

THE UNAVOIDABLE ARRANGEMENTS OF PSEUDOCIRCLES

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ABSTRACT. A fact closely related to the classical Erdős-Szekeres theorem is that cyclic arrangements are the only *unavoidable* simple arrangements of pseudolines: for each fixed $m \geq 1$, every sufficiently large simple arrangement of pseudolines has a cyclic subarrangement of size m . In the same spirit, we show that there are three unavoidable arrangements of pseudocircles.

1. INTRODUCTION

A seminal result by Erdős and Szekeres [3] states that for every fixed integer $k \geq 3$, every sufficiently large set of points in general position in \mathbb{R}^2 contains k points in *convex position*, that is, k points that are the vertices of a convex k -gon. In the dual setting, a set (or *arrangement*) of lines in \mathbb{R}^2 is *in general position* if no two are parallel, and no three lines intersect at a common point. An arrangement of k lines is *in convex position* if one of its cells is bounded by a k -gon, which necessarily contains a segment from each of the k lines. The dual version of the Erdős-Szekeres theorem for lines is that for every fixed $k \geq 3$, every sufficiently large arrangement of lines in general position contains k lines in convex position (see [1]).

It must be noted that the point and line versions of the Erdős-Szekeres theorem are not dual to each other in \mathbb{R}^2 , but they are so in the projective plane \mathbb{RP}^2 . A *pseudoline* is a noncontractible simple closed curve in \mathbb{RP}^2 . An *arrangement of pseudolines* is a set of pseudolines that cross each other exactly once. An arrangement of pseudolines is *simple* if no three pseudolines have a common point. Two arrangements of pseudolines are *isomorphic* if the cell complexes they induce in \mathbb{RP}^2 are isomorphic.

A simple arrangement of pseudolines is *cyclic* if its pseudolines can be labelled $1, 2, \dots, m$, so that each pseudoline $i \in [m]$ intersects the pseudolines in $\{1, 2, \dots, m\} \setminus \{i\}$ in increasing order (see Figure 1). Cyclic arrangements of pseudolines have a cell whose boundary contains a segment from each pseudoline, and so they are the natural counterpart in \mathbb{RP}^2 to arrangements of lines in convex position in \mathbb{R}^2 .

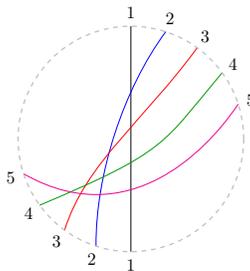


FIGURE 1. A cyclic arrangement of five pseudolines.

The following statement is the version of the Erdős-Szekeres theorem for simple arrangements of pseudolines. In the spirit of [15], this shows that cyclic arrangements are the only *unavoidable* arrangements of pseudolines.

Theorem 1 ([10, 13]). *For each fixed $m \geq 1$, every sufficiently large simple arrangement of pseudolines has a cyclic subarrangement of size m .*

We prove an analogue of Theorem 1 for arrangements of pseudocircles. A *pseudocircle* is a simple closed curve in the sphere \mathbb{S}^2 . We use Grünbaum’s original notion that an *arrangement of pseudocircles* is a set of pseudocircles that pairwise intersect at exactly two points, at which they cross, and no three pseudocircles have a common point [9]. This notion is still adopted nowadays [14], and some more general notions are also used in the literature [7, 11]. Two arrangements of pseudocircles are *isomorphic* if the cell complexes they induce in \mathbb{S}^2 are isomorphic.

In Figure 2 we illustrate three arrangements $\mathcal{C}_5^1, \mathcal{C}_5^2, \mathcal{C}_5^3$, and it is clear how to generalize them to arrangements $\mathcal{C}_m^1, \mathcal{C}_m^2$, and \mathcal{C}_m^3 of size m , for any $m \geq 1$ (we describe formally these arrangements below). These are the unavoidable arrangements of pseudocircles.

