A Tiling Problem and the Frobenius Number

D. Labrousse and J.L. Ramírez Alfonsín

Abstract In this paper, we investigate tilings of tori and rectangles with rectangular tiles. We present necessary and sufficient conditions for the existence of an integer C such that any torus, having dimensions greater than C, is tiled with two given rectangles (C depending on the dimensions of the tiles). We also give sufficient conditions to tile a *sufficiently* large n-dimensional rectangle with a set of (n-dimensional) rectangular tiles. We do this by combining the periodicity of some particular tilings and results concerning the so-called *Frobenius number*.

Key words: Tiling, torus, rectangle, Frobenius number

1 Introduction

Let a and b be positive integers. Let $R(a,b)$ be the 2-dimensional rectangle of sides a and b and let $T(a,b)$ be the 2-dimensional torus. We think of $T(a,b)$ as a rectangle where their parallel sides are identified in the usual way. We will say that a torus T (or a rectangle R) can be *tiled* with *tiles* (i.e., smaller 2-dimensional rectangles) $R_1,\dots,R_k$ if T (or R) can be filled entirely with copies of R_i, $1 \le i \le k$ where rotations are not allowed.

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Question 1. Does there exist a function $C_T = C_T(x,y,u,v)$ (resp. $C_R = C_R(x,y,u,v)$) such that for all integers $a,b \geq C_T$ (resp. $a,b \geq C_R$) the torus $T(a,b)$ (resp. rectangle $R(a,b)$) can be tiled with copies of the rectangles $R(x,y)$ and $R(u,v)$ for given positive integers x,y,u and v?

The special case of Question 1 for $R(a,b)$ when $x=4, y=6, u=5$ and $v=7$ was posed in the 1991 William Lowell Putnam Examination (Problem B-3). In this case, Klosinski *et. al.* [9] gave a lower bound for C_R. Their method was based on knowledge of the *Frobenius number*. The *Frobenius number*, denoted by $g(s_1,\ldots,s_n)$, of a set of relatively prime positive integers $s_1,\ldots,s_n$, is defined as the largest integer that is not representable as a nonnegative integer combination of $s_1,\ldots,s_n$. It is well known [15] that

$$g(s_1,s_2) = s_1s_2 - s_1 - s_2. \tag{1}$$

It turns out that the computation of a similar (simple) formula when $n \geq 3$ is much more difficult. In fact, finding $g(s_1,\ldots,s_n)$, for general n, is a hard problem from the computational point of view (we refer the reader to [13] for a detailed discussion on the Frobenius number). Let us notice that equality (1) can be interpreted in terms of *1-dimensional tilings* as follows:

all sufficiently large interval can be tiled by two given intervals whose lenghts are relatively primes.

Klosinski *et. al.* [9] used equation (1), with particular values for s_1 and s_2, to show that $R(a,b)$ can be tiled with $R(4,6)$ and $R(5,7)$ if $a,b \geq 2214$. We improve the latter by showing (see Remark 1) that if $a,b \geq 198$ then $R(a,b)$ can be tiled with $R(4,6)$ and $R(5,7)$. This lower bound is not optimal, Narayan and Schwenk [10] showed that it is enough to have $a_1,a_2 \geq 33$ by presenting tilings with more complicated patterns (allowing rotations of both tiles) which is not the case here. We also mention that Barnes [1, Theorem 2.1] used algebraic arguments to show the existence of C_R if some *complex set points* conditions are verified but explicit value for C_R was not given.

In the same spirit of the subjects treated in the volume *Unusual Applications of Number Theory* [11], we explore the connection between tilings and the Frobenius number. We show how plane *periodic* tilings can be *pertubed* with tilings, obtained via the Frobenius number, leading to a positive answer to Question 1. We hope these new methods will motivate further investigations.

The paper is organized as follows. In the next section, we shall give neccessary and sufficiente conditions on integers x,y,u,v for the existence of $C_T(x,y,u,v)$ (see Theorem 3). In Section 3, we give various results in relation with a generalization of C_R for n-dimensional rectangles (see Theorem 5). In particular, the knowledge of an upper bound for $g(s_1,\ldots,s_n)$ is used to show that a n-dimensional rectangle $R(a_1,\ldots,a_n)$ can be tiled with a given set of tiles if $a_j > r^{2n}$ for all $1 \leq j \leq n$ where r is the largest length among all the tiles (see Corollary 1). We finally give some results concerning tilings of n-dimensional cubes.

2 Tiling tori

It is known [5, 8] that $R(a,b)$ can be tiled with $R(x,y)$ if and only if either x divides one side of R and y divides the other or xy divides one side of R and the other side can be expressed as a nonnegative integer combination of x and y. This shows that a rectangle $R(a,b)$ can be tiled with $R(1,n)$ if and only n divide either a or b. It is clear that this condition is also sufficient for tiling $T(a,b)$ (since a tiling of $R(a,b)$ is also a tiling of $T(a,b)$) but it is not necessary, see for instance Figure 1.

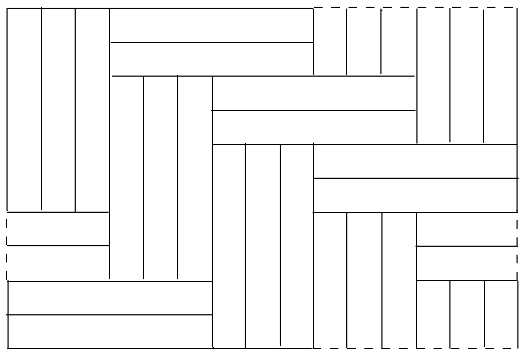

Fig. 1 A tiling of $T(15,10)$ with $R(1,6)$ and $R(6,1)$

Proposition 1. *Let n be a prime integer. Then, $T(a,b)$ can be tiled with $R(1,n)$ if and only if n divides either a or b.*

Proof. . If n divides either a or b then there is a trivial tiling of $T(a,b)$. If $T(a,b)$ is tiled with $R(1,n)$ then n must divides ab and since it is prime then n must divides either a or b. $\square$

In 1995, Fricke [6] gave the following characterization for tiling a rectangle with two squares.

Theorem 1. *[6] Let a,b,x and y be positive integers with $\gcd(x,y) = 1$. Then, $R(a,b)$ can be tiled with $R(x,x)$ and $R(y,y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a,b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.*

The conditions of Theorem 1 are again sufficients for tiling $T(a,b)$ but they are not necessary, that is, there are tilings of $T(a,b)$ with $R(x,x)$ and $R(y,y)$ not verifying the above conditions (and thus not tiling $R(a,b)$), see for instance Figure 2.

