \frac{d+1}{2} \rceil, \\ \leq f_k(C_d(n-2)) & \textit{if } d \geq 4,\ n \geq 2d + \lceil \frac{d+1}{2} \rceil + 1, \\ \leq f_k(C_d(n-l-2)) & \textit{if } d \geq 4,\ 2d+3+l(d-2) \leq n < 2d+3+(l+1)(d-2) \end{cases}$$

where $C_d(n)$ denotes the d–dimensional cyclic polytope *with n vertices.*

Moreover, $H_k(n,d) \geq f_k(P_d(n))$ for $n \leq 2d+1, d \geq 2$ and in particular $H_1(7,3) = 15$ and $H_2(7,3) = 10$.

1.1.2. *Minimal Radon partitions.* Recall that given a set of points in general position $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$, with $n \geq d+2$, A, B is a *Radon partition* of X if $X = A \cup B$, $A \cap B = \emptyset$, and $conv(A) \cap conv(B) \neq \emptyset$.

It happens that $H_{d-1}(n,d)$ is very useful to count minimal Radon partitions. More specifically, let $X = A \cup B$ be any partition of X, we define $r_X(A,B)$ as the number of $(d+2)$–size subsets $S \subset X$ such that $conv(A \cap S) \cap conv(B \cap S) \neq \emptyset$, that is, as the number of *minimal (size) Radon partitions* induced by A and B.

We define the functions

$$r(X) := \max_{\{(A,B) | A \cup B = X\}} r_X(A,B) \quad \text{and} \quad r(n,d) := \min_{X \subset \mathbb{R}^d, |X| = n} r(X).$$

Our second main result establishes the connection of minimal Radon partitions with $H_{d-1}(n,d)$.

Theorem 2. *Let $d, n \geq 1$ be integers. Then,*

$$r(n,d) = H_{d'-1}(n,d') \textit{ where } d' = n-d-2.$$

We shall prove this by using the duality between *Gale transforms* and projective transformations. Theorem 2 might be useful to study of a problem due to Pach and Szegedy [12].

1.1.3. *Pach and Szegedy's question.* In [12], Pach and Szegedy investigated the probability that a triangle induced by 3 randomly and independently selected points in the plane contains the origin in its interior. They remarked [12, Last paragraph] that in order to generalize their arguments to 3-space the following problem should be solved.

Question 1. *Given n points in general position in the plane, coloured red and blue, maximize the number of multicoloured 4-tuples with the property that the convex hull of its red elements and the convex hull of its blue elements have at least one point in common. In particular, show that when the maximum is attained, the number of red and blue elements are roughly the same.*

This question may be studied in any dimension. However, if the dimension and the number of points are very similar then optimal partitions can be unbalanced. For example, one may consider $d+2$ points in $\mathbb{R}^d$ with one point contained in the simplex spanned by the remaining $d+1$ points. The optimal partition will have 1 red point and $d+1$ blue points and becomes arbitrarily unbalanced as d goes to infinity. Nevertheless, it is not clear whether for a large set of points with respect to the dimension it is also possible that very unbalanced partitions optimize the maximum number of induced Radon partitions.

On this direction, we present the following result providing support for a positive answer of Question 1.

Theorem 3. *Let $X \subset \mathbb{R}^2$ be a set of points in general position with $|X| = n \geq 9$. Then, for any $Y \subset X$ with $|Y| = |X|-1$ and any partition A, B of Y such that $r_Y(A,B) = r(Y)$, we have that $|A|, |B| \leq \lfloor \frac{n-1}{2} \rfloor + 1$.*

1.1.4. *Tolerance.* Let us define the following function

> $\lambda(t,d)$:= the smallest number λ such that for any set X of λ points in $\mathbb{R}^d$ there exists a partition of X into two sets A,B and a subset $P \subseteq X$ of cardinality $\mu - i$, for some $0 \leq i \leq t$, such that $conv(A \backslash y) \cap conv(B \backslash y) \neq \emptyset$ for every $y \in P$ and $conv(A \setminus y) \cap conv(B \setminus y) = \emptyset$ for every $y \in X \setminus P$.

We shall show the following interesting relationship between $n(t,d)$ and $\lambda(t,d)$.

Theorem 4. *Let $t \geq 0$ and $d \geq 1$ be integers. Then,*

$$n(t,d) = \max_{m \in \mathbb{N}} \{m \mid \lambda(t, m-d-1) \leq m\}$$

and

$$\lambda(t,d) = \min_{m \in \mathbb{N}} \{m \mid m \leq n(t, m-d-1)\}.$$

This theorem can be considered as a generalization of a result due to Larman [9] obtained when $t = 0$.

The parameter $\lambda(t,d)$ can be thought of as a generalization of the *tolerant* Radon theorem stating that there is a minimal positive integer $N = N(t,d)$ so that any set $X \subset \mathbb{R}^d$ with $|X| = N$ allows a partition into two pairwise disjoint subsets $X = A \cup B$ such that after deleting any t points from X the convex hulls of remaining parts intersect, i.e.,

$$conv(A \setminus Y) \cap conv(B \setminus Y) \neq \emptyset \text{ for any } Y \subset X,\ |Y| = t.$$

The information on $\lambda(d,t)$ sheds light on the understanding of the tolerant Radon theorem as well as a more general version known as the *tolerant Tverberg theorem*, see [7, 18].

1.1.5. *Arrangements of (pseudo)hyperplanes.* A *projective* d-arrangement of n pseudo-hyperplanes $\mathcal{H}(d,n)$ is a finite collection of pseudo-hyperplanes in the projective space $\mathbb{P}^d$ such that no point belongs to every hyperplane of $\mathcal{H}(d,n)$. Any such arrangement, $\mathcal{H}$ decomposes $\mathbb{P}^d$ into a d–dimensional cell complex. A cell of dimension d is usually called a *tope* of the arrangement $\mathcal{H}$. The *size* of a tope is the number of pseudo-hyperplanes bordering it.

A classic research topic is to study the combinatorics of the topes in arrangements of hyperplanes. For instance, it is known [16, 17] that arrangements of n hyperplanes (that is, realizable oriented matroids) always admit n topes of size $d+1$ (a simplex). In [14], Richter proved that the number of topes simplices in an arrangement of $4k$ pseudo hyperplanes in $\mathbb{P}^3$ is at most $3k+1$ for $k \geq 2$. Finding a sharp lower bound for the number of simplices in the non-realizable case is an open problem for $d \geq 3$. Las Vergnas conjectured that in fact every arrangement of (pseudo) hyperplanes in $\mathbb{P}^d$ admits at least one simplex. In [15], Roudneff proved that the number of *complete topes* (a tope touching all the hyperplanes) of the *cyclic arrangement* on dimension d with n hyperplanes, is at least $\sum\limits_{i=0}^{d-2} \binom{n-1}{i}$ and conjectured [15, Conjecture 2.2] that for every d-arrangement of $n > 2d+1 > 5$ (pseudo)hyperplanes has at most this number of complete topes; see [11] for the proof of this conjecture for an infinite family of arrangements.

It happens that the function $H_0(n,d)$ is very helpful to investigate the size's behavior of topes in arrangements of (pseudo)hyperplanes. Here, we may consider the following questions :

> Are there simple arrangements of n (pseudo)hyperplanes in $\mathbb{P}^d$ in which every tope is of at most *certain size* ?
>
> Which arrangements of n (pseudo)hyperplanes in $\mathbb{P}^d$ contain a tope of at least *certain size* ?

We partially answer these questions for small values of d.

1.2. **Paper's organization.** The structure of the paper is as follows: in next section we give some easy values and bounds for both $H_0(n,d)$ and $H_{d-1}(n,d)$ (Propositions 1 and 1).

In Section 3, we discuss the treatment of the function $n(d,t)$ in the oriented matroid setting. We also recall several notions and results on oriented matroids and, specifically, on the special class of *Lawrence oriented matroids* (LOM) that are needed for the rest of the paper.

In Section 4, we present several upper bounds based on specific constructions of LOM (Theorems 5, 6, 7, 8). The latter yield to the proofs of Theorem 1 and Corollary 1 also presented in this section.

After recalling the relationship between Gale transforms and projective transformations, we prove Theorem 2 in Section 5. We also present values and bounds for $r(n,2)$ (Theorem

10) that we use to prove Theorem 3. We finally prove our Tolerant results (Theorem 4) at the end of this section.

In Section 6, we present some results concerning the size of *topes* in arrangements of (pseudo)hyperplanes. Finally, in an Annex, we present the proof of a result (Theorem 9) improving the upper bound given in Theorem 7 when d is even.

2. Some basic results

The well-known Upper Bound Theorem (UBT) [10] states that for all $1 \leq k \leq d$,

$$f_{k-1}(P) \leq f_{k-1}(C_d(n))$$

among all simplicial (convex) polytopes with n vertices $P \subset \mathbb{R}^d$ where $C_d(n)$ is the d–dimensional *cyclic polytope* with n vertices, usually defined as the convex hull of n distinct points in the moment curve $x(t) := (t, t^2, \dots, t^d)$.

For $d \geq 2$ and $0 \leq k \leq d-1$, the number of k-faces of $C_d(n)$ with n vertices is given by

$$f_k(C_d(n)) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \binom{j}{d-k-1}\binom{n-d+j-1}{j} + \sum_{j=\lfloor \frac{d}{2} \rfloor+1}^{d} \binom{j}{d-k-1}\binom{n-j-1}{d-j} \tag{4}$$

Since $H_0(n,d)$ is the maximal number of projective vertices obtained from any set of n points in $\mathbb{R}^d$ then, by the UBT, the number of k-faces of a projective polytope on $H_0(n,d)$ vertices is bounded by the number of k-faces of $C_d(H_0(n,d))$. We thus have that

$$H_k(n,d) \leq f_k(C_d(H_0(n,d))) \text{ for all } n \geq 1. \tag{5}$$

Analogously, the Lower Bound Theorem [1, 2] states that for all $1 \leq k \leq d-1$,

$$f_k(P_d(n)) \leq f_k(P)$$

among all simplicial (convex) polytopes $P \subset \mathbb{R}^d$ with n vertices, where $P_d(n)$ is a d-dimensional *stacked polytope* with n vertices, defined as a polytope formed from a simplex by repeatedly gluing another simplex onto one of its facets.