Theorem 2. *For each fixed $m \geq 1$, every sufficiently large arrangement of pseudocircles has a subarrangement isomorphic to $\mathcal{C}_m^1, \mathcal{C}_m^2$, or \mathcal{C}_m^3 .*

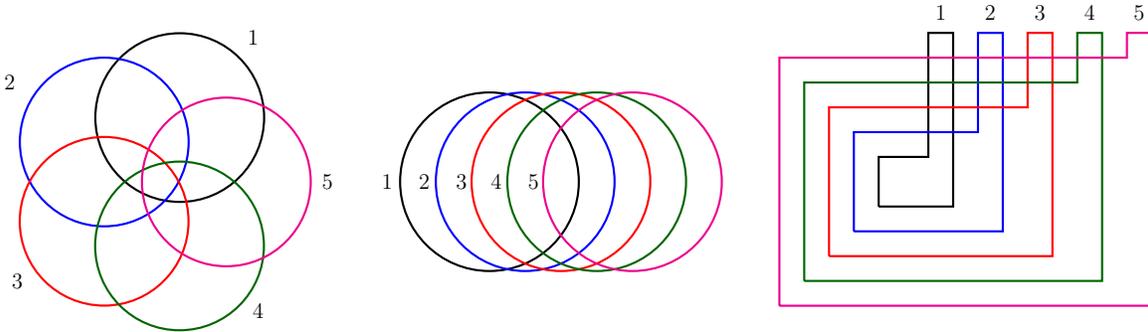


FIGURE 2. The arrangements \mathcal{C}_5^1 (left), \mathcal{C}_5^2 (center), and \mathcal{C}_5^3 (right).

We note that Theorem 2 is best possible, as no other type of arrangement of pseudocircles can be added to the collection $\{\mathcal{C}_m^1, \mathcal{C}_m^2, \mathcal{C}_m^3\}$. This follows since for all integers m, n with $m \leq n$, all subarrangements of \mathcal{C}_n^1 (respectively, $\mathcal{C}_n^2, \mathcal{C}_n^3$) of size m are isomorphic to \mathcal{C}_m^1 (respectively, $\mathcal{C}_m^2, \mathcal{C}_m^3$). Thus it is accurate to say that these are (up to isomorphism) *the* unavoidable arrangements of pseudocircles.

We now give a formal description of the arrangements $\mathcal{C}_m^1, \mathcal{C}_m^2$, and \mathcal{C}_m^3 for every $m \geq 1$. The pseudocircles of \mathcal{C}_m^1 are labelled $1, 2, \dots, m$, and pseudocircle i is the unit radius disk centered at $((4/5) \cos(2\pi i/m), (4/5) \sin(2\pi i/m))$. The pseudocircles of \mathcal{C}_m^2 are labelled $1, 2, \dots, m$, and pseudocircle i is the unit radius disk centered at $((2i-1)/m, 0)$. The pseudocircles of \mathcal{C}_m^3 are labelled $1, 2, \dots, m$, and pseudocircle i is the axis-parallel 6-gon with vertices $(-i, -i), (-i, i), (2i-1, i), (2i-1, m+1), (2i, m+1)$, and $(2i, -i)$. For good measure, in order to verify the accuracy of these descriptions, we did not prepare the depictions in Figure 2 “by hand”, but with the vector graphics drawing program TikZ [18], using these prescriptions with $m = 5$.

For the rest of the paper, for brevity we refer to an arrangement of pseudocircles simply as an *arrangement*.

2. REDUCING THEOREM 2 TO TWO KINDS OF ARRANGEMENTS

There are, up to isomorphism, only two arrangements of size 3. Following [6], these are the *Krupp arrangement* and the *NonKrupp arrangement*. We refer the reader to Figure 3. Note that the Krupp arrangement is isomorphic to \mathcal{C}_3^1 , and the NonKrupp arrangement is isomorphic to \mathcal{C}_3^2 and \mathcal{C}_3^3 . If an arrangement \mathcal{P} of size at least 3 has all its 3-subarrangements isomorphic to the Krupp arrangement (respectively, to the NonKrupp arrangement), then we say that \mathcal{P} is *Krupp-packed* (respectively, *NonKrupp-packed*).

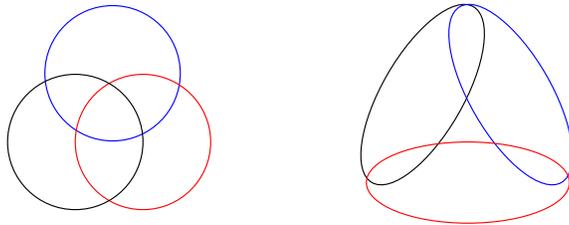


FIGURE 3. The Krupp arrangement (left) and the NonKrupp arrangement (right).