Remila [14] studied tilings of $T(a,b)$ with two *bars* (that is, when the rectangles are of the form $R(1,y)$ and $R(u,1)$) where rotations are not allowed. In [14, Section 8] the problem of investigating tilings of tori with two general rectangles

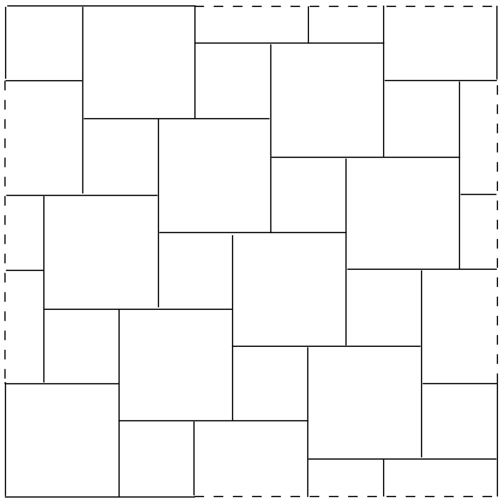

Fig. 2 A tiling of $T(13,13)$ with $R(2,2)$ and $R(3,3)$

(not neccesarily bars) was posed. By using the algebraic approach (via polynomials and ideals) first introduced by Barnes [1, 2], Clivio found [4, Theorem 6.2] the existence of a value C such that for any n-dimensional torus T, having dimensions at least C, there exist necessary and sufficient conditions for T to be tiled with two given n-dimensional rectangles. In particular, for the 2-dimensional case, Clivio's result reads as follows.

Theorem 2. *[4, Theorem 6.2] For arbitrary rectangles $R(x,y)$ and $R(u,v)$ there exists integer C such that for every $T(a,b)$ with $a,b \geq C$, $T(a,b)$ can be tiled with $R(x,y)$ and $R(u,v)$ if and only if* $\gcd\left(\frac{uv}{\gcd(u,a)\gcd(v,b)}, \frac{xy}{\gcd(x,a)\gcd(y,b)}\right) = 1$.

Theorem 2 gives a characterization of *sufficiently large* tori to be tiled with two given rectangles. An estimation of value C was not given in [4] (even for $n = 2$). Clivio remarked that if the volumes of the two given rectangles $R(x,y)$ and $R(u,v)$ (and, in general, the two given n-dimensional rectangles) are relatively primes, that is, if $\gcd(xy,uv) = 1$, then the condition of Theorem 2 always holds.

Proposition 2. *[4, Proposition 6.1, Step 2] Let u,v,x,y and s be positive integers with* $\gcd(xy,uv) = \gcd(s,xy) = \gcd(s,uv) = 1$ *and such that $T(s,s)$ is tiled with $R(xy,xy)$ and $R(uv,uv)$. Then, $T(a,b)$ can be tiled with $R(x,y)$ and $R(u,v)$ if $a,b \geq s(xy)(uv)$.*

This yield to the following lower bound (by taking $s = xy + uv$)

$$C_T \geq (xy+uv)xyuv. \tag{2}$$

We might improve the latter by using a complete different technique.

Theorem 3. *Let u,v,x and y be positive integers such that* $\gcd(xy,uv)=1$. *Then, $T(a,b)$ can be tiled with $R(x,y)$ and $R(u,v)$ if*

$$a,b \geq \min\{n_1(uv+xy)+1, n_2(uv+xy)+1\}$$

where $n_1=\max\{ux,vy\}$ and $n_2=\max\{vx,uy\}$.

We notice that the above lower bound improves the one given in equation (2) by a factor of $\max\{ux,vy\}$. For instance, if we take $R(3,5)$ and $R(4,2)$ then $n_1=12$, $n_2=20$ and Theorem 3 gives $C_T \geq 12(15+8)+1=277$ while equation (2) gives $C_T \geq (15+8)(15)(8)=2760$. The latter lower bound can be improved since, by Proposition 2, $C_T \geq 120s$ where $\gcd(s,15)=\gcd(s,8)=1$ and such that $T(s,s)$ is tiled with $R(15,15)$ and $R(8,8)$. It is clear that such integer s must be at least 11 and thus obtainig $C_T \geq 1320$ (which still worst than our lower bound).

Theorem 3 implies the following characterization.

Theorem 4. *Let u,v,x and y be positive integers. Then, there exists $C_T(x,y,u,v)$ such that any $T(a,b)$ with $a,b \geq C_T$ can be tiled with $R(x,y)$ and $R(u,v)$ if and only if*$\gcd(xy,uv)=1$.

Proof. The sufficiency follows from Theorem 3. For the necessity, suppose, by contradiction, that $\gcd(xy,uv)=d>1$. Since $T(a,b)$ can be tiled with $R(x,y)$ and $R(u,v)$ then $ab=l_1(xy)+l_2(uv)$ for some nonnegative integers l_1,l_2 and any $a,b \geq C_T$. Since $\gcd(xy,uv)=d$ then d divides ab for any $a,b \geq C_T$. In particular, d divides pq for any pair of primes $p,q > C_T$ which is a contradiction. □

In order to prove Theorem 3, we may consider a special Euclidean plane tiling $\mathscr{T}^*$ formed with two rectangles $R(x,y)$ and $R(u,v)$ with sides parallel to the real axes, as shown in Figure 3 (we always suppose that the sides u and x are horizontal and the sides y and v are vertical).

Let u and v be positive integers. A plane tiling $\mathscr{T}$ is said to be *horizantally periodic* with *horizontal period*, denoted by $h_{\mathscr{T}}$, equals to u (resp. *vertically periodic* with *vertical period*, denoted by $v_{\mathscr{T}}$, equals to v) if $\mathscr{T}+(u,0)$ (resp. $\mathscr{T}+(0,v)$) is a congruent tansform mapping $\mathscr{T}$ into itself. A tiling $\mathscr{T}$ is *periodic* if it is both horizontally and vertically periodic.

Lemma 1. *Let $\mathscr{T}^*$ be the plane tiling given in Figure 3 with $R(x,y)$ and $R(u,v)$. Then, $\mathscr{T}^*$ is periodic with $h_{\mathscr{T}^*}=v_{\mathscr{T}^*}=uv+xy$.*

Proof. Without loss of generality, we assume that the lower leftmost corner of one copy of $R(x,y)$ is placed at $(0,0)$. It is clear that the coordinate of the lower leftmost corner of any other copy of $R(x,y)$ is given by $p(x,v)+q(u,-y)$ with $p,q \in \mathbf{Z}$. And thus, the translation $\mathscr{T}^*+(px+qu,pv-qy)$ is a congruent transform mapping $\mathscr{T}^*$ into itself. In particular, by taking $p=y$ and $q=v$ (resp. by taking $p=u$ and $q=-x$) we have that $\mathscr{T}^*+(vu+yx,0)$ (resp. $\mathscr{T}^*+(0,xy+uv)$) is a congruent transform mapping $\mathscr{T}^*$ into itself. Therefore, $\mathscr{T}^*$ is periodic with $h_{\mathscr{T}^*}=v_{\mathscr{T}^*}=uv+xy$. □