For $d \geq 2$ and $0 \leq k \leq d-1$, the number of k-faces of $P_d(n)$ with n vertices is

$$f_k(P_d(n)) = \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{if } 0 \leq k \leq d-2, \\ (d-1)n - (d+1)(d-2) & \text{if } k = d-1. \end{cases} \tag{6}$$

As above, we may deduce that

$$f_k(P_d(H_0(n,d))) \leq H_k(n,d). \tag{7}$$

Proposition 1. *Let $d \geq 2, n \geq 1$ be integers. Then,*

$$H_0(n,d) \begin{cases} = n & \text{if } n \leq 2d+1, \\ < n & \text{if } n \geq 2d + \lceil \frac{d+1}{2} \rceil. \end{cases}$$

Proof. Let $n \leq 2d+1$. By the lower bound of $\nu(d)$ given in (2), it follows that any set of points of cardinality n can be mapped to the vertices of a convex polytope by a permissible projective transformations, and thus $H_0(n,d) = n$. If $n \geq 2d + \lceil \frac{d+1}{2} \rceil$ then by the upper bound of $\nu(d)$ given in Equation (2), there exists a set of n points that cannot be mapped to the vertices of a convex polytope by any permissible projective transformation, and thus $H_0(n,d) \leq n-1$.

□

We have the following easy consequence of Proposition 1 and Inequality (5).

Proposition 2. *Let $d \geq 2, n \geq 1$ and $1 \leq k \leq d-1$ be integers. Then,*

$$H_k(n,d) \begin{cases} \leq f_k(C_d(n)) & \text{if } n \leq 2d+1, \\ \leq f_k(C_d(n-1)) & \text{if } n \geq 2d + \lceil \frac{d+1}{2} \rceil. \end{cases}$$

Moreover, if $n \leq 2d+1, d \geq 2$ then, by Propostion 1, $H_0(n,d) = n$ and so $f_k(P_d(H_0(n,d))) = f_k(P_d(n))$ obtaining, by Equation (7), that

$$H_k(n,d) \geq f_k(P_d(n)) \tag{8}$$

3. Oriented matroid setting

Let us briefly give some basic notions and definitions on oriented matroid theory needed for the rest of the paper. We refer the reader to [3] for background on oriented matroid theory.

3.1. **Oriented matroid preliminaries.**

Let M be an oriented matroid on a finite set E. The matroid M is *acyclic* if it does not contain positive circuits (otherwise, M is called *cyclic*). A *reorientation* of M on $A \subseteq E$ is performed by changing the signs of the elements in A in all the circuits of M. It is easy to check that the new set of signed circuits is also the set of circuits of an oriented matroid, usually denoted by ${}_{-A}M$. A reorientation is *acyclic* if ${}_{-A}M$ is acyclic. An element $e \in E$ of an acyclic oriented matroid is *interior* if there exists a signed circuit $C = (C^+, C^-)$ with $C^- = \{e\}$.

Cordovil and Da Silva [4] proved that a permissible projective transformation on a set n points in $\mathbb{R}^d$ corresponds to an acyclic reorientation of its oriented matroid of affine dependencies M of rank $r = d+1$ and that the converse also holds.

As a consequence of Cordovil and Da Silva's result it is evident that the natural generalization of $n(t,d)$ in terms of oriented matroids is given by the following function:

> $\bar{n}(t,d) :=$ the largest integer m such that for any uniform oriented matroid M of rank $d+1$ with m elements there is an acyclic reorientation of M with at most t interior elements.

We notice that $n(t,d) = \bar{n}(t,d)$ in the case when M is *realizable.*

In this section we shall provide examples of uniform oriented matroids with the property that in any of their acyclic reorientations there are at least $t+1$ interior elements. These examples provide upper bounds on $\bar{n}(t,d)$. With this aim, we will briefly outline some facts about *Lawrence oriented matroids*. For further details and proofs on this special class of matroids see [3, 13].

3.2. **Lawrence oriented matroid.**

A *Lawrence oriented matroid* (LOM) M of rank r on the totally ordered set $E = \{1, \dots, n\}$, $r \leq n$, is a uniform oriented matroid obtained as the union of r uniform oriented matroids $M_1, \dots, M_r$ of rank 1 on $(E, <)$. LOMs can also be defined via the signature of their bases, that is, via their chirotope χ. Indeed, the chirotope χ corresponds to some LOM, M_A, if and only if there exists a matrix $A = (a_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n$ with entries from $\{+1, -1\}$ (where the i-th row corresponds to the chirotope of the oriented matroid $\mathcal{M}_i$) such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i}$$

where B is an ordered r-tuple, $j_1 \leq \cdots \leq j_r$, of elements of E.

Remark 2. The following statements about LOMs hold:

(a) Acyclic LOMs are realizable as configurations of points (since they are unions of realizable oriented matroids).

(b) LOMs are closed under minors and duality.

(c) The LOM corresponding to the reorientation of an element $c \in E$, ${}_{\bar{c}}M_A$ is obtained by reversing the sign of all the coefficients of a column c in A.

For a proof of this remark see [3].

From now on, we will denote $A = A_{r,n}$ as a matrix with entries $a_{i,j} \in \{+1, -1\}$, $1 \leq i \leq r$, $1 \leq j \leq n$. Some of the following definitions and lemmas, which highlight the properties of A and facilitate the study of this type of matroid, were introduced and proved in [13].

A *Top Travel*, denoted as TT, in A is a subset of the entries of A,

$$\{[a_{1,1}, a_{1,2}, \dots, a_{1,j_1}], [a_{2,j_1}, a_{2,j_1+1}, \dots, a_{2,j_2}], \dots, [a_{s,j_{s-1}}, a_{s,j_{s-1}+1}, \dots, a_{s,j_s}]\},$$

where $[a_{l,j_{l-1}}, \dots, a_{l,j_l}]$ are the entries in line l, with the following characteristics:

1. $a_{i,j_{i-1}} \times a_{i,j} = 1, \quad \forall \quad j_{i-1} \leq j < j_i$;
2. $a_{i,j_{i-1}} \times a_{i,j_i} = -1$; and
3. either
    - (a) $1 \leq s < r$; then $j_s = n$ or
    - (b) $s = r$ and $j_s \leq n$.

A *Bottom Travel*, denoted as BT, in A is a subset of the entries of A,

$$\{[a_{r,n}, a_{r,n-1}, \dots, a_{r,j_r}], [a_{r-1,j_r}, a_{r-1,j_r-1}, \dots, a_{r-1,j_{r-1}}], \dots, [a_{s,j_{s-1}}, a_{s,j_{s-1}+1}, \dots, a_{s,j_s}]\},$$

with the following characteristics:

1. $a_{i,j_{i+1}} \times a_{i,j} = 1, \quad \forall \quad j_i < j \leq j_{i+1}$;

(2) $a_{i,j_{i+1}} \times a_{i,j_i} = -1$; and

(3) either

  (a) $1 < s \leq r$; then $j_s = 1$ or

  (b) $s = 1$ and $1 \leq j_s$.

Every matrix A has exclusively one TT and one BT and they carry surprising information about M_A.

Remark 3. Let A be a $r \times n$-matrix, then the following statements are equivalent:

(a) M_A is *cyclic*;

(b) TT ends at $a_{r,s}$ for some $1 \leq s < n$; and

(c) BT ends at $a_{1,s'}$ for some $1 < s' \leq n$.

For a proof of this remark see [13].

Let $a_{i,k-1}, a_{i,k}, a_{i,k+1} \in TT$ we say that TT and BT are *parallel* at column k if either $a_{i,k-1}, a_{i,k}, a_{i,k+1} \in BT$ or $a_{i+1,k-1}, a_{i+1,k}, a_{i+1,k+1} \in BT$, with $2 \leq k \leq n-1$, $1 \leq i \leq r$.

Remark 4. [13] Let A be a $r \times n$-matrix then k is an interior element of M_A if and only if

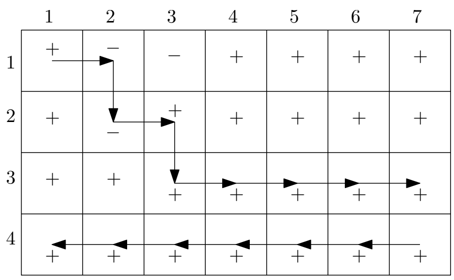
(a) $BT = (a_{r,n}, \ldots, a_{1,2}, a_{1,1})$ for $k = 1$,

(b) $TT = (a_{1,1}, \ldots, a_{r,n-1}, a_{r,n})$ for $k = n$,

(c) TT and BT are parallel at k for $2 \leq k \leq n-1$.

Remark 4 implies that we can identify acyclic reorientations and interior elements of an oriented matroid M_A by studying the behaviour of the TT and BT in the re-orientations of A.

Example 1. Let M_A be the LOM associated to the matrix A described in Figure 1. M_A is acyclic, and 4, 5 and 6 are interior elements.



	1	2	3	4	5	6	7
1	+	−	−	+	+	+	+
2	+	−	+	+	+	+	+
3	+	+	+	+	+	+	+
4	+	+	+	+	+	+	+

Figure 1. Top and Bottom travels in matrix A.

Furthermore, all possible re-orientations of the matroid can be identified with yet another simple object;

A *Plain Travel* in A, denoted as PT, is a subset of the entries of A which satisfies:

$$PT = \{[a_{1,1}, a_{1,2}, \ldots, a_{1,j_1}], [a_{2,j_1}, a_{2,j_1+1}, \ldots, a_{2,j_2}], \ldots, [a_{s,j_{s-1}}, a_{s,j_{s-1}+1}, \ldots, a_{s,j_s}]\}$$

with $2 \leq j_{i-1} < j_i \leq n$ for all $1 \leq i \leq r$, $1 < s \leq r$ and $j_s = n$.

Remark 5. There is a bijection between the set of all plain travels of A and the set of all acyclic reorientations of M_A, it is defined by associating to each PT the set of column indices of A that should be reoriented in order to transform A into a new matrix $\mathcal{A}$ whose TT is identical to PT. For a proof of this remark see [13].

The *chessboard* $B[A]$ of A is another useful object that can be constructed from its entries. It is defined as a black and white board of size $(r-1) \times (n-1)$, such that the square $s(i,j)$ has its upper left hand corner at the intersection of row i and column j; a square $s(i,j)$, with $1 \leq i \leq r-1$ and $1 \leq j \leq n-1$ will be said to be *black* if the product of the entries $a_{i,j}, a_{i,j+1}, a_{i+1,j}, a_{i+1,j+1}$ is -1, and *white* otherwise. See Figure 2 for an example.

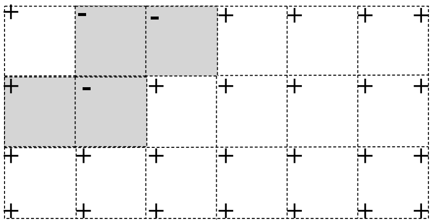


FIGURE 2. The chessboard $B[A]$ of the matrix A described in Figure 1.

Remark 6. [13] The following statements establish a link between chessboards, reorientations of a LOM, and BT and TT:

(a) $B[A]$ is invariant under reorientations of M_A.