We now state Theorem 2 for Krupp-packed and for NonKrupp-packed arrangements. As we shall see shortly, the general version of Theorem 2 easily follows as a consequence.

Lemma 3. *Theorem 2 holds for Krupp-packed arrangements.*

Lemma 4. *Theorem 2 holds for NonKrupp-packed arrangements.*

We use Ramsey theory in our arguments. We recall that the *order* of a hypergraph is its number of vertices, and $r_k(\ell_1, \ell_2, \dots, \ell_n)$ denotes the Ramsey number for complete k -uniform hypergraphs. That is, if each k -edge of a complete k -uniform hypergraph of order at least $r_k(\ell_1, \ell_2, \dots, \ell_n)$ has colour i for some $i \in [n]$, then there is an $i \in [n]$ and a complete subhypergraph of order ℓ_i , all of whose k -edges have colour i .

Proof of Theorem 2, assuming Lemmas 3 and 4. Let $m \geq 1$ be a fixed integer. Assuming Lemmas 3 and 4, it follows that there is an integer p such that every Krupp-packed or NonKrupp-packed arrangement of size at least p has a subarrangement isomorphic to \mathcal{C}_m^1 , \mathcal{C}_m^2 , or \mathcal{C}_m^3 . Let \mathcal{Q} be an arrangement of size $q = r_3(p, p)$. Regard \mathcal{Q} as a complete 3-uniform hypergraph, and colour a 3-edge blue (respectively, red) if the pseudocircles in the 3-edge form an arrangement isomorphic to the Krupp arrangement (respectively, to the NonKrupp arrangement).

Since $q = r_3(p, p)$ it follows from Ramsey's theorem that \mathcal{Q} has a subarrangement \mathcal{P} of size p that is either Krupp-packed or NonKrupp-packed. The assumption on p implies that \mathcal{P} , and hence \mathcal{Q} , contains a subarrangement isomorphic to \mathcal{C}_m^1 , \mathcal{C}_m^2 , or \mathcal{C}_m^3 . \square

We finish this section by proving Lemma 3. The rest of the paper is devoted to the proof of Lemma 4.

The key fact we use in the proof of Lemma 3 is the bijection between simple arrangements of pseudolines and Krupp-packed arrangements (which are known in the literature as *arrangements of great pseudocircles*). This correspondence makes use of the wiring diagram representation of an arrangement of pseudolines [4, 8]. We now proceed to recall this bijection. For further details we refer the reader to the lively description in [5, Section 2.3].

Let \mathcal{L} be an arrangement of n pseudolines. Construct a wiring diagram representation \mathcal{W} of \mathcal{L} , drawing \mathcal{W} inside a rectangle R , so that flipping R around its horizontal symmetry axis

takes wire endpoints to wire endpoints. We refer the reader to Figure 4(a) for an illustration. Now let \mathcal{W}' be the wiring diagram obtained by flipping R along its horizontal symmetry axis (Figure 4(b)). We identify the right wire endpoints of \mathcal{W} with the left wire endpoints of \mathcal{W}' , obtaining a diagram \mathcal{W}'' whose n wires pairwise cross in exactly two points. In our running illustration, \mathcal{W}'' is enclosed in the dotted rectangle of Figure 4(c).

The obtained diagram \mathcal{W}'' is a wiring diagram representation of an arrangement of pseudocircles \mathcal{P} . The arrangement \mathcal{P} is obtained by joining with an arc the two endpoints of each wire in \mathcal{W}'' , as illustrated in Figure 4(c). In this way the simple arrangement of pseudolines \mathcal{L} induces the arrangement of pseudocircles \mathcal{P} , and it is easy to check that \mathcal{P} is Krupp-packed. Moreover (this the nontrivial direction of the bijection), every Krupp-packed arrangement of pseudocircles is isomorphic to an arrangement of pseudocircles induced in this manner by a simple arrangement of pseudolines (see [7, Section 3.2], or [17, Section 6.1.4]).