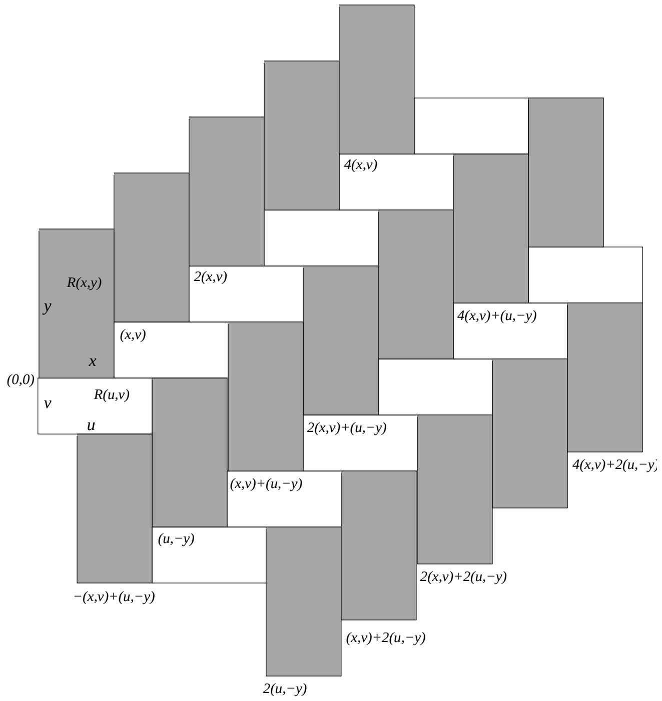

Fig. 3 Tiling T^* of the plane

Proposition 3. *Let $p,q \geq 1$ be integers. Then, $T(ph_{\mathscr{T}^*}, qv_{\mathscr{T}^*})$ can be tiled with $R(x,y)$ and $R(u,v)$.*

Proof. Without loss of generality, we assume that the lower leftmost corner of one copy of $R(x,y)$ is placed at $(0,0)$. Let B be the rectangle formed by lines $x_1 = 0$, $x_2 = ph_{\mathscr{T}^*}$, $y_1 = 0$ and $y_2 = qv_{\mathscr{T}^*}$. By definition of horizontally period, if a rectangle R is split by a line x_1 into two parts, r_1 (the part lying inside B) and r_2 (the part lying outside B) then the corresponding traslated rectangle is also split by line x_2 into two parts r_1' (the part lying outside B) and r_2' (the part lying inside B) where r_1 is congruent to r_1' and r_2 is congruent to r_2' (similarly for the split rectangles by lines y_1 and y_2), this is illustrated in Figure 4 when $p = q = 1$ and $x = 3, y = 5, u = 4, v = 2$. Thus, the tiling induced by the copies inside B where their opposites sides are identified gives the desired tiling of $T(ph_{\mathscr{T}^*}, qv_{\mathscr{T}^*})$. □

Proposition 4. *Let x,y,u and v be positive integers. Then, $T(ph_{\mathscr{T}^*} + sux, qv_{\mathscr{T}^*} + tvy)$ can be tiled with $R(x,y)$ and $R(u,v)$ for all integers $p,q \geq 1$ and $s,t \geq 0$.*

Proof. Let E_1 (resp. E_2) be the row formed by sticking together u (resp. x) copies of $R(x,y)$ (resp. $R(u,v)$) and let F_1 (resp. F_2) be the column formed by sticking together v (resp. y) copies of $R(x,y)$ (resp. $R(u,v)$), see Figure 5.

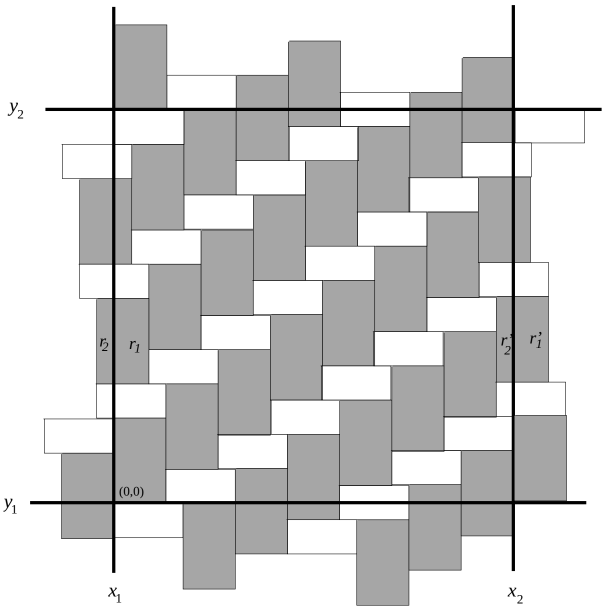

Fig. 4 Rectangle B formed with $R(3,5)$ and $R(4,2)$

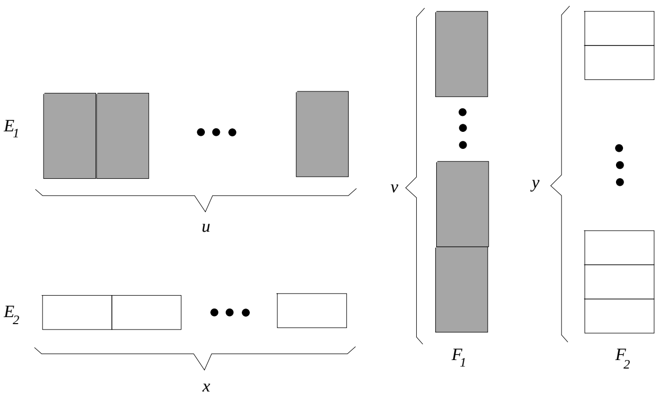

Fig. 5 Blocks of rows and columns

Given the constructed rectangle B as in Proposition 3 (that induces a tiling of $T(ph_{\mathscr{T}^*}, qv_{\mathscr{T}^*})$), we shall construct a rectangle B' that will induce a tiling of $T(ph_{\mathscr{T}^*} + sux, qv_{\mathscr{T}^*} + tvy)$. We will do this as follows (each step of the construction is illustrated with the case when $p = q = s = t = 1$, $x = 3, y = 5, u = 4, v = 2$).

Let E (resp. F) be the set of rectangles R of $\mathscr{T}^*$ such that either R shares its left-hand side border (resp. its bottom border) with the right-hand side border (resp. the top border) of B or R is cut by the right-hand side border (resp. the top border) of B. Let $\bar{B}$ be the union of the rectangles inside B together with sets E and F. We place $\bar{B}$ in the plane such that the leftmost bottom corner of one copy of $R(x,y)$ is placed at $(0,0)$. Figure 6 illustrates the construction of $\bar{B}$.

Now, for each rectangle R of E we stick s copies of E_1 if $R = R(x,y)$ (or s copies of E_2 if $R = R(u,v)$) to the right-hand side of R. And, analogously, for each rectangle R of F we stick t copies of F_1 if $R = R(x,y)$ (or t copies of F_2 if $R = R(u,v)$) above R, we do this in Figure 7.

Let B' be the rectangle formed by lines $x_1 = 0, y_1 = 0, x_3 = ph_{\mathscr{T}^*} + sux$ and $y_3 = qv_{\mathscr{T}^*} + tvy$ (notice that if $s = t = 0$ then $x_3 = x_2$ and $y_3 = y_2$). The rectangle formed by lines x_2, x_3, y_2 and y_3 (lying inside B' in its rightmost top corner) is of size $(sux) \times (tvy)$ and it can be tiled by placing tv, rows each formed by sticking together su copies of E_1, this is done in Figure 8.

We have the following two observations concerning B'.

(a) By definition of horizontal (resp. vertical) periodicity of $\mathscr{T}^*$, the intersection of x_1 and y_3 (resp. of y_1 and x_3) is a leftmost bottom corner of a copy of $R(x,y)$.