(b) If in a pair of consecutive columns there is one black square between the top travel TT and the bottom travel BT, they follow symmetrically opposite paths through the entries of the matrix; in other words, if TT makes a single horizontal movement from $a_{i,j}$ to $a_{i,j+1}$ and continues its movement forward in the same row (i.e $a_{i,j} = a_{i,j+1}$) , then BT goes from $a_{i+h,j+1}$ to $a_{i+h,j}$ and moves vertically to $a_{i+h-1,j}$ (i.e. $a_{i+h,j+1} \neq a_{i+h,j}$), with $h \geq 1$, and vice versa.

4. UPPER BOUNDS FOR $\bar{n}(d,t)$

4.1. **Small dimension d.** We first show that $\bar{n}(t,d) \leq 2d+1+t$ for $d = 2,3$ and every $t \geq 0$. The following remark will be very useful throughout this section.

Remark 7. Any acyclic reorientation of a rank 2 oriented matroid on n elements has $n-2$ interior elements.

Given a matrix $A = A_{n,r}$, let $A^{+}_{i,j}$ be the sub-matrix of A that results after removing rows $i+1,\dots,r$ and columns $j+1,\dots,n$. Similarly, let $A^{-}_{i,j}$ be the sub-matrix of A resulting after the removal of rows $1,\dots,i-1$ and columns $1,\dots,j-1$.

Theorem 5. $\bar{n}(t,2) < t+6$ *for every integer* $t \geq 0$.

Proof. Let $A = A_{3,t+6}$ be such that the corresponding chessboard $B[A]$ has exactly one black square for each column and let PT be any plain travel in A. We shall prove that the corresponding $\mathcal{A}$ in which PT is the Top Travel has at least $t+1$ interior elements. Let j be the smallest number such that column j is not an interior element in $\mathcal{A}$ and there are not vertical movements in column j of PT neither of BT. If j does not exists, as PT and BT can make at most 2 vertical movements each, then $\mathcal{A}$ would have at least $t+2$ interior elements. Hence, we may suppose that j exists. By the rules of Proposition 6, PT and BT make a vertical movement in column $j+1$ and $j-1$, respectively. Notice by the definition of j that PT and BT arrives in column j at row 1 and 3, respectively, otherwise j would be an interior element. Then, each interior element of $A^{+}_{2,j-1}$ and $A^{-}_{2,t+6-j}$ is an interior element of $\mathcal{A}$. Therefore, $A^{+}_{2,j-1}$ has $j-3$ interior elements and $A^{-}_{2,t+6-j}$ has $t+6-j-2$ interior elements by Remark 7, concluding the proof. □

Given a matrix $A = A_{r,n}$, we say that a chess board $B[A]$ has the *sequence* $(x_1, x_2, \dots, x_{r-1})$ if the square $s(i,j)$ is black if and only if $\sum_{k=0}^{i-1} x_k + 1 \leq j \leq \sum_{k=0}^{i} x_k$ with $1 \leq i \leq r-1$ and $1 \leq j \leq n-1$, where we define $x_0 = 0$.

Example 2. Figure 4 illustrates a chessboard with sequence $(2,3,2,3)$.

Theorem 6. $\bar{n}(t,3) < t+8$ *for every integer* $t \geq 0$.

Proof. Let $A = A_{4,t+8}$ be such that the corresponding chessboard $B[A]$ has a sequence $(2, t+3, 2)$ and let PT be any plane travel in A. We prove that the corresponding $\mathcal{A}$ in which PT is the Top Travel has at least $t+1$ interior elements. Let j be the smallest number such that column j is not an interior element in $\mathcal{A}$ and there are not vertical movements in column j of PT neither of BT. If j does not exist, as PT and BT can make at most 3 vertical movements each, then $\mathcal{A}$ would have at least $t+2$ interior elements. Hence, we may suppose that j exists. By the rules of Proposition 6, PT and BT make a vertical movement in column $j+1$ and $j-1$, respectively.

By the definition of j, TT arrives in column j at row 1 or BT arrives in column j at row 4, otherwise j would be an interior element. Suppose without loss of generality that TT arrives in column j at row 1. Then, BT arrives in column j at row 2 or 3 (see Figure 3). Notice that each interior element of $A^{+}_{3,j-1}$ and $A^{-}_{2,t+8-j}$ is an interior element of $\mathcal{A}$. If BT arrives in column j at row 3, by the rules of Proposition 6, $A^{+}_{3,j-1}$ has $j-4$ interior

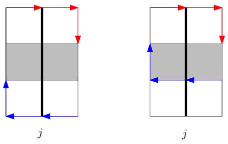


FIGURE 3. BT in blue and TT in red.

elements and $A^-_{2,t+8-j}$ has $t+5-j$ interior elements, concluding the proof in this case. If BT arrives in column j at row 2, by the rules of Proposition 6 and by Remark 7, one can check that $A^+_{3,j-1}$ has $j-3$ interior elements and $A^-_{2,t+8-j}$ has at least $t+4-j$ interior elements, concluding the proof. ☐

4.2. **High dimension d.** In what follows, we will consider different matrices $A = A_{r,h(r)}$ where $h(r)$ is a strictly increasing function. If $B[A]$ has the sequence $(x_1, x_2, \dots, x_{r-1})$, we will consider functions, $h(r)$, where $h(1) = 1$, $\sum\limits_{k=1}^{m-1} x_k + 1 \le h(m) \le \sum\limits_{k=1}^{m} x_k + 1$ if $2 \le m \le r-1$ and $\sum\limits_{k=1}^{r-1} x_k + 1 \le h(r)$.

For every $1 \le m \le r-1$, we will say that the element $a_{m,h(m)}$ is the *m–th corner* of A.

Example 3. Figure 4 illustrates a chessboard with $h(r) = 2(r-1) + \lceil \frac{r}{2} \rceil$.

The following lemmas will be helpful ingredients for our proposes.

Lemma 1. *Let $A = A_{r,h(r)}$ be a matrix with $r \ge 3$ such that $B[A]$ has the sequence $(x_1, x_2, \dots, x_{r-1})$, with $x_i \ge 1$, $1 \le i \le r-1$. Suppose that TT always passes strictly above all corners after the 1–st corner, then*

(i) *$a_{i,h(2)} \in BT$ for some $i \le \max\{2, 2r - h(r-1) + h(2) - 3\}$,*

(ii) *$a_{i,h(2)} \in BT$ for some $i \le \max\{2, 2r - h(r) + h(2) - 1\}$ if TT and BT do not share steps from columns $h(2)$ to $h(r)$.*

Proof. We will proceed by induction on $r \ge 3$. Using the rules of Proposition 6, one can check that the lemma holds for $r = 3$. Suppose that the result holds for $r-1$ and we show it for $r \ge 4$. If BT arrives at the m–th corner for $3 \le m \le r-1$, the result follows by the induction hypothesis. Moreover, if BT arrives at the 2–th corner, the result follows. Similarly, the result follows if BT arrives at $a_{i,h(m)}$ for $2 \le m \le r-1$ and $i \le m$. Then, we may suppose that BT always passes strictly below the m–th corner for every $2 \le m \le r-1$.

Notice that TT makes exactly $h(r) - h(2)$ horizontal movements and at most $r-1$ vertical movements, from right to left, to arrive at $a_{1,h(2)}$. We know, by the rules of construction of TT that for each vertical movement we must also count one horizontal movement. So, TT makes at least $h(r)-h(2)-(r-1)$ single horizontal movements, from right to left, until $a_{1,h(2)}$ is attained.

On the other hand, as TT always passes strictly above all corners after the 1–st corner and BT always passes strictly below the m–th corner for every $2 \leq m \leq r-1$, then TT and BT do not share steps from columns $h(2)$ to $h(r-1)$. However, TT and BT could share at most $h(r)-h(r-1)-2$ steps from columns $h(r-1)+1$ to $h(r)$ (see Figure 4).

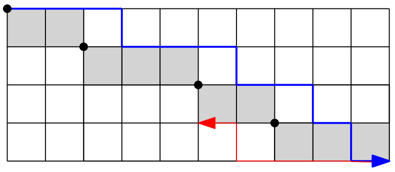


FIGURE 4. The chessboard $B[A]$ with sequence $(2,3,2,3)$ and $h(r) = 2(r-1)+\lceil \frac{r}{2} \rceil$. The points represent the corners of $A = A_{5,h(5)}$. We observe that TT and BT share the final step.

Therefore, by the rules of Proposition 6, for each single horizontal movement that TT does not share with BT, BT makes a vertical movement. So, BT makes at least $h(r)-h(2)-(r-1)-(h(r)-h(r-1)-2) = h(r-1)-h(2)-r+3$ vertical movements, from right to left, until column $h(2)$ is attained. Hence, BT arrives at $a_{i,h(2)}$ for some $i \leq \max\{2, r-(h(r-1)-h(2)-r+3)\}$ concluding the first part of the proof. If TT and BT do not share steps from columns $h(2)$ to $h(r)$, then BT makes at least $h(r)-h(2)-(r-1)$ vertical movements, from right to left, until column $h(2)$ is attained, concluding that BT arrives at $a_{i,h(2)}$ for some $i \leq \max\{2, r-(h(r)-h(2)-(r-1))\}$. ☐

Lemma 2. *Let $A = A_{r,h(r)}$ be a matrix and suppose that TT always passes strictly above all corners after the 1–st corner. Then the following holds:*

(i) *If $B[A]$ has a sequence $(x_1, x_2, \ldots, x_{r-1})$, $x_i \geq 2$ for odd i, $x_j \geq 3$ for even j and $h(m) = \sum_{k=0}^{m-1} x_k + 1$ for every $1 \leq m \leq r$, then $a_{1,1}, a_{1,2} \in BT$ when $r \geq 4$. When $r = 3$, $a_{1,1}, a_{1,2} \in BT$, or column $h(r)$ is an interior element and $a_{2,1}, a_{1,1} \in BT$.*

(ii) *If $B[A]$ has a sequence $(x_1, x_2, \ldots, x_{r-1})$, $x_i \geq 3$ for odd i, $x_j \geq 2$ for even j and $h(m) = \sum_{k=0}^{m-1} x_k + 1$ for every $1 \leq m \leq r$, then $a_{1,1}, a_{1,2} \in BT$ when $r \geq 3$.*

(iii) *If $B[A]$ has a sequence $(2, t+3, 2, t+1, t+1, \ldots, t+1)$ for some $t \geq 2$, $h(2) = t+3$, $h(3) = t+6$ and $h(m) = (t+1)(m-3)+7$ for $4 \leq m \leq r$ for every $4 \leq m \leq r$, then $a_{i,t+3} \in BT$ for some $i \leq 2$ when $r \geq 4$ (see Figure 6).*

(iv) *If $B[A]$ has a sequence $(t+1, \ldots, t+1, 2, t+3, 2)$ for some $t \geq 2$, $h(2) = t+3$, $h(r) = (t+1)(r-3)+7$ and TT and BT do not share steps from columns $h(2)$ to $h(r)$, then $a_{i,t+3} \in BT$ for some $i \leq 2$ when $r \geq 5$.*

(v) *If* $B[A]$ *has a sequence* $(2,4,2,3,2,3,\ldots)$ *and* $h(m) = 2(m-1) + \lceil \frac{m}{2} \rceil + 1$ *for every* $2 \leq m \leq r$*, then* $a_{i,4} \in BT$ *for some* $i \leq 2$ *when* $r \geq 4$*.*

(vi) *If* $r \geq 6$ *is even,* $B[A]$ *has a sequence* $(2,3,2,3,\ldots,2,4,2)$*,* $h(2) = 4$*,* $h(r) = 2(r-1) + \lceil \frac{r}{2} \rceil + 1$ *and* TT *and* BT *do not share steps from columns* $h(2)$ *to* $h(r)$*, then* $a_{i,4} \in BT$ *for some* $i \leq 2$*.*

Proof. We prove (i) and (ii) for $B[A]$ with sequences $(2,3,2,3\ldots)$ and $(3,2,3,2\ldots)$, respectively, since the general case holds as a consequence.