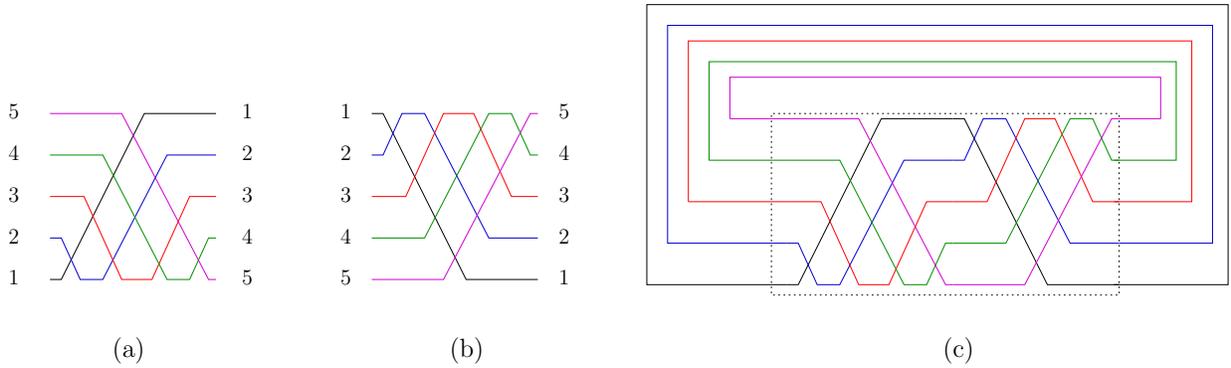


FIGURE 4. A cyclic arrangement of 5 pseudolines (shown in its wiring diagram representation) induces an arrangement of pseudocircles isomorphic to \mathcal{C}_5^1 .

Proof of Lemma 3. Let $m \geq 1$ be a fixed integer. Let \mathcal{P} be a Krupp-packed arrangement of size $p := r_3(m, m)$, and let \mathcal{S} be a simple arrangement of pseudolines that induces \mathcal{P} . Since $p = r_3(m, m)$, then by [16, Proposition 1.4] it follows that \mathcal{S} has a cyclic subarrangement \mathcal{L} of size m . It is readily verified that if an arrangement of pseudocircles is induced by a cyclic arrangement of pseudolines of size m , then it is isomorphic to \mathcal{C}_m^1 . From this it follows that the subarrangement of \mathcal{P} induced by the pseudolines in \mathcal{L} is isomorphic to \mathcal{C}_m^1 . \square

3. INTERSECTION CODES

In the proof of Lemma 4 we use intersection codes, as developed in [12, 14]. This framework, in which one naturally encodes combinatorially essential information of an arrangement, can be seen as a generalization of the axiomatization of oriented matroids based on hyperline sequences [2], and its essence goes back to the work of Gauss on planar curves in the 1830s.

Let $\mathcal{P} = \{1, \dots, n\}$ be an arrangement. For each $i \in \mathcal{P}$ choose a *reference point* p_i not contained in any other pseudocircle, and also choose one of the two possible orientations for i , so that for each pseudocircle we can naturally speak of a left side and a right side. Suppose that as we traverse i starting at p_i following the chosen orientation, we intersect j from the left (respectively, right) side of j . We record this by writing j^+ (respectively, j^-).

By keeping track of the order in which the intersections occur, we obtain the *code* of i in \mathcal{P} . Thus the code of each i is a permutation of $\bigcup_{j \in [n] \setminus \{i\}} \{j^+, j^-\}$. If we omit the superscripts $+$ and $-$, we obtain the *unsigned code* of i .

For instance, suppose that \mathcal{P} is \mathcal{C}_5^2 in Figure 2. Choose the counterclockwise orientation for each pseudocircle. If we choose as reference point for 3 its leftmost point, then its code is $1^+2^+4^-5^-5^+4^+2^-1^-$, and its unsigned code is 12455421.

We make essential use of the following.

Proposition 5 ([12, Section 3],[14, Section 2]). *Let \mathcal{P}, \mathcal{Q} be arrangements, both of which have their pseudocircles labelled $1, 2, \dots, n$, where an orientation and a reference point has been chosen for each pseudocircle in \mathcal{P} and each pseudocircle in \mathcal{Q} . Suppose that for each $i \in [n]$, the code of i in \mathcal{P} is the same as the code of i in \mathcal{Q} . Then \mathcal{P} and \mathcal{Q} are isomorphic.*