(b) If a rectangle is split by line x_1 into two parts r_1 (part lying inside B') and r_2 (part lying outside B') then the corresponding traslated rectangle is split by line x_3 into two parts, r'_1 (part lying outside B') and r'_2 (part lying inside B') where r_1 is congruent to r'_1 and r_2 is congruent to r'_2 (similarly for the split rectangles by lines y_1 and y_3).

Therefore, by the above observations, the desired tiling of $T(ph_{\mathscr{T}^*} + sux, qv_{\mathscr{T}^*} + tvy)$ is obtained by identifying opposite sides of B'. □

Proposition 5. *Let x, y, u and v be positive integers such that* $\gcd(xy, uv) = 1$. *Then,* $\gcd(xy + uv, vx) = \gcd(xy + uv, uy) = 1$.

Proof. We first show that if $\gcd(xy, uv) = 1$ then $\gcd(u, x) = \gcd(u, y) = \gcd(v, x) = \gcd(v, y) = 1$. Indeed, if $\gcd(x, u) = d > 1$ then there exists an integer $k > 1$ with $k|d$. So, k divides both x and u and thus $k|\gcd(xy, uv)$ implying that $\gcd(xy, uv) > 1$ which is a contradiction (similar for the other cases).

We shall now show that $\gcd(xy + uv, vx) = 1$ (the case $\gcd(uv + xy, uy) = 1$ can be done similarly). Let us suppose that $\gcd(uv + xy, vx) = k > 1$ and thus k divides both $uv + xy$ and vx. Let $p > 1$ be a prime such that p divides k. Then p also divides both $uv + xy$ and vx, and since p is prime then we have that p divides either v or x.

Case 1) If p divides v then $p|uv$ and since $p|(uv + xy)$ then either $p|x$ (but since $p|v$ then $p|\gcd(x, v)$ implying that $\gcd(x, v) > 1$ which is a contradiction) or $p|y$ (but since $p|v$ then $p|\gcd(y, v)$ implying that $\gcd(y, v) > 1$ which is a contradiction).

Case 2) If p divides x then $p|xy$ and since $p|(uv + xy)$ then either $p|u$ (but since $p|x$ then $p|\gcd(x, u)$ implying that $\gcd(x, u) > 1$ which is a contradiction) or $p|v$ (but since $p|x$ then $p|\gcd(x, v)$ implying that $\gcd(x, v) > 1$ which is a contradiction). □

We may now prove Theorem 3.

Proof of Theorem 3. Let $C = \max\{g(h_{\mathscr{T}^*}, ux) + h_{\mathscr{T}^*} + 1, g(v_{\mathscr{T}^*}, vy) + v_{\mathscr{T}^*} + 1\}$ (notice that the Frobenius numbers are well defined by Proposition 5). Let us suppose that $C = g(h_{\mathscr{T}^*}, ux) + h_{\mathscr{T}^*} + 1$ (similarly, in the case $C = g(v_{\mathscr{T}^*}, uy) + v_{\mathscr{T}^*} + 1$).

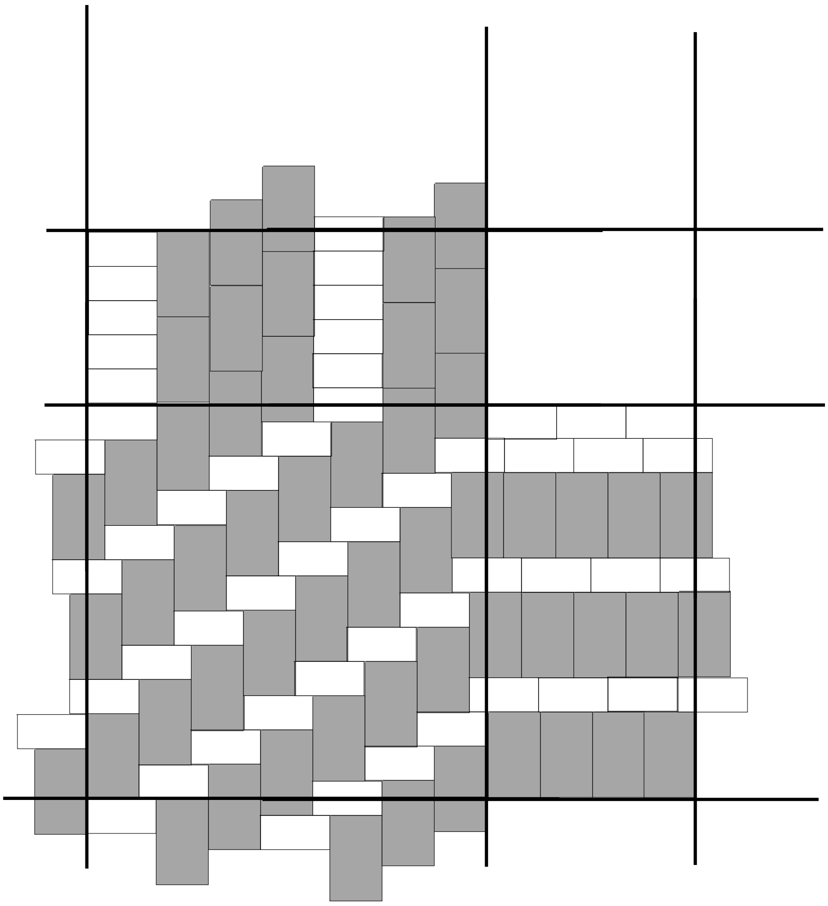

Fig. 6 Rectangle $\bar{B}$

Then, by definition of the Frobenius number there exist integers $p,s \geq 0$ such that $N = ph_{\mathscr{T}^*} + sux$ for any integer $N \geq g(h_{\mathscr{T}^*}, ux) + 1$. Thus there exist integers $p \geq 1$ and $s \geq 0$ such that $N = ph_{\mathscr{T}^*} + sux$ for any integer $N \geq g(h_{\mathscr{T}^*}, vx) + h_{\mathscr{T}^*} + 1$. So, since $p \geq 1$ then, by Proposition 4, $T(a,b)$ can be tiled with $R(x,y)$ and $R(u,v)$ if $a,b \geq \max\{g(h_{\mathscr{T}^*}, ux) + h_{\mathscr{T}^*} + 1, g(v_{\mathscr{T}^*}, vy) + v_{\mathscr{T}^*} + 1\}$ or equivalentely, by equation (1), if $a,b \geq \max\{vx(uv+xy)+1, uy(uv+xy)+1\}$.