(i) By the sequence of $B[A]$, we observe that $h(m) = 2(m-1) + \lceil \frac{m}{2} \rceil$ for $1 \leq m \leq r$. Then, as $2r - h(r-1) + h(2) - 3 = 4 - \lceil \frac{r-1}{2} \rceil \leq 2$ when $r \geq 4$, we obtain by Lemma 1 (i) that $a_{i,h(2)} \in BT$ for some $i \leq 2$. Since $a_{1,i} \in TT$ for $i \leq 4$ and $h(2) = 3$, we conclude by the rules of Proposition 6 that $a_{1,1}, a_{1,2} \in BT$. If $r = 3$, one can check that $a_{1,1}, a_{1,2} \in BT$, or column 6 is an interior element and $a_{1,1}, a_{2,1} \in BT$ (see Figure 5).

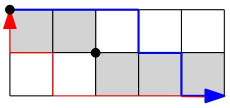


FIGURE 5. Case $r = 3$ when column 6 is an interior element and $a_{1,1}, a_{2,1} \in BT$. The points represent the corners.

(ii) By the sequence of $B[A]$, we observe that $h(m) = 2(m-1) + \lceil \frac{m+1}{2} \rceil$ for $1 \leq m \leq r$. Then, as $2r - h(r-1) + h(2) - 3 = 5 - \lceil \frac{r}{2} \rceil \leq 3$, we obtain by Lemma 1 (i) that $a_{i,h(2)} \in BT$ for some $i \leq 3$. Since $a_{1,i} \in TT$ for $i \leq 5$ and $h(2) = 4$, we conclude by the rules of Proposition 6 that $a_{1,1}, a_{1,2} \in BT$.

(iii) As $2r - h(r-1) + h(2) - 3 = 2$ when $r = 4$ and $2r - h(r-1) + h(2) - 3 = 2r - (t+1)(r-4) - 7 + t \leq 7 - r \leq 2$ when $r \geq 5$ and $t \geq 2$, we obtain by Lemma 1 (i) that $a_{i,h(2)} \in BT$ for some $i \leq 2$.

(iv) As $2r - h(r) + h(2) - 1 = 2r - (t+1)(r-3) + t - 5 \leq 6 - r \leq 1$ when $r \geq 5$ and $t \geq 2$, we obtain by Lemma 1 (ii) that $a_{i,h(2)} \in BT$ for some $i \leq 2$.

(v) As $2r - h(r-1) + h(2) - 3 = 4 - \lceil \frac{r-1}{2} \rceil \leq 2$ when $r \geq 4$, we obtain by Lemma 1 (i) that $a_{i,h(2)} \in BT$ for some $i \leq 2$.

(vi) As $2r - h(r) + h(2) - 1 = 2r - 2(r-1) - \frac{r}{2} + 2 = 4 - \frac{r}{2} \leq 1$ when $r \geq 6$ is even, we obtain by Lemma 1 (ii) that $a_{i,h(2)} \in BT$ for some $i \leq 2$. ☐

From now on, denote as A^+_m and A^-_m the matrices $A^+_{m,h(m)}$ and $A^-_{m,h(m)}$, respectively (see Figure 6).

We are now ready to tackle the case when $d \geq 4$ and $t \geq 2$.

Theorem 7. $\bar{n}(t,d) < 2d + (t-1)(d-2) + 3$ *for integers* $d \geq 4$ *and* $t \geq 2$*.*

Proof. Let $A = A_{r,h(r)}$ be a matrix where $h(r)$ is defined as $h(2) = t+3$, $h(3) = t+6$, $h(m) = (t+1)(m-3)+7$ for $4 \leq m \leq r$ and $B[A]$ with sequence $(2, t+3, 2, t+1, t+1, \dots, t+1)$ for $t \geq 2$ (see Figure 6). We shall show by induction on r that for every $r \geq 2$ and for any plain travel PT in A, the corresponding $\mathcal{A}$ in which PT is the Top Travel has at least $t+1$ interior elements. In particular, as $h(r) = (t+1)(d-2)+7 = 2d+(t-1)(d-2)+3$ for $r \geq 5$, we will prove the theorem for $d \geq 4$ and $t \geq 2$.

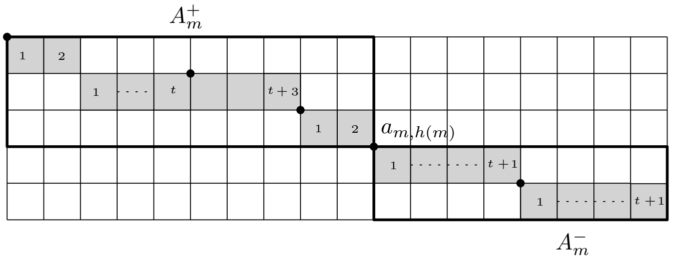


FIGURE 6. A matrix $A = A_{6,h(6)} = A_{6,19}$ and the sub-matrices A_4^+ and A_4^-. The chessboard $B[A]$ has sequence $(2, t+3, 2, t+1, t+1)$ for $t = 3$. The points represent the corners of A associated to the function $h(r)$ of Theorem 7.

We observe that $\mathcal{A}$ has $t+1$ interior elements when $r = 2$ since $\mathcal{A}$ is a $2 \times (t+3)$ matrix (Remark 7). For $r = 3$ and 4, the result follows by Theorems 5 and 6, respectively, since the chessboards considered in these theorems coincide with $B[A]$. Thus, assume that the theorem holds for $r-1$ and we show it for $r \geq 5$. Suppose that the m–st corner is the last corner that PT meets in A, for some $1 \leq m \leq r-1$. If PT always passes strictly below the i–st corner for $i > m$, then there would be at least $t+2$ interior elements in $\mathcal{A}$ (from columns $h(r-1)$ to $h(r)$). Hence, we may suppose that PT always passes strictly above the i–st corner for $i > m$. We have the following cases.

Case $m \leq r-3$. First suppose that $m = 1$. Then $a_{i,t+3} \in BT$ for some $i \leq 2$ by Lemma 2 (iii), concluding by the rules of Proposition 6 that there are at least $t+1$ interior elements in $\mathcal{A}$. Now suppose that $2 \leq m \leq r-3$. As $\mathcal{A}_m^-$ has at least 4 rows, applying Lemma 2 (i) (when $m = 3$) and Lemma 2 (ii) (when $m \neq 3$) on sub-matrix $\mathcal{A}_m^-$, we obtain that $a_{m,h(m)}, a_{m,h(m)+1} \in BT$. Thus, the theorem holds by induction hypothesis on $\mathcal{A}_m^+$ since BT restricted in $\mathcal{A}_m^+$ is also the Bottom travel of $\mathcal{A}_m^+$ and each interior element of $\mathcal{A}_m^+$ is an interior element of $\mathcal{A}$.

Case $m = r-2$. As $\mathcal{A}_m^-$ has 3 rows, $a_{m,h(m)}, a_{m,h(m)+1} \in BT$, or column $h(r)$ is an interior element and $a_{m+1,h(m)}, a_{m,h(m)} \in BT$ by Lemma 2 (i). If $a_{m,h(m)}, a_{m,h(m)+1} \in BT$, the theorem holds by induction hypothesis on $\mathcal{A}_m^+$. If column $h(r)$ is an interior element and $a_{m+1,h(m)}, a_{m,h(m)} \in BT$, notice that each interior element of $\mathcal{A}_m^+$ is an interior element of $\mathcal{A}$, except for column $h(m)$. Thus, $\mathcal{A}$ has at least t interior elements from columns 1 to $h(m)$ by induction hypothesis on $\mathcal{A}_m^+$ and one interior element in column $h(r)$.

Case $m = r-1$. First suppose that BT arrives at the k-th corner for some $2 \leq k \leq r-1$. As each interior element of $\mathcal{A}_k^+$ and $\mathcal{A}_m^-$ is an interior element of $\mathcal{A}$, except for (maybe) columns $h(k)$ and $h(m)$, $\mathcal{A}$ has at least t interior elements from columns 1 to $h(k)$ by induction hypothesis on $\mathcal{A}_k^+$ and at least $t-1 \geq 1$ interior elements from columns $h(m)$ to $h(r)$ by Remark 7 (since $\mathcal{A}_m^-$ is a $2 \times (t+2)$ matrix), concluding the proof in this case. Similarly, the proof holds if BT arrives at $a_{i,h(k)}$ for $2 \leq k \leq r-2$ and $i \leq k$. Now suppose that BT passes always below the i–st corner, for every $i \geq 2$. In particular, as BT does not arrives at the m-th corner, every interior element of $\mathcal{A}_m^-$ is an interior element of $\mathcal{A}$, concluding by Remark 7 that $\mathcal{A}$ has t interior elements from columns $h(m)+1$ to $h(r)$. So, we may suppose that TT and BT do not share steps from columns 1 to $h(2)$, otherwise the theorem holds. Also, if $a_{m,h(m)-1} \in PT$, then column $h(m)$ is an interior element and the theorem holds. So, we may suppose that $a_{m,h(m)-1} \notin PT$. Hence, $a_{r,h(m)-2} \in BT$ by the rules of Proposition 6. Let $\mathcal{A}^{'}$ be the matrix obtained by turning the matrix $\mathcal{A}$ upside down. We observe that BT and PT are the Top and Bottom Travels of $\mathcal{A}^{'}$, respectively. Let us define the i–st corners of $\mathcal{A}^{'}$ as $a_{r-i+1,h(r-i+1)}$ for $i \neq 2$ and define the 2–st corner of $\mathcal{A}^{'}$ as $a_{m,h(m)-1}$.