We now introduce some notation to describe the codes of \mathcal{C}_m^2 and \mathcal{C}_m^3 in a compact manner. Let i, m be integers such that $1 \leq i \leq m$. We use $[1^+:i^+)$ to denote the string $1^+2^+ \dots (i-1)^+$. In a similar spirit, we use $(i^-:1^-]$ to denote $(i-1)^-(i-2)^- \dots 1^-$; we use $(i^-:m^-]$ to denote $(i+1)^-(i+2)^- \dots m^-$; and we use $[m^+:i^+)$ to denote $m^+ \dots (i+2)^+(i+1)^+$. Finally, we use $[1^+1^-:i^+i^-)$ to denote $1^+1^-2^+2^- \dots (i-1)^+(i-1)^-$. Note that $[1^+:1^+)$, $(1^-:1^-]$, $[1^+1^-:1^+1^-)$, $(m^-:m^-]$, and $[m^+:m^+)$ are all empty strings.

Using this terminology, based on the descriptions of the arrangements \mathcal{C}_m^2 and \mathcal{C}_m^3 in Section 1, we have the following.

Observation 6. *Orient all pseudocircles $1, 2, \dots, m$ of \mathcal{C}_m^2 counterclockwise, and for each pseudocircle choose as reference point its leftmost point. Then for each $i \in [m]$, the code of pseudocircle i in \mathcal{C}_m^2 is $[1^+:i^+)(i^-:m^-] [m^+:i^+) (i^-:1^-]$.*

Observation 7. *Orient all pseudocircles $1, 2, \dots, m$ of \mathcal{C}_m^3 clockwise, and for each pseudocircle choose as reference point its bottom right corner. Then the code of each i in \mathcal{C}_m^3 is $[1^+1^-:i^+i^-)(i^-:m^-] [m^+:i^+)$.*

We close this section with a remark on NonKrupp-packed arrangements. We say that an arrangement is *bad* if its pseudocircles can be labelled $1, \dots, n$ so that with a reference point p_i for $i = 1, \dots, n$, the following holds. For each pseudocircle $i = 1, 2, \dots, n-2$, the unsigned code of i in the subarrangement $\{i, i+1, \dots, n\}$ is $(i+1)(i+1)(i+2)(i+2) \dots nn$. Up to isomorphism, there is only one bad NonKrupp-packed arrangement of size 4, namely the arrangement \mathcal{X}_4 shown in Figure 5. This is easily checked by hand, or by an inspection of [7, Figure 2], which contains all arrangements of size 4.

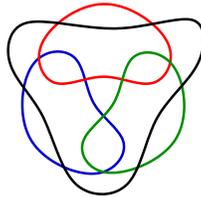


FIGURE 5. The arrangement \mathcal{X}_4 .

In a bad NonKrupp-packed arrangement of size 5, all 4-subarrangements would then be isomorphic to \mathcal{X}_4 . A routine case analysis by hand shows that no arrangement of size 5 satisfies this property. We highlight this remark, as we use it in the proof of Lemma 4.

Observation 8. *There is no bad NonKrupp-packed arrangement of size 5 (or larger).*

4. PROOF OF LEMMA 4

First we identify a property shared by \mathcal{C}_m^2 and \mathcal{C}_m^3 . To motivate this, we refer the reader to \mathcal{C}_5^2 and \mathcal{C}_5^3 , shown in Figure 2. If we perform the relabelling $i \mapsto i-1$ to the pseudocircles in either of these arrangements, the parts of 1, 2, 3, 4 in one component of $\mathbb{S}^2 \setminus \{0\}$ are pairwise disjoint, and appear in a rainbow-like fashion: the unsigned code of 0 is 12344321. We say that an arrangement is *rainbow* if its pseudocircles can be labelled $0, 1, \dots, n$ so that (I) one of the components of $\mathbb{S}^2 \setminus \{0\}$ contains no intersections among the pseudocircles $1, 2, \dots, n$; and (II) the unsigned code of pseudocircle 0 is $12 \cdots nn \cdots 21$.

Lemma 4 follows easily from the next two statements.

Proposition 9. *For each fixed integer $n \geq 5$, every sufficiently large NonKrupp-packed arrangement has a rainbow subarrangement of size n .*

Proposition 10. *For each fixed integer $m \geq 5$, every sufficiently large NonKrupp-packed rainbow arrangement contains a subarrangement isomorphic to \mathcal{C}_m^2 or \mathcal{C}_m^3 .*

Before proving these propositions, for completeness we give the proof of Lemma 4.