We finally observe that in the construction of $\mathscr{T}^*$ we assume that the sides v and y are vertical and the sides u and x are horizontal but we could construct a similar tiling with the sides u and y vertical and the sides v and x horizontal. In this case, by applying the same arguments as above, we obtain that $T(a,b)$ can be tiled with $R(x,y)$ and $R(v,u)$ if $a,b \geq \max\{ux(uv+xy)+1, vy(uv+xy)+1\}$, and the result follows. □

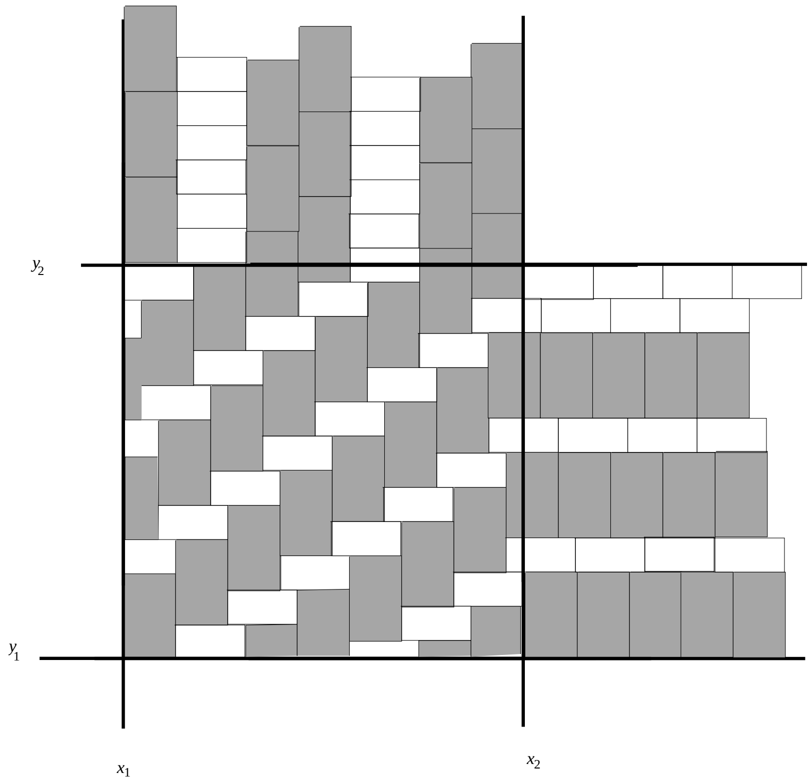

Fig. 7 Extended tiling

3 Tiling rectangles

Let $a_1,\ldots,a_n$ be positive integers. We denote by $R = R(a_1\ldots,a_n)$ the n-dimensional rectangle of sides a_i, that is, $R = \{(x_1,\ldots,x_n) \in \mathbf{R}^n | 0 \le x_i \le a_i, i = 1,\ldots,n\}$. A n-dimensional rectangle R is said to be *tiled* with *tiles* (n-dimensional rectangles) $R_1,\ldots,R_k$ if R can be filled entirely with copies of R_i, $1 \le i \le k$ (rotations are not allowed).

Our main result in this section is given by Theorem 5 (below) stating that a *sufficiently* large n-dimensional rectangle can be tiled with a set of $n+k-1$ tiles if any k-subset of the set of 1-coordinates (set of the first lengths) of the tiles are relatively primes and the set of j-coordinates (set of the j^{th} lengths) of the tiles are pairwise relatively prime for each $j = 2,\ldots,n$. We shall use again the Frobenius number and for, we need the following result.

Proposition 6. *Let* $a_1,\ldots,a_n$ *be positive integers such that* $\gcd(a_i,a_j) = 1$ *for all* $1 \le i \ne j \le n$*. Then,*

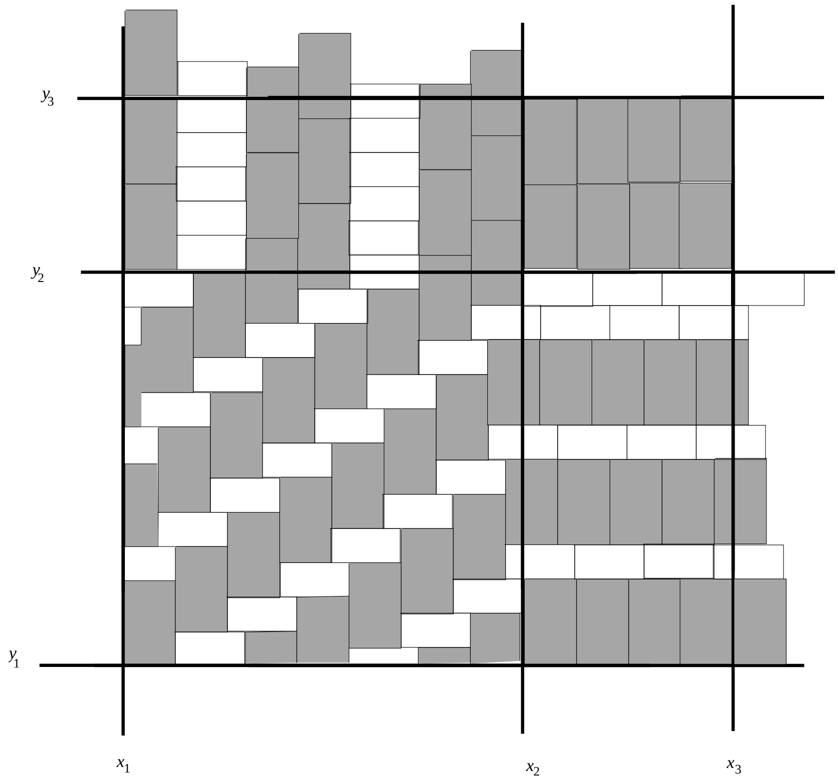

Fig. 8 Rectangle B' inducing a tiling of $T^*(33,35)$ with $R(3,5)$ and $R(4,2)$

$$\gcd\left(\frac{a_{i_1}\cdots a_{i_\ell}}{a_{i_\ell}},\ldots,\frac{a_{i_1}\cdots a_{i_\ell}}{a_{i_1}}\right)=1$$

for any $\{i_1<\ldots<i_\ell\}\subseteq\{1,\ldots,n\}$.

We leave the reader to prove this proposition by induction on ℓ.

Theorem 5. *Let* $k\geq 2$ *and* $n\geq 1$ *be integers. Let* $R_i(x_1^i,\ldots,x_n^i)$*,* $i=1,\ldots,n+k-1$ *be rectangles formed with integers* $x_j^i\geq 2$ *such that*

(a) $\gcd(x_1^{i_1},\ldots,x_1^{i_k})=1$ *for any* $\{i_1,\ldots,i_k\}\subset\{1,\ldots,n+k-1\}$ *and*

(b) $\gcd(x_j^{i_1},x_j^{i_2})=1$ *for any* $\{i_1,i_2\}\subset\{1,\ldots,n+k-1\}$ *and any* $j=2,\ldots,n$.

Let $g_1=\max\{g(x_{i_1}^1,\ldots,x_{i_k}^1)|\{i_1,\ldots,i_k\}\subset\{1,\ldots,n+k-1\}\}$ *and*

$$g_\ell=\max\left\{g\left(\frac{x_\ell^{i_1}\cdots x_\ell^{i_{\ell+k-2}}}{x_\ell^{i_{\ell+k-2}}},\ldots,\frac{x_\ell^{i_1}\cdots x_\ell^{i_{\ell+k-2}}}{x_\ell^{i_1}}\right)\ \middle|\ \{i_1,\ldots,i_{\ell+k-2}\}\subseteq\{1,\ldots,n+k-1\}\right\}$$

for each $\ell=2,\ldots,n$*. Then,*

$R(a_1,\dots,a_n)$ can be tiled with tiles $R_1,\dots,R_{n+k-1}$ if $a_j > \max_{1\le \ell\le n}\{g_\ell\}$ for all j.