Notice that BT always passes strictly above all corners of $\mathcal{A}^{'}$ after the 1–st corner of $\mathcal{A}^{'}$. Moreover, $B[A']$ has the same sequence and the same 2–st corner as that considered in Lemma 2 (iv). Hence, as TT and BT do not share steps from columns $h(1)$ to $h(m)-1$, we know by Lemma 2 (iv) that $a_{i,h(m)-1} \in PT$ for some $i \geq r-1$, but this is a contradiction since we had assumed that $a_{m,h(m)-1} \notin PT$ and clearly $a_{r,h(m)-1} \notin PT$, concluding the proof. □

Figure 7 (a) shows that the chessboard considered in Theorem 7 cannot be used to prove $\bar{n}(d,1) < 2d + (t-1)(d-2) + 3$ for $d = 4$ and $t = 1$. In fact, this example can be generalized in order to show that this chessboard cannot be used to prove $\bar{n}(d,t) < 2d+3$ for $d \geq 4$.

Theorem 8. *$\bar{n}(1,d) < 2d + \lceil \frac{d+1}{2} \rceil + 1$ for any integer $d \geq 4$.*

Proof. Let $A = A_{r,h(r)}$ be a matrix where $h(r)$ is defined as $h(m) = 2(m-1) + \lceil \frac{m}{2} \rceil + 1$ for every $2 \leq m \leq r$ and $B[A]$ has sequence $(2,4,2,3,2,3,\ldots)$. We shall show by induction on r that for every $r \geq 2$ and for any plain travel PT in A, the corresponding $\mathcal{A}$ in which PT is the Top Travel has at least 2 interior elements. In particular, as $h(r) = 2d + \lceil \frac{d+1}{2} \rceil + 1$ for $r \geq 5$, we will prove the theorem for $d \geq 4$.

We observe that $\mathcal{A}$ has at least 2 interior elements when $r = 2$ since $\mathcal{A}$ is a 2×4 matrix (Remark 7). For $r = 3$ and 4, the result follows by Theorems 5 and 6, respectively (applying them for $t = 1$), since the chessboards considered in these theorems coincide with $B[A]$. Thus, assume the theorem holds for $r-1$ and we show it for $r \geq 5$. Suppose that the m–st corner is the last corner that PT meets in A, for some $1 \leq m \leq r-1$. If PT always passes strictly below the i–st corner for $i > m$, then there would be at least 3 interior elements in $\mathcal{A}$ (from columns $h(r-1)$ to $h(r)$). Hence, we may suppose that PT always passes strictly above the i–st corner for $i > m$. If $m = 1$, then $a_{i,4} \in BT$ for some $i \leq 2$ by Lemma 2 (v), concluding by the rules of Proposition 6 that there are at least 2 interior elements in $\mathcal{A}$. We omit the proof of the cases $2 \leq m \leq r-2$, since they are analogous to those in the proof of Theorem 7. Now, consider the case $m = r-1$.

If r is odd, $B[A]$ has a sequence $(2,4,2,3,2,3,\dots,2,3)$ and one can verify that there are at least 2 interior elements in $\mathcal{A}$ or $a_{m,h(m)}, a_{m,h(m)+1} \in BT$. In both cases the theorem holds. Now suppose that r is even and then $B[A]$ has a sequence $(2,4,2,3,2,3,\dots,3,2)$. If $a_{m,h(m)}, a_{m,h(m)+1} \in BT$, the result follows by induction hypothesis on $\mathcal{A}^+_m$. Then, suppose from now that the above does not hold. Hence, each interior element of $\mathcal{A}^-_m$ is an interior element of $\mathcal{A}$ concluding by Remark 7 that $\mathcal{A}$ has one interior element from columns $h(m)+1$ to $h(r)$ (since $\mathcal{A}^-_m$ is a 2×3 matrix). First suppose that BT arrives at the k-th corner for some $2 \leq k \leq r-1$. As each interior element of $\mathcal{A}^+_k$ is an interior element of $\mathcal{A}$, except for (maybe) column $h(k)$, $\mathcal{A}$ has at least one interior element from columns 1 to $h(k)$ by induction hypothesis on $\mathcal{A}^+_k$ and since $\mathcal{A}$ has one interior element from columns $h(m)+1$ to $h(r)$, the theorem holds in this case. Similarly, the proof holds if BT arrives at $a_{i,h(k)}$ for $2 \leq k \leq r-2$ and $i \leq k$. Now suppose that BT passes always below the i–st corner, for every $i \geq 2$. If TT and BT share steps from columns 1 to $h(2)$, then $\mathcal{A}$ has at least one interior element from columns 1 to $h(2)$ and since $\mathcal{A}$ has one interior element from columns $h(m)+1$ to $h(r)$, the theorem holds. Then, we may suppose that TT and BT does not share steps from columns 1 to $h(2)$. Also, if $a_{m,h(m)-1} \in PT$, then column $h(m)$ is an interior element and the theorem holds. So, we may suppose that $a_{m,h(m)-1} \notin PT$. Hence, $a_{r,h(m)-2} \in BT$ by the rules of Proposition 6. Let $\mathcal{A}'$ be the matrix obtained by turning the matrix $\mathcal{A}$ upside down. We observe that BT and PT are the Top and Bottom Travels of $\mathcal{A}'$, respectively.

Let define the i–st corners of $\mathcal{A}'$ as $a_{r-i+1,h(r-i+1)}$ for $i \neq 2$ and define the 2–st corner of $\mathcal{A}'$ as $a_{m,h(m)-1}$. Notice that BT always passes strictly above all corners of $\mathcal{A}'$ after the 1–st corner of $\mathcal{A}'$. Moreover, $B[A']$ has the same sequence and the same 2–st corner as that considered in Lemma 2 (vi). Hence, as TT and BT do not share steps from columns $h(1)$ to $h(m)-1$, we know by Lemma 2 (vi) that $a_{i,h(m)-1} \in PT$ for some $i \geq r-1$, but this is a contradiction since we had assumed that $a_{m,h(m)-1} \notin PT$ and clearly $a_{r,h(m)-1} \notin PT$, concluding the proof. ☐

The chessboard considered in Theorem 7 cannot be extended to a chessboard with sequence $(2, t+3, 2, 3, 2, 3, \dots)$ in order to prove $\bar{n}(d,t) < 2d + \lceil \frac{d+1}{2} \rceil + t$. Figures 7 (b) and (c) provide examples of this phenomena for $d=5, t=2$, and for $d=4, t=3$, respectively. In fact, these examples can be generalized in order to show that this chessboard cannot be used to prove $\bar{n}(d,t) < 2d + \lceil \frac{d+1}{2} \rceil + t$ for odd $d \geq 5, t \geq 2$ and for even $d \geq 4, t \geq 3$.

The upper bound given in Theorem 7 can be improved when d is even.

Theorem 9. $\bar{n}(t,d) < 2d + (t-1)\frac{d}{2} + 3$ *for any integer* $d \geq 4$, *d-even and* $t \geq 2$.

This theorem can be proved by making one final tweak to the chessboard defined previously. The latter is a bit technical and requires some extra work in the same flavour as above. This will be done in the Annex.

4.3. Proof of Theorem 1.

Proof of Theorem 1. Recall that $n(t,d)=\bar{n}(t,d)$ and that $H_0(n(t,d),d)=n(t,d)-t$.

- $d=1,\ n\geq 2$. We clearly have that $H_0(n,1)=2$ since every convex set in dimension 1 has as support only two vertices.
- $d=2,\ n\geq 5$. By Equality (11), we have $n(t,2)=5+t$ for any integer $t\geq 0$. Therefore, by Remark 1, we have $H_0(t+5,2)=5$ for any integer $t\geq 0$ or, equivalently, $H_0(n,2)=5$ for any integer $n\geq 5$.
- $d=3,\ n\geq 7$. By Theorem 6, we have $n(t,3)\leq 7+t$ for any integer $t\geq 0$. Therefore, by Remark 1, we have $H_0(t+7,3)\leq 7$ for any integer $t\geq 0$ or, equivalently, $H_0(n,3)\leq 7$ for any integer $n\geq 7$.
- $d\geq 2,\ n\leq 2d+1$. By Proposition 1, $H_0(n,d)=n$.
- $d\geq 4,\ n\geq 2d+\lceil\frac{d+1}{2}\rceil$. By Proposition 1, $H_0(n,d)<n$.
- $d\geq 4,\ n\geq 2d+\lceil\frac{d+1}{2}\rceil+1$. By Theorem 8 $\bar{n}(1,d)<2d+\lceil\frac{d+1}{2}\rceil+1$. Therefore, $H_0(n,d)<n-1$.
- $d\geq 4,\ 2d+3+l(d-2)\leq n<2d+3+(l+1)(d-2),\ l\geq 1$. By Theorem 7, $\bar{n}(d,t)<2d+(t-1)(d-2)+3$ for every $t\geq 2$. Since $H_0(n(d,t),d)=\bar{n}(t,d)-t$ then, for a given n, it would be enough to work out t such that $2d+(t-2)(d-2)+3\leq n<2d+(t-1)(d-2)+3$, in order to conclude $H_0(n,d)<n-t$.

  It is not hard to see that by taking $t=\lfloor\frac{n-2d-3}{d-2}\rfloor+1\geq 2$ we obtain that $H_0(n,d)\leq n-(\lfloor\frac{n-2d-3}{d-2}\rfloor+2)$ for every $n\geq 3d+1$. This can be expressed as; if $2d+3+l(d-2)\leq n<2d+3+(l+1)(d-2)$ for some $l\geq 1$, then $H_0(n,d)\leq n-(l+2)$.

☐

Proof of Corollary 1. The desired inequalities are obtained by combining inequalities (5) and (7) and the values and upper bounds given in Propositions 1 and 2 and Theorem 1.

The lower bound is obtained by combining Equality (6) and Inequality (8).

☐

Question 2. *Let $d\geq 1$ and $1\leq k\leq d-1$. Is it true that $H_k(n,d)\geq H_k(n-1,d)$?*

We believe that the answer is positive.

5. Minimal Radon partitions

In order to prove Theorem 2 we need to take a geometric detour on the relationship between faces of convex polytopes, simplices embracing the origin and Radon partitions. There is an old tradition of using Gale transforms to study facets of convex polytopes [8] by studying simplices embracing the origin. This equivalence was further extended by Larman [9] to studying Radon partitions of points in space.

Let us recall that a *projective transformation* $T : \mathbb{R}^d \to \mathbb{R}^d$ is a function such that $T(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$, where A is a linear transformation of $\mathbb{R}^d$, $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$, is such that at least one of $c \neq 0$ or $\delta \neq 0$. T is said to be *permissible* for a set $X \subset \mathbb{R}^d$ if and only if $\langle c, x\rangle + \delta \neq 0$ for all $x \in X$.