Proof of Lemma 4, assuming Propositions 9 and 10. Obviously it suffices to prove Theorem 2 for every integer $m \geq 5$. Let $m \geq 5$ be a fixed integer. By Proposition 10, there is an integer $n := n(m)$ such that every NonKrupp-packed rainbow arrangement contains a subarrangement isomorphic to \mathcal{C}_m^2 or \mathcal{C}_m^3 . By Proposition 9, there is an integer $q := q(n)$ such that every NonKrupp-packed arrangement has a rainbow subarrangement of size n . Thus every NonKrupp-packed arrangement of size at least q contains a subarrangement isomorphic to \mathcal{C}_m^2 or \mathcal{C}_m^3 . \square

Proof of Proposition 9. Let $n \geq 5$ be a fixed integer. Let $p := r_3(n, n) + 1$ and $q := r_3(p, p, p)$. We let $\mathcal{Q} = \{1, \dots, q\}$ be a NonKrupp-packed arrangement, and show that \mathcal{Q} contains a rainbow subarrangement of size n .

Choose an arbitrary reference point and orientation for each pseudocircle in \mathcal{Q} . The NonKrupp-packedness of \mathcal{Q} implies that if j, k, ℓ are pseudocircles in \mathcal{Q} such that $j < k < \ell$, then the unsigned code of j in the subarrangement $\{j, k, \ell\}$ is either (i) $kk\ell\ell$; or (ii) $\ell\ell k k$; or (iii) $k\ell\ell k$; or (iv) $\ell k k \ell$.

Regard \mathcal{Q} as a complete 3-uniform hypergraph, and assign to each 3-edge $\{j, k, \ell\}$ with $j < k < \ell$ the colour (i), (ii), (iii), or (iv), depending on which of these scenarios holds. By Ramsey's theorem, \mathcal{Q} has a subarrangement $\mathcal{P} = \{1', 2', \dots, p'\}$, with $1 \leq 1' < \dots < p' \leq q$, all of whose 3-edges are of the same colour.

Suppose that all 3-edges of \mathcal{P} are of colour (i). Then for each $i = 1, \dots, p-2$, the unsigned code of i' in the subarrangement $\{i', (i+1)', \dots, p'\}$ is $(i+1)'(i+1)' \cdots p'p'$. Thus \mathcal{P} is a bad arrangement of size $p > 5$, contradicting Observation 8. Thus not all 3-edges of \mathcal{P} can be of colour (i). An analogous argument shows that not all 3-edges can be of colour (ii).

If all 3-edges of \mathcal{P} are of colour (iv) then by relabelling the pseudocircles in the reverse order we obtain an arrangement in which all 3-edges are of colour (iii). Thus we may assume that all 3-edges of \mathcal{P} are of colour (iii). In particular, the unsigned code of $1'$ in $\{1', \dots, p'\}$ is $2'3' \cdots p'p' \cdots 3'2'$. Using that $p-1 = r_2(n, n)$, an application of Ramsey's theorem shows that there exist i'_1, i'_2, \dots, i'_n , with $2' \leq i'_1 < i'_2 < \dots < i'_n \leq p'$ such that one of the connected components of $\mathbb{S}^2 \setminus \{1'\}$ contains no intersections among the pseudocircles i'_1, \dots, i'_n . The arrangement $\{1', i'_1, \dots, i'_n\}$ is rainbow. To see this, it suffices to relabel $1'$ with 0, and i'_j with j , for each $j = 1, \dots, n$. \square

Proof of Proposition 10. Let $m \geq 5$ be a fixed integer. Let $q = r_3(m, m, m)$, and $n = r_3(q, q)$. Let $\mathcal{N}_0 = \{0, 1, 2, \dots, n\}$ be a NonKrupp-packed rainbow arrangement. We show that \mathcal{N}_0 contains a subarrangement isomorphic to either \mathcal{C}_m^2 or to \mathcal{C}_m^3 .

Our first goal is to show that we may assume that the layout of \mathcal{N}_0 is as shown on the right hand side of Figure 6. To achieve this, first we note that by performing a self-homeomorphism of the sphere we may assume that the pseudocircle 0 is the union of the Greenwich Meridian and the 180th Meridian, in particular passing through the north pole N and the south pole S . We orient 0 in the direction from S to N following the Greenwich Meridian, as on the left hand side of Figure 6.