Notice that when $k=2$ the number of tiles is $n+1$ which is the minimum required since, by Theorem 1, two square tiles do not suffice to tile all sufficiently large rectangles. Also, notice that if $k=2$ condition (a) becomes condition (b) with $j=1$ and when $k>2$ the number of tiles is increased but condition (a) is less restrictive that condition (b), we justify this below (see second paragraph after Corollary 1). We finally remark that the Frobenius numbers g_i used in Theorem 5 are well defined by Proposition 6.

In order to understand how the Frobenius number is used, we show how the constructive proof proceeds in the special case when $n=2$ and $k=2$ (the complete proof, given below, will be done by induction on n). Let us consider a rectangle $R(a_1,a_2)$ and tiles $R_i(x_1^i,x_2^i)$ with $i=1,\dots,3$. Since $\gcd(x_1^i,x_1^j)=1$ then if $a_1>g_1$ we have $a_1=ux_1^i+vx_1^j$ for all $1\le i\ne j\le 3$. So, we can form a rectangle $R_{ij}=R(a_1,x_2^ix_2^j)$ by sticking together u copies of R_i and v copies of R_j along the first coordinate, and then by replacing each R_i (resp. R_j) by a colmumn of x_2^j (resp. of x_2^i) copies of R_i (resp. R_j). Now, since $\gcd(x_2^i,x_2^j)=1$ for all $1\le i\ne j\le 3$ then, by Proposition 6, $\gcd(x_2^1x_2^2,x_2^1x_2^3,x_2^2x_2^3)=1$. So, if $a_2>g(x_2^1x_2^2,x_2^1x_2^3,x_2^2x_2^3)$ we have $a_2=ux_2^1x_2^2+vx_2^1x_2^3+wx_2^2x_2^3$. Therefore, $R(a_1,a_2)$ can be tiled with R_1,R_2,R_3 by sticking together u copies of R_{12}, v copies of R_{13} and w copies of R_{23} along the second coordinate.

Remark 1. $R(a,b)$ can be tiled with $R(4,6)$ and $R(5,7)$ if $a,b>197$.

Proof. We apply the above argument with $R_1(6,4),R_2(5,7)$ and $R_3(7,5)$ obtaining that $g_1=\max\{g(6,5),g(6,7),g(5,7)\}=\max\{19,29,23\}=29$ and $g_2=\max\{g(28,20,35)\}=197$. □

We denote by $(R;q)$ the rectangle obtained from $R(x_1,\dots,x_n)$ by sticking together q copies of R along the n^{th}-axis, that is, $(R;q)=R(x_1,\dots,x_{n-1},qx_n)$. We also denote by $\bar{R}$ the $(n-1)$-dimensional rectangle obtained from $R(x_1,\dots,x_n)$ by setting $x_n=0$, that is, $\bar{R}=\bar{R}(x_1,\dots,x_{n-1})$.

Proof of Theorem 5. We shall use induction on the dimension n with a fixed $k\ge 2$. For $n=1$ we have that $\gcd(x_1^{i_1},\dots,x_1^{i_k})=1$ for any $\{i_1,\dots,i_k\}\subset\{1,\dots,n+k-1\}$. Since $a_1>g_1$ then, by definition of the Frobenius number, any integer $a_1>g(x_1^{i_1},\dots,x_1^{i_k})$ is of the form $a_1=\sum_{j=1}^{k}u_jx_1^{i_j}$ where u_j is a nonnegative integer. Thus, the 1-dimensional rectangle $R(a_1)$ (that is, the interval $[0,a_1]$) can be tiled by sticking together tiles $(R^{i_1};u_1),\dots,(R^{i_k};u_k)$ (that is, by sticking together intervals $[0,u_1x_1^{i_1}],\dots,[0,u_kx_1^{i_k}]$) .

Let us suppose that it is true for $n-1\ge 1$ and we prove it for n. Let x_j^i be a positive integer for each $j=1,\dots,n$ and each $i=1,\dots,n+k-1$ with $\gcd(x_1^{i_1},\dots,x_1^{i_k})=1$ for any $\{i_1,\dots,i_k\}\subset\{1,\dots,n+k-1\}$ and $\gcd(x_j^{i_1},x_j^{i_2})=1$ for any $\{i_1,i_2\}\subset\{1,\dots,n+k-1\}$ and any $j=2,\dots,n$. Let $R_i=R_i(x_1^i,\dots,x_n^i)$,

$i = 1,\ldots,n+k-1$ and $a_j > \max\{g_1, g_2, \ldots, g_n\}$. By induction, $R(a_1,\ldots,a_{n-1})$ can be tiled with tiles $\bar{R}_{i_1},\ldots,\bar{R}_{i_{n+k-2}}$ for any $\{i_1 < \cdots < i_{n+k-2}\} \subset \{1,\ldots,n+k-1\}$ since $a_j > \max\{g_1, g_2, \ldots, g_{n-1}\}$ for any $1 \le j \le n-1$.

We claim that $R(a_1,\ldots,a_{n-1}, x_n^{i_1}\cdots x_n^{i_{n+k-2}})$ can be tiled with tiles $R_{i_1},\ldots,R_{i_{n+k-2}}$ for any $\{i_1 < \cdots < i_{n+k-2}\} \subset \{1,\ldots,n+k-1\}$. Indeed, if we consider the rectagle $R(a_1,\ldots,a_{n-1})$ embedded in $\mathbf{R}^n$ with $x_n = 0$ then, by replacing each tile $\bar{R}_{i_j}$ (used in the tiling of $R(a_1,\ldots,a_{n-1})$) by $(R_{i_j}; \frac{x_n^{i_1}\cdots x_n^{i_n}}{x_n^{i_j}})$ we obtain a tiling of $R(a_1,\ldots,a_{n-1}, x_n^{i_1}\cdots x_n^{i_{n+k-2}})$ with tiles $R_{i_1},\ldots,R_{i_{n+k-2}}$.

Now, since $a_n > g_n$ then $a_n = w_{n+k-1}(\frac{x_n^1\cdots x_n^{n+k-1}}{x_n^{n+k-1}}) + \cdots + w_1(\frac{x_n^1\cdots x_n^{n+k-1}}{x_n^1})$ where each w_i is a nonnegative integer. By the above claim, rectangle $R'_j = (a_1,\ldots,a_{n-1}, \frac{x_n^1\cdots x_n^{n+k-1}}{x_n^j})$ can be tiled with tiles $\{R_1,\ldots,R_{n+k-1}\} \setminus R_j$ for each $j = 1,\ldots,n+k-1$. Thus, $R(a_1,\ldots,a_{n-1},a_n)$ can be tiled with $R_1,\ldots,R_{n+k-1}$ by sticking together tiles $(R'_1;w_1),\ldots,(R'_{n+k-1};w_{n+k-1})$ along the n^{th}-axis. $\square$

Example 1. Let $R_1 = (22,3,3), R_2 = (14,5,5), R_3 = (21,2,2), R_4 = (15,7,7)$ and $R_5 = (55,11,11)$. In this case, we have $k = n = 3$.