A *Gale transform* $\bar{X}$ of a finite set of points $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ such that the dimension of their affine span is r is defined as $\bar{X} = \{\bar{x}_j = (\alpha_{j,1}, \dots \alpha_{j,n-r-1})\}_{j=1}^n$, where $\{a_i = (\alpha_{1,i}, \dots, \alpha_{n,i})\}_{i=1}^{n-r-1}$ is a basis of the $(n-r-1)$-dimensional space of *affine dependences* of X, $D(X) = \{\alpha = (\alpha_1, \dots, \alpha_n)|\ \sum_{i=1}^n \alpha_i x_i = 0,\ \sum_{i=1}^n \alpha_i = 0\}$. It is emphasized that $\bar{X}$ is *a* Gale transform of X, rather than *the* Gale transform of X, because the resulting points depend on the specific choice of basis for $D(X)$. Still, different Gale transforms of the same set of points are linearly equivalent.[8]

A *Gale diagram* $\hat{X}$ of X is a set of points in $\mathbb{S}^{n-r-2}$ obtained by *normalizing* a Gale transform, that is: $\hat{X} = \{\hat{x_i} = \frac{\bar{x}_i}{\|\bar{x}_i\|} | \bar{x}_i \in \bar{X}, \bar{x}_i \neq 0\} \cup \{\hat{x_i} = \bar{x}_i | \bar{x}_i \in \bar{X}, \bar{x}_i = 0\}$.

Remark 8. Let $X = \{x_1, \dots, x_n\}$ be a set of n points in $\mathbb{R}^d$ and let $\hat{X}$ (resp. $\bar{X}$) be its Gale diagram (resp. Gale transform), then the following statements hold.

(a) The n points of X are in general position in $\mathbb{R}^d$ if and only if the n-tuple $\hat{X}$ $(\bar{X})$ consists of n points in linearly general position in $\mathbb{R}^{n-d-1}$.

(b) Faces of *conv*(X) are in one-to-one correspondence with simplices of $\hat{X}$ (resp. $\bar{X}$) that contain 0 in their convex hull. More precisely, $Y \subset X$ is a face of *conv*(X) if and only if $0 \in$ *relint conv*$(\hat{X} \setminus \hat{Y})$ (resp. $0 \in$ *relint conv*$(\bar{X} \setminus \bar{Y})$).

(c) X is projectively equivalent to a set of points Y (by a permissible projective transformation) if and only if there is a non-zero vector $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{1, -1\}^n$ $(\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n)$ such that $\hat{y_i} = \epsilon_i \hat{x_i}$ $(\bar{y_i} = \lambda_i \bar{x}_i)$.

We refer the reader to [8] for the proofs of this remark.

Let $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ be a set of points in general position where $n \geq d + 2$, and $X = A \cup B$ be a disjoint partition of X. We define a *Partitioned Affine Projection* (PAP) $\tilde{X}$ of X into the unit d-sphere as follows; $\tilde{X} = \{\tilde{x}_i = \mathbf{I}(x_i)\frac{(x_i;1)}{\|(x_i;1)\|} | x_i \in X\} \subset \mathbb{S}^d$, where $\mathbf{I}(x_i) = 1$ if $x_i \in A$, $\mathbf{I}(x_i) = -1$ when $x_i \in B$, and $(x_i; 1)$ is the $d + 1$ dimensional vector whose first d entries are identical to those of x_i and last entry is 1.

Remark 9. A, B is a Radon partition of X if and only if $0 \in$ *conv*$(\tilde{A} \cup \tilde{B})$. The proof follows straightforwardly, using linear algebra.

Proof of Theorem 2. By Remark 9, we know that if we consider the PAP of X into $\mathbb{S}^d$, $\tilde{X}$, we have that *conv*$(A \cap S) \cap$ *conv*$(B \cap S) \neq \emptyset$ if and only if $0 \in$ *conv*$(\tilde{S})$.

Let $\tilde{X}_\epsilon = \{\epsilon_1\tilde{x}_1, \dots, \epsilon_n\tilde{x}_n\}$ for $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{1, -1\}^n$. We define $\rho(\tilde{X}_\epsilon)$ as the number of subsets $\tilde{S} \subset \tilde{X}_\epsilon$ such that $0 \in$ *conv*$(\tilde{S})$, $\rho(\tilde{X}) = \max_{\epsilon \in \{1,-1\}^n} \rho(\tilde{X}_\epsilon)$, and $\rho(n, d) = \min_{\{\tilde{X} \subset \mathbb{S}^d, |\tilde{X}|=n\}} \rho(\tilde{X})$.

It easy to check that

$$r(n,d)=\rho(n,d).$$

Notice that $\tilde{X}\subset\mathbb{S}^d\subset\mathbb{R}^{d+1}$ while $X\subset\mathbb{R}^d$.

Now, recall that the set $\tilde{X}\subset\mathbb{S}^d$ can be considered to be the Gale diagram of a set of points in $X'\subset\mathbb{R}^{n-d-2}$ where each $\tilde{X}_\epsilon=\{\epsilon_1\tilde{x}_1,\dots,\epsilon_n\tilde{x}_n\}$ corresponds to a permissible projective transformation of X'. Therefore, each $(d+2)$–subset $\tilde{S}\subset\tilde{X}$ such that $0\in conv(\tilde{S})$ is in one to one correspondence with a co-facet of X' (i.e. the corresponding set $X'\setminus S'$ is a facet of X'). Hence, finding $\rho(n,d)$ is equivalent to finding $H_{d'-1}(n,d')$ where $d'=n-d-2$. □

Corollary 2. *Let $d\geq 1$ be an integer. Then,*

$$r(n,d)\begin{cases} =2 & \textit{if } n=d+3,\\ =5 & \textit{if } n=d+4,\\ \leq 10 & \textit{if } n=d+5,\\ \leq f_{n-d-3}(C_{n-d-2}(n)) & \textit{if } n\geq 2d+3,\\ \leq f_{n-d-3}(C_{n-d-2}(n-1)) & \textit{if } n\leq \frac{5d+8}{3},\\ \leq f_{n-d-3}(C_{n-d-2}(n-2)) & \textit{if } n\leq \frac{5d+6}{3},\\ \leq f_{n-d-3}(C_{n-d-2}(n-(l+2))),\quad 1\leq l & \textit{if } d+4+\frac{d-3}{l+2}<n\leq d+4+\frac{d-1}{l+1}.\end{cases}$$

Moreover, if $n\geq 2d+3,\ d\geq 2$ then

$$r(n,d)\geq (n-d-3)n-(n-d-1)(n-d-4).$$

Proof. The values and upper bounds can be obtained by combining Theorems 1 and 2. Moreover, by combining Theorem 2 with Equations (6), (8) we have

$$r(n,d)=H_{d'-1}(n,d')\geq f_{d'-1}(P_{d'}(n))=(d'-1)n-(d'+1)(d'-2)$$

when if $n\leq 2d'+1,\ d\geq 2$ where $d'=n-d-2$.

The latter gives the desired lower bound of $r(n,d)$. □

Theorem 10. *Let $n\geq 4$ be an integer. Then,*

$$r(n,2)\begin{cases} =2 & \textit{if } n=5,\\ =5 & \textit{if } n=6,\\ =10 & \textit{if } n=7,\\ \leq 2\binom{\frac{n-1}{2}+2}{\frac{n-1}{2}-2} & \textit{if } n\geq 7, n\textit{-odd},\\ \leq \binom{\frac{n}{2}+2}{\frac{n}{2}-2}+\binom{\frac{n}{2}+1}{\frac{n}{2}-3} & \textit{if } n\geq 8, n\textit{-even}.\end{cases}$$

Moreover, if $n\geq 7$ then $r(n,2)\geq 2(2n-9)$.

Proof. • If $n=5,6$ then the values are obtained directly from Equation (2).

• If $n=7$ then from (2) we have $r(7,2)\leq 10$ and from Corollary 2 we have $r(7,2)\geq (7-5)7-(7-3)(7-6)=14-4=10$, and the equality follows.

• By the forth inequality in Corollary 2, we have $r(n,2) \leq f_{n-5}(C_{n-4}(n))$ for any $n \geq 7$. Now by taking $k = d-1$ in the formula (4), we have

$$f_{d-1}(C_d(n)) = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - \lfloor \frac{d}{2} \rfloor - 1}{\lceil \frac{d}{2} \rceil - 1}.$$

Therefore, by taking $d = n-4$, we have

$$r(n,2) \leq \binom{n - \lceil \frac{n-4}{2} \rceil}{\lfloor \frac{n-4}{2} \rfloor} + \binom{n - \lfloor \frac{n-4}{2} \rfloor - 1}{\lceil \frac{n-4}{2} \rceil - 1}. \tag{9}$$

The upper bounds for the cases $n \geq 8$, n-even and $n \geq 7$, n-odd are obtained from (9).

• The lower bound for $r(n,2)$ when $n \geq 7$ is a straightforward calculation from the lower bound given in Corollary 2.

□

Other bounds for specific values of n and d can easily be obtained by using Equations (2) and Corollary 2, for instance,

$$17 \leq r(9,3) \leq 27.$$

The following result is a straightforward consequence of the bounds of Theorem 10.

Corollary 3. *The order of* $r(n,2)$ *is between* $o(n)$ *and* $o(n^4)$.

5.1. **Pach and Szegedy's question.** We may now prove Theorem 3.

Proof of Theorem 3. Let $X \subset \mathbb{R}^2$ be a set of $n \geq 9$ points in general position. Let $\tilde{X}_\epsilon \subset \mathbb{S}^2$ be it's corresponding PAP, and $X' \subset \mathbb{R}^{n-4}$ be a point configuration whose Gale diagram is $\tilde{X}_\epsilon$.

Let $Y \subset X$ with $|Y| = n-1$ and let A, B be a partition that attains the maximum number of induced minimal Radon partitions for the set Y, that is $r(Y) = r_Y(A,B)$.

Recall that $\nu(d,k) \geq d + \lfloor \frac{d}{k} \rfloor + 1$ for $k \geq 2$ (Equation (3)). Therefore, by taking $k = \lfloor \frac{n-4}{2} \rfloor$ and $d = n-4$, we obtain that

$$\nu\left(n-4, \left\lfloor \frac{n-4}{2} \right\rfloor\right) \geq n-1 \tag{10}$$

for any integer $\lfloor \frac{n-4}{2} \rfloor \geq 2$, that is, for $n \geq 8$.

This implies that $n-1$ points in $\mathbb{R}^{n-4}$ can always be mapped by a permissible projective transformation onto the vertices of a $\lfloor \frac{n-4}{2} \rfloor$–neighbourly polytope. The latter implies that Y' is in the projective class of a neighborly polytope. Using Remark 8(b), the neighborliness of Y' translates in $\tilde{Y}_\epsilon \subset \mathbb{S}^2$ having the following property: for every subset $S \subset \tilde{Y}_\epsilon$ such that $|S| \leq \lfloor \frac{n-4}{2} \rfloor$, $0 \in conv(\tilde{Y}_\epsilon \setminus S)$. Hence, no plane through the origin H is such that $|H^+ \cap \tilde{Y}_\epsilon| \geq (n-1) - \lfloor \frac{n-4}{2} \rfloor$. Therefore, for all planes through the origin H, $|H^+ \cap \tilde{Y}_\epsilon| < (n-1) - \lfloor \frac{n-4}{2} \rfloor$. The latter directly implies that in Y both $|A| < \lfloor \frac{n-1}{2} \rfloor + 1$ and $|B| < \lfloor \frac{n-1}{2} \rfloor + 1$, as desired. □

Remark 10. Let $X \subset \mathbb{R}^2$ be a set of $n \geq 9$ points. Then, by applying similar arguments as those used in the above proof, it can be deduced that if X' is in the projective class of a neighborly polytope, then for any partition A, B of X such that $r_X(A,B) = r(X)$ we have that $|A|, |B| \leq \lfloor \frac{n-1}{2} \rfloor + 1$.

Question 3. *Could this approach be extended to investigate balanced 2-partitions in higher dimensions ?*

Unfortunately, our method is not suitable to study balanced 3-partitions since the translation into Gale diagrams involves partitions of points by a hyperplane (allowing to consider 2-partitions only).

5.2. **Tolerance result.** We may now prove Theorem 4.

Proof of Theorem 4. Let X be such that it has a partition into two sets A,B and a subset $P \subseteq X$ of cardinality $\mu - i$, for some $0 \leq i \leq t$, such that $conv(A \setminus y) \cap conv(B \setminus y) \neq \emptyset$ for every $y \in P$ and $conv(A \setminus y) \cap conv(B \setminus y) = \emptyset$ for every $y \in X \setminus P$. By Remark 9, we know that if we consider the PAP of X into $\mathbb{S}^d$, $\tilde{X}$, we have that $conv(A \setminus y) \cap conv(B \setminus y) \neq \emptyset$ if and only if $0 \in conv((\tilde{A} \setminus \tilde{y}) \cup (\tilde{B} \setminus \tilde{y}))$.

Now let $\rho(t,d)$ be the smallest number such that for all sets $\tilde{X}$ of cardinality ρ in $\mathbb{S}^d$, there exists a partition of $\tilde{X}$ into two sets $\tilde{A}$,$\tilde{B}$ and a subset $\tilde{P} \subseteq \tilde{X}$ of cardinality $\rho - i$, for some $0 \leq i \leq t$, such that $0 \in conv((\tilde{A} \setminus \tilde{y}) \cup (\tilde{B} \setminus \tilde{y}))$ for every $\tilde{y} \in \tilde{P}$ and $0 \notin conv((\tilde{A} \setminus \tilde{y}) \cup (\tilde{B} \setminus \tilde{y}))$ for every $\tilde{y} \in \tilde{X} \setminus \tilde{P}$, then $\lambda(t,d) = \rho(t,d)$. That is, one can seamlessly go from a tolerant partition to a *tolerant* configuration of points in the sphere.

For the next part we will need to establish a relationship between $n(t,d)$ and $\rho(t,d)$. This relationship arises from the connection between projective transformations of points and antipodal functions of their Gale diagrams, as has already been explored in Theorem 2.

Let y be a point strictly in the interior of $conv(X)$. Recall that if we consider the Gale diagram of X, $\hat{X} \subset \mathbb{S}^{n-d-1}$, by Remark 8.(b), as p is not a face of $conv(X)$, $0 \notin conv(\hat{X} \setminus \hat{y})$. Also Remark 8.(c) draws the connection between projective transformations of X and taking diametrically opposite points in $\hat{X}$.

Thus, if we consider the Gale diagram of a set of $n = n(t,d)$ points $\hat{X}$, we must have that for some $\hat{X}_\epsilon = \{\epsilon_1 \hat{x}_1, \ldots, \epsilon_n \hat{x}_n\}$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{1,-1\}^n$, there is a set of at most $n - i$ points, for some $0 \leq i \leq t$, $\hat{P}_\epsilon$ such that $0 \in conv(\hat{X}_\epsilon \setminus \hat{y})$ for $\hat{y} \in \hat{P}_\epsilon$. Thus $\rho(t, n-d-1) \leq n$, and the necessary partition is given by the signs of the epsilons.

Conversely, let $\hat{X}$ be a set of points $\rho = \rho(t,d')$ points, then the Gale transform of these points, X will be such that there is a set of at most t points, such that they are in the interior of $conv(X)$. This is $\rho \leq n(t, \rho - d' - 1)$.

As argued at the beginning of the proof, in both inequalities we can straight forwardly substitute ρ for λ obtaining

$$n(t,d) = \max_{m\in\mathbb{N}}\{m \mid \lambda(t, m-d-1) \leq m\} \text{ and } \lambda(t,d) = \min_{m\in\mathbb{N}}\{m \mid m \leq n(t, m-d-1)\}.$$

as desired. ☐

6. Arrangements of (pseudo)hyperplanes

The so-called Topological Representation Theorem, due to Folkman and Lawrence [5], states that loop-free oriented matroids of rank $d+1$ on n elements (up to isomorphism) are in one-to-one correspondence with arrangements of pseudo-hyperplanes in the projective space $\mathbb{P}^{r-1}$ (up to topological equivalence).

A d-arrangement of n pseudo-hyperplanes is called *simple* if $n \geq d$ and every intersection of d pseudo-hyperplanes is a unique distinct point. It is known that simple arrangements correspond to uniform oriented matroids. It is well known that a tope corresponds to an acyclic reorientation (projective transformations) having as interior elements precisely those pseudo-hyperplanes not bordering the tope.

By the above discussion, we may redefine $\bar{n}(t,d)$ in terms of hyperplane arrangements:

> $\bar{n}(t,d) :=$ the largest integer n such that any simple arrangement of n (pseudo)hyperplanes in $\mathbb{P}^d$ contains a tope of size at least $m-t$.

Proposition 3. *Every simple arrangement of at least* 5 *pseudo-lines in* $\mathbb{P}^2$ *has a tope of size at least* 5*, that is,*

$$5 + t \leq \bar{n}(t,2) \textit{ for every integer } t \geq 0.$$

Proof. The proof is by induction on the set of n (pseudo) lines. By Equation (2), any arrangement of 5 (pseudo) lines in $\mathbb{P}^2$ has a tope of size 5 and thus the proposition holds for $n=5$. We suppose the result true for $n' < n$ and will prove that any arrangement H of $n \geq 6$ (pseudo) lines in $\mathbb{P}^2$ has a tope of size at least 5. Let $l \in H$, then by induction $H \setminus l$ has a tope T of size at least 5 in $\mathbb{P}^2$. If l does not touch T then T is a tope of H of size at least 5 in $\mathbb{P}^2$. Otherwise, l divides T into two topes, and since H is simple then one of these two topes is of size at least 5. ☐

Combining Proposition 3 and Theorem 5 for $d+2$, we obtain:

$$\bar{n}(t,2) = 5 + t \text{ for any integer } t \geq 0. \tag{11}$$

For the case $d=3$, Theorem 6 implies that $\bar{n}(t,3) \leq 7+t$ for any integer $t \geq 1$, that is, for any $n \geq 7$ there exists a simple arrangement of n (pseudo)planes in $\mathbb{P}^3$ with every tope of size at most 7. This supports the following:

Conjecture 1. $\bar{n}(t,3) = 7+t$ for any integer $t \geq 1$.

Furthermore, we may ask the following general questions:

Question 4. *Let $d \geq 2$ and $t \geq 0$ be integers. Is it true that $\bar{n}(t,d) = 2d+1+t$? In other words, is it true that any simple arrangement of $n \geq 2d+1$ (pseudo)hyperplanes in $\mathbb{P}^d$ contains a tope of size at least $2d+1$ and conversely, for any $n \geq 2d+1$ there exists a simple arrangement of n (pseudo)hyperplanes in $\mathbb{P}^d$ with every tope of size at most $2d+1$?*

Or, alternatively,

Question 5. *Let $d \geq 2$ and $t \geq 0$ be integers. Is there a constant $c(d) \geq 1$ such that $\bar{n}(t,d) = 2d+1+c(d)t$?*

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ANNEX

We shall prove Theorem 9. First, we need the following.

Lemma 3. *Let* $A = A_{r,h(r)}$ *be a matrix and suppose that* TT *always passes strictly above all corners after the* 1*–st corner. Then the following holds:*

(i) *If* $r \geq 5$ *is odd,* $B[A]$ *has a sequence* $(2, t+3, 2, t+1, 2, t+1, \dots, 2, t+1)$ *for some* $t \geq 2$*,* $h(2) = t+3$ *and* $h(m) = 2\lceil \frac{m-1}{2} \rceil + (t+1)\lfloor \frac{m-1}{2} \rfloor + 3$ *for every* $3 \leq m \leq r$*, then* $a_{i,t+3} \in BT$ *for some* $i \leq 2$*.*

(ii) *If* $r \geq 4$ *is even,* $B[A]$ *has a sequence* $(t+1, 2, t+1, 2, \dots, t+1, 2, t+1)$ *for some* $t \geq 2$ *and* $h(r) = 2\lfloor \frac{m-1}{2} \rfloor + (t+1)\lceil \frac{m-1}{2} \rceil + 1$ *for every* $1 \leq m \leq r$*, then* $a_{1,t} \in BT$*. Moreover, if* TT *and* BT *do not share steps from columns* $h(2)$ *to* $h(r)$*, then* $a_{1,t+1} \in BT$*.*

(iii) *If* $r \geq 5$ *is odd,* $B[A]$ *has a sequence* $(t+1, 2, t+1, 2, t+1 \dots, 2, t+3, 2)$ *for some* $t \geq 2$*,* $h(2) = t+3$*,* $h(r) = 2\lfloor \frac{m-1}{2} \rfloor + (t+1)\lceil \frac{m-1}{2} \rceil + 1$ *for every* $3 \leq m \leq r$ *and* TT *and* BT *do not share steps from columns* $h(2)$ *to* $h(r)$*, then* $a_{i,t+3} \in BT$ *for some* $i \leq 2$*.*

Proof. (i) As $2r - h(r-1) + h(2) - 3 = 2r - (r - 1 + \frac{(t+1)(r-3)}{2} + 3) + t = \frac{2r+2t-4-(t+1)(r-3)}{2} \leq \frac{9-r}{2} \leq 2$ when $r \geq 5$ is odd and $t \geq 2$. Then $a_{i,h(2)} \in BT$ for some $i \leq 2$ by Lemma 1 (i).