$$\begin{aligned}
g_1 &= \max\{g(22,14,21), g(22,14,15), g(22,14,55), g(22,21,15), g(22,21,55), g(22,15,55),\\
&\qquad g(14,21,15), g(14,21,55), g(14,15,55), g(21,15,55)\}\\
&= \max\{139,91,173,181,243,97,288,151,179\} = 288.
\end{aligned}$$

With $\ell = 2$ we obtain

$$\begin{aligned}
g_2 &= \max\{g(3\cdot5, 3\cdot2, 5\cdot2), g(3\cdot5, 3\cdot7, 5\cdot7), g(3\cdot5, 3\cdot11, 5\cdot11), g(3\cdot2, 3\cdot7, 2\cdot7)\\
&\qquad g(3\cdot2, 3\cdot11, 2\cdot11), g(3\cdot7, 3\cdot11, 7\cdot11), g(5\cdot2, 5\cdot7, 2\cdot7), g(5\cdot2, 5\cdot11, 2\cdot11),\\
&\qquad g(5\cdot7, 5\cdot11, 7\cdot11), g(2\cdot7, 2\cdot11, 7\cdot11)\}\\
&= \max\{g(15,6,10), g(15,21,35), g(15,33,55), g(6,21,14), g(6,33,22),\\
&\qquad g(21,33,77), g(10,35,14), g(10,55,22), g(35,55,77), g(14,22,77)\}\\
&= \max\{29,139,227,43,71,331,81,133,603,195\} = 603.
\end{aligned}$$

And with $\ell = 3$

$$\begin{aligned}
g_3 &= \max\{g(3\cdot5\cdot2, 3\cdot5\cdot7, 3\cdot2\cdot7, 5\cdot2\cdot7), g(3\cdot5\cdot2, 3\cdot5\cdot11, 3\cdot2\cdot11, 5\cdot2\cdot11),\\
&\qquad g(3\cdot5\cdot7, 3\cdot5\cdot11, 3\cdot7\cdot11, 5\cdot7\cdot11), g(5\cdot2\cdot7, 5\cdot2\cdot11, 5\cdot7\cdot11, 2\cdot7\cdot11)\}\\
&= \max\{g(30,105,42,70), g(30,165,66,110), g(105,165,231,385), g(70,110,385,154)\}\\
&= \max\{383,619,2579,1591\} = 2579.
\end{aligned}$$

Therefore, Theorem 5 implies that $R(a_1,a_2,a_3)$ can be tiled with tiles $R_1,\ldots,R_5$ if $a_1,a_2,a_3 > \max\{g_1,g_2,g_3\} = \{288,603,2579\} = 2579$.

Corollary 1. *Let $k \geq 2$ and $n \geq 1$ be integers and let $R_i(x_1^i,\dots,x_n^i)$, $i = 1,\dots,n+k-1$ be rectangles formed with integers $x_j^i \geq 2$ verifying conditions (a) and (b) of Theorem 5. Then,*

$$R(a_1,\dots,a_n) \text{ can be tiled with tiles } R_1,\dots,R_{n+k-1} \text{ if } a_j > r^{2n} \text{ for all } j$$

where r is the largest length among all the tiles R_i.

Proof. The following upper bound for the Frobenius number, due to Wilf [13, Theorem 3.1.9], states that

$$g(b_1,\dots,b_n) \leq b_n^2 \tag{3}$$

where $b_1 < \cdots < b_n$ are relatively prime integers. In our case, this gives

$$g_\ell \leq (z_\ell^\ell)^2$$

where $z_\ell = \max\{x_\ell^1,\dots,x_\ell^{n+k-1}\}$ for each $\ell = 1,\dots,n$. Therefore, by Theorem 5, we have that $R(a_1,\dots,a_n)$ can be tiled with tiles $R_1,\dots,R_{n+k-1}$ if

$$a_i > r^{2n} \geq \max_{1\leq\ell\leq n}\{z_\ell^{2\ell}\} \geq \max_{1\leq\ell\leq n}\{g_\ell\}, \tag{4}$$

where r is the largest length among tiles $R_1,\dots,R_{n+k-1}$. □

Notice that the lower bound given in the above corollary depends on the lower bound given by equation (3) and thus it is not necessary optimal. For instance, in the above example, Corollary 1 would give $a_1,a_2,a_3 > 55^6$ while $a_1,a_2,a_3 > 2579$ is sufficient as shown in the example.

In [3, Theorem 3], it was announced (without proof) Theorem 5 for the case when $k = 2$, that is, when each set consisting of the j^{th} lengths of the tiles, are pairwise relatively prime. The latter is sometimes restrictive, for instance, the above example cannot be considered under these conditions. Indeed, any permutation of the coordinates (lengths) of tiles in this case will give a pair of j^{th}-coordinates not relatively primes for some $1 \leq j \leq 3$.

Katona and Szász [7] also investigated conditions for tiling n-dimensional rectangles by applying a generalization of the well-known Marriage theorem. They showed [7, Theorem 2 and Theorem 3] that $R(a_1,\dots,a_n)$ can be tiled with tiles $R_1,\dots,R_m$ if

$$a_j > 3^{km2^{mk}} r^{2^{kn}+2} \text{ for all } j$$

where r is the largest length among all the tiles and $k \geq 1$ is the cardinality of *special* sets constructed from the lengths of the tiles. In particular, when $k = 1$ (the smallest cardinality possible) the above inequality gives

$$a_j > 3^{m2^m} a^{2^n+2}. \tag{5}$$

It is clear that this lower bound is exponentially worst than the one given by Corollary 1.

3.1 Cube tiles

Theorem 6. *[3, Theorem 4] All sufficiently large n-dimensional rectangle R can be tiled by any given set of $n+1$ cubes with pairwise relatively prime edge lengths.*

We notice that this theorem is a particular case of Theorem 5 by taking $k=2$ and $x^i_j = a_i$ for each $i = 1,\ldots,n+1$ and all $1 \le j \le n$ where $1 < a_1 < a_2 < \cdots < a_{n+1}$ are pairwise relatively prime integers, $n \ge 1$. Moreover, Theorem 5 implies that $R(\underbrace{a,\ldots,a}_{n})$ can be tiled with $R(\underbrace{a_1,\ldots,a_1}_{n}),\ldots,R(\underbrace{a_{n+1},\ldots,a_{n+1}}_{n})$ if

$$a > g(A_1,\ldots,A_{n+1}) \tag{6}$$

where $A_i = P/a_i$ with $P = \prod_{j=1}^{n+1} a_j$. It turns out that the above lower bound can be stated explicitly by using the following formula, due to Tripathi [16], when $1 < a_1 < a_2 < \cdots < a_{n+1}$ are pairwise relatively prime integers,

$$g(A_1,\ldots,A_{n+1}) = nP - \sum_{i=1}^{n+1} A_i. \tag{7}$$

The above lower bound is not optimal in general. For instance, by combining equations (6) and (7), we obtain that $R(a,a)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ if $a \ge 7p-6$ where p is an odd integer with $3 \not| \, p$. The following result improves the latter.

Theorem 7. *Let $p > 4$ be an odd integer with $3 \not| \, p$. Then, $R(a,a)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ if $a \ge 3p+2$.*

We refer the reader to [12] where a collection of some unpublished work, due to D.A. Klarner, in relation with Theorem 7 can be found.