(ii) As $2r - h(r-1) + h(2) - 3 = 2r - (r - 2 + (t+1)\frac{(r-2)}{2} + 1) + t - 1 \leq 5 - \frac{r}{2} \leq 3$ when $r \geq 4$ is even and $t \geq 2$, we obtain by Lemma 1 (i) that $a_{i,h(2)} \in BT$ for some $i \leq 3$. Then, as $a_{1,i} \in PT$ for $i \leq t+3$, we conclude by the rules of Proposition 6 that $a_{1,t} \in BT$. If TT and BT do not share steps from columns $h(2)$ to $h(r)$, as $2r - h(r) + h(2) - 1 = 2r - (r - 2 + (t+1)\frac{(r)}{2} + 1) + t + 1 \leq 4 - \frac{(r)}{2} \leq 2$ when $r \geq 4$ is even and $t \geq 2$, we obtain by Lemma 1 (i) that $a_{i,h(2)} \in BT$ for some $i \leq 2$. Then, as $a_{1,i} \in PT$ for $i \leq t+3$, we conclude by the rules of Proposition 6 that $a_{1,t+1} \in BT$.

(iv) As $2r - h(r) + h(2) - 1 = 2r - (r - 1 + \frac{(t+1)(r-1)}{2} + 3) + t + 2 = \frac{2r+2t-(t+1)(r-1)}{2} \leq \frac{7-r}{2} \leq 1$ when $r \geq 5$ is odd and $t \geq 2$. Then $a_{i,h(2)} \in BT$ for some $i \leq 2$ by Lemma 1 (ii). ☐

We will use the following remark.

Remark 11. Let $B[A_1]$ and $B[A_2]$ be with sequences $(x_1, x_2, \dots, x_{r-1})$ and $(y_1, y_2, \dots, y_{r-1})$, $y_i \geq x_i$, $1 \leq i \leq r-1$, respectively. If for any plain travel PT in A_1, the corresponding $\mathcal{A}_1$ in which PT is the Top Travel has k interior elements, then for any plain travel PT in A_2, the corresponding $\mathcal{A}_2$ in which PT is the Top Travel has k interior elements.

We may now prove Theorem 9.

Proof of Theorem 9. Let $A = A_{r,h(r)}$ be a matrix where $h(r)$ is defined as $h(2) = t+3$, $h(m) = 2\lceil \frac{m-1}{2} \rceil + (t+1)\lfloor \frac{m-1}{2} \rfloor + 3$ for every $3 \leq m \leq r$ and $B[A]$ has sequence $(2, t+3, 2, t+1, 2, t+1, \dots, 2, t+1)$ for $t \geq 2$. We shall show by induction on r that for odd $r \geq 3$ and for any plain travel PT in A, the corresponding $\mathcal{A}$ in which PT is the Top Travel has at least $t+1$ interior elements. In particular, as $h(r) = r-1+(t+1)\frac{r-1}{2}+3 = 2d+(t-1)\frac{d}{2}+3$ for odd $r \geq 5$, we will prove the theorem for $d \geq 4$ and $t \geq 2$.

For $r = 3$, the result follows by Theorem 5, then assume that the theorem holds for $r-1$ and we show it for odd $r \geq 5$. Suppose that the m–st corner is the last corner that PT meets in A, for some $1 \leq m \leq r-1$. If PT always passes strictly below the i–st corner for $i > m$, then there would be at least $t+2$ interior elements in $\mathcal{A}$ (from columns $h(r-1)$ to $h(r)$). Hence, we may suppose that PT always passes strictly above the i–st corner for $i > m$.

The case $m = 1$ holds by Lemma 3 (i). The case $m = 2$ holds applying Lemma 2 (ii) on sub-matrix $\mathcal{A}_m^-$ and then Remark 7. The case $m = 3$ holds applying Lemma 2 (i) on sub-matrix $\mathcal{A}_m^-$ and then induction hypothesis on $\mathcal{A}_m^+$. We have the following cases.

Case m odd and $5 \leq m \leq r-2$. Applying Lemma 2 (i) on $\mathcal{A}_m^-$, $a_{m,h(m)}, a_{m,h(m)+1} \in BT$, or column $h(r)$ is an interior element and $a_{m+1,h(m)}, a_{m,h(m)} \in BT$. If $a_{m,h(m)}, a_{m,h(m)+1} \in BT$, the theorem holds by induction hypothesis on $\mathcal{A}_m^+$. If column $h(r)$ is an interior element and $a_{m+1,h(m)}, a_{m,h(m)} \in BT$, each interior element of $\mathcal{A}_m^+$ is an interior element of $\mathcal{A}$, except for column $h(m)$, then $\mathcal{A}$ has at least t interior elements from columns 1 to $h(m)$ by induction hypothesis on $\mathcal{A}_m^+$ and one interior element in column $h(r)$.

Case m even and $4 \leq m \leq r-3$. As $\mathcal{A}_m^-$ has at least 4 rows, $a_{m,h(m)}, a_{m,h(m)+1} \in BT$ by Lemma 2 (ii). Then, as each interior element of $\mathcal{A}_m^+$ is an interior element of $\mathcal{A}$, applying Theorem 8 and Remark 11 to $\mathcal{A}_m^+$, we obtain that $\mathcal{A}$ has at least 2 interior elements from columns 1 to $h(m)$. Suppose first that TT and BT do not share steps from columns $h(2)$ to $h(r)$. Then, by Lemma 3 (ii), $a_{m,h(m)+t} \in BT$, concluding that $\mathcal{A}$ has $t-1$ interior elements from columns $h(m)+1$ to $h(m)+t-1$. Now, suppose that TT and BT share at least one step. Then, by Lemma 3 (ii), $a_{m,h(m)+t-1} \in BT$, concluding that $\mathcal{A}$ has $t-2$ interior elements from columns $h(m)+1$ to $h(m)+t-2$. By the election of m, TT and BT share at least one step from column $h(m)+1$ to $h(r)$, say $a_{i,j}, a_{i,j+1}$ for some $i \leq r$ and some $h(m)+1 \leq j \leq h(r)$. If $i = r$, then $\mathcal{A}$ has one interior element in column $j+1$ concluding the proof in this case, then we may suppose that $i < r$. Hence, the squares in between of TT and BT after column $h(m)$ must be white, concluding by the rules of Proposition 6 that $a_{m,h(m)+t+1} \in BT$ and the theorem holds also in this case.

Case $m = r-1$. First suppose that BT arrives at the k-th corner for some $2 \leq k \leq r-1$. As each interior element of $\mathcal{A}_k^+$ and $\mathcal{A}_m^-$ is an interior element of $\mathcal{A}$, except for (maybe) columns $h(k)$ and $h(m)$, $\mathcal{A}$ has at least t interior elements from columns 1 to $h(k)$ by induction hypothesis on $\mathcal{A}_k^+$ and at least $t-1 \geq 1$ interior elements from columns $h(m)$ to $h(r)$ by Remark 7 (since $\mathcal{A}_m^-$ is a $2 \times (t+2)$ matrix), concluding the proof of this case. Similarly, the proof holds if BT arrives at $a_{i,h(k)}$ for $2 \leq k \leq r-2$ and $i \leq k$. Now suppose that BT passes always below the i–st corner, for every $i \geq 2$. In particular, as BT does not arrives at the m-th corner, every interior element of $\mathcal{A}_m^-$ is an interior element of $\mathcal{A}$, concluding by Remark 7 that $\mathcal{A}$ has t interior elements from columns $h(m)+1$ to $h(r)$. So, we may suppose that TT and BT do not share steps from columns 1 to $h(2)$, otherwise the theorem holds. Also, if $a_{m,h(m)-1} \in PT$, then column $h(m)$ is an interior element and the theorem holds. So, we may suppose that $a_{m,h(m)-1} \notin PT$. Hence, $a_{r,h(m)-2} \in BT$ by the rules of Proposition 6. Let $\mathcal{A}^{'}$ be the matrix obtained by turning the matrix $\mathcal{A}$ upside down. We observe that BT and PT are the Top and Bottom Travels of $\mathcal{A}^{'}$, respectively. Let define the i–st corners of $\mathcal{A}^{'}$ as $a_{r-i+1,h(r-i+1)}$ for $i \neq 2$ and define the 2–st corner of $\mathcal{A}^{'}$ as $a_{m,h(m)-1}$. Notice that BT always passes strictly above all corners of $\mathcal{A}^{'}$ after the 1–st corner of $\mathcal{A}^{'}$. Moreover, $B[A']$ has the same sequence and the same 2–st corner as that considered in Lemma 3 (iii). Hence, as TT and BT do not share steps from columns $h(1)$ to $h(m)-1$, we know by Lemma 3 (iii) that $a_{i,h(m)-1} \in PT$ for some $i \geq r-1$, but this is a contradiction since we had assumed that $a_{m,h(m)-1} \notin PT$ and clearly $a_{r,h(m)-1} \notin PT$, concluding the proof. ☐

The chessboard considered in Theorem 9 cannot be used to prove $\bar{n}(d,1) < 2d+(t-1)\frac{d}{2}+3$ for odd d. Figure 7 (b) gives an example for $d = 5$ and $t = 2$, This example can be generalized in order to show that this kind of chessboard cannot be used to prove $\bar{n}(d,1) < 2d+(t-1)\frac{d}{2}+3$ for odd $d \geq 5$ and $t \geq 2$.

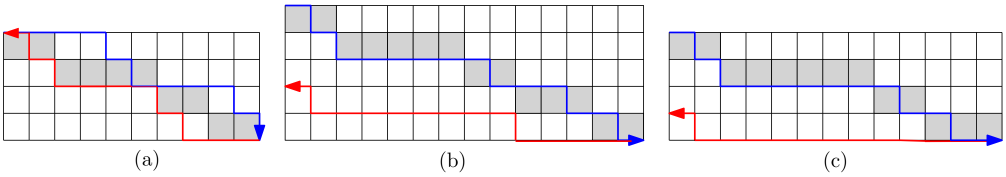

FIGURE 7. Figures (a), (b) and (c) show the matrices $A_{5,11}$, $A_{6,15}$ and $A_{5,14}$ respectively, with chessboards $(2, t+3, 2, 2)$ for $t = 1$ (a), $(2, t+3, 2, 3, 2)$ for $t = 2$ (b) and $(2, t+3, 2, 3)$ for $t = 3$ (c). We observe that for the pair of Top and Bottom travels described in these matrices, $A_{5,11}$ has only one interior element (column 1), $A_{6,15}$ has only two interior elements (columns 13 and 15) and $A_{5,14}$ has only three interior elements (columns 11, 13 and 14).

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