Proposition 7. *Let L,a,b,c and r be positive integers with $b|r$ and such that $r = x_1a + x_2c$ for some integers $x_1,x_2 \ge 0$ and $Lc = y_1a + y_2b$ for some integers $y_1,y_2 \ge 0$. Then, $R(r+ac, r+ac)$ and $R(Lc+kab, Lc+kab)$ can be tiled with $R(a,a), R(b,b)$ and $R(c,c)$ for any integer $k \ge 1$.*

Proof. Suppose that $b|r$. By Theorem 1, we have that

- $R(r,r)$ can be tiled with $R(b,b)$
- $R(ac,ac)$ can be tiled with $R(a,a)$,
- $R(ac,r)$ can be tiled with $R(a,a)$ and $R(c,c)$,
- $R(Lc,Lc)$ can be tiled with $R(c,c)$,
- $R(Lc,kab)$ can be tiled with $R(a,a)$ and $R(b,b)$ and
- $R(kab,kab)$ can be tiled with $R(a,a)$ (or with $R(b,b)$).

The results folllow by sticking together copies of the tilings of the above rectangles as shown in Figure 9. □

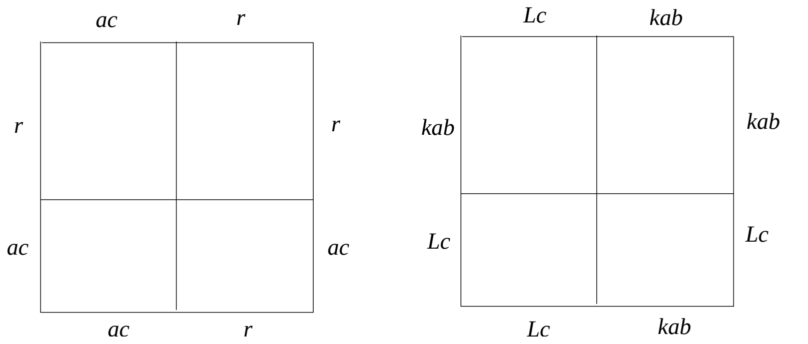

Fig. 9 Compositions of tilings

Proof of Theorem 7. By Theorem 1, $R(f,f)$ can be tiled with $R(2,2)$ and $R(3,3)$ if $f \equiv 0 \bmod 2$ or $f \equiv 0 \bmod 3$. So, we only need to show that $R(f,f)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ if $f \geq 3p+2$ when f is odd and $f \equiv 1$ or $2 \bmod 3$. Since $3 \not| p$ then $p \equiv i \bmod 3$ with $i = 1$ or 2.

Let $s = p - i + 3t \geq p+1$ for any integer $t \geq 1$. Since $s > g(2,p) = p-2$ then there exist nonnegative integers u and v such that $s = 2u + pv$. So, by Proposition 7 (with $a = 2, b = 3, r = s$ and $c = p$), we have that $R(s+2p, s+2p) = R(3(p+t) - i, 3(p+t) - i)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ for any integer $t \geq 1$. Or equivalently, $R(f,f)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ for any integer $f \geq 3p+1$ with $f \equiv -i \bmod 3$.

Also, since $p = 3t + i$ with $i = 1$ or 2 for some integer $t \geq 1$ then for $p > 3$ we have that $p = (t-1)3 + 2(2)$ and so, by Proposition 7 (with $a = 2, b = 3, r = s, c = p$ and $L = 1$), we have that, $R(p+6k, p+6k) = R(3(t+2k)+i, 3(t+2k)+i)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ for any integer $k \geq 1$. Or equivalently, $R(f,f)$ can be tiled with $R(2,2), R(3,3)$ and $R(p,p)$ for any odd integer $f \geq p+6$ with $f \equiv i \bmod 3$. □

Corollary 2. *$R(a,a)$ can be tiled with (a) $R(2,2), R(3,3)$ and $R(5,5)$ if and only if $a \neq 1,7$ and with (b) $R(2,2), R(3,3)$ and $R(7,7)$ if and only if $a \neq 1,5,11$.*

Proof. (a) It is clear that $R(1,1)$ and $R(7,7)$ cannot be tiled with $R(2,2), R(3,3)$ and $R(5,5)$. By Theorem 7, we have that $R(a,a)$ can be tiled with $R(2,2), R(3,3)$ and $R(5,5)$ if $a \geq 3p+2 = 17$ and, by Theorem 1, $R(a,a)$ can be tiled with $R(2,2)$ and $R(3,3)$ if $a \equiv 0 \bmod 2$ or $a \equiv 0 \bmod 3$. These leave us with the cases when $a = 5, 11$ and 13. The case $a = 5$ is trivial. $R(11,11)$ can be tiled with $R(2,2), R(3,3)$ and $R(5,5)$ since, by Theorem 7, the result is true for any odd integer $a \geq p+6 = 11$ and $a \equiv 2 \bmod 3$. Finally, $R(13,13)$ can be tiled as it is illustrated in Figure 10.

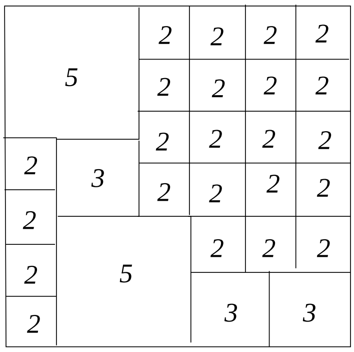

Fig. 10 Tiling $R(13,13)$ with $R(2,2), R(3,3)$ and $R(5,5)$

(b) It is clear that $R(1,1)$, $R(5,5)$ and $R(11,11)$ cannot be tiled with $R(2,2), R(3,3)$ and $R(7,7)$. By Theorem 7, we have that $R(a,a)$ can be tiled with $R(2,2), R(3,3)$ and $R(7,7)$ if $a \geq 3p+2 = 23$ and, by Theorem 1, $R(a,a)$ can be tiled with $R(2,2)$ and $R(3,3)$ if $a \equiv 0 \bmod 2$ or $a \equiv 0 \bmod 3$. These leave us with the cases when $a = 7, 13, 17$ and 19. The case $a = 7$ is trivial. $R(13,13)$ and $R(19,19)$ both can be tiled since, by Theorem 7, the result is true for any odd integer $a \geq p+6 = 13$ with $a \equiv 1 \bmod 3$. Finally, $R(17,17)$ can be tiled as it is illustrated in Figure 11. $\square$

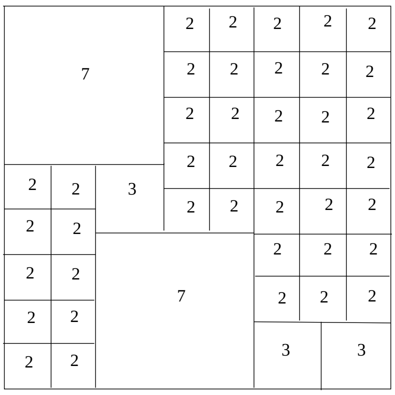

Fig. 11 Tiling $R(17,17)$ with $R(2,2), R(3,3)$ and $R(7,7)$

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