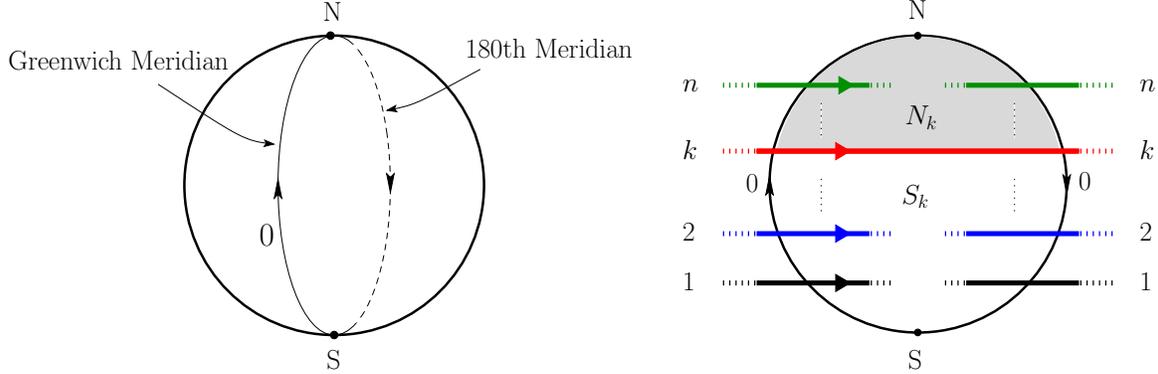


FIGURE 6. Initial setup in the proof of Proposition 10. On the left hand side we illustrate pseudocircle 0. On the right hand side we illustrate the eastern hemisphere, which contains all intersections among the pseudocircles $1, \dots, n$. We illustrate the regions N_k (grey) and S_k (white) of pseudocircle k .

Since \mathcal{N}_0 satisfies rainbowness Property (II), we may assume that as we traverse the Greenwich Meridian from S to N we intersect the pseudocircles $1, 2, \dots, n$ in this order, and as we traverse the 180th Meridian from N to S , we intersect them in the order $n, \dots, 2, 1$. Since \mathcal{N}_0 satisfies rainbowness Property (I), then either the eastern or the western hemisphere (say the eastern one) contains all the intersections among the pseudocircles in $\mathcal{N} := \{1, 2, \dots, n\}$. We orient all pseudocircles in \mathcal{N} so that as we traverse 0 along the Greenwich Meridian from S to N the code we obtain is $1^- 2^- \dots n^-$ (that is, these pseudocircles hit 0 from its left hand side). Thus as we traverse the 180th Meridian from N to S the code is $n^+ (n-1)^+ \dots 1^+$. Each pseudocircle $k \in \mathcal{N}$ decomposes the eastern hemisphere into two parts, a part N_k that contains N , and a part S_k that contains S . Thus the setup is as illustrated in Figure 6.

Since \mathcal{N} is NonKrupp-packed, for any three pseudocircles j, k, ℓ in \mathcal{N} , either both intersections of j and ℓ occur in N_k , or they both occur in S_k . Regard \mathcal{N} as a complete 3-uniform hypergraph, and assign to a 3-edge $\{j, k, \ell\}$ with $j < k < \ell$ the colour N (respectively, S) if the intersections of j and ℓ lie on N_k (respectively, S_k). Since $|\mathcal{N}| = n = r_3(q, q)$, then by Ramsey's theorem \mathcal{N} has a subarrangement $\mathcal{Q} = \{1', 2', \dots, q'\}$ of size q , with $1' < 2' < \dots < q'$, all of whose 3-edges are of the same colour. By symmetry we may assume that all 3-edges of \mathcal{Q} have colour N . To avoid unnecessary cluttered notation, we relabel the pseudocircles in \mathcal{Q} as $1, 2, \dots, q$.

Let j, k, ℓ be such that $1 \leq j < k < \ell \leq q$. Since the intersections between j and ℓ occur in N_k , that is, above k , and \mathcal{Q} is NonKrupp-packed, then there are only three possibilities for how j and ℓ can intersect each other, namely as shown in Figure 7(a), (b), and (c). Thus the code of k in $\{j, k, \ell\}$ is either (1) $j^+ \ell^- \ell^+ j^-$, or (2) $j^+ j^- \ell^- \ell^+$, or (3) $\ell^- \ell^+ j^+ j^-$, respectively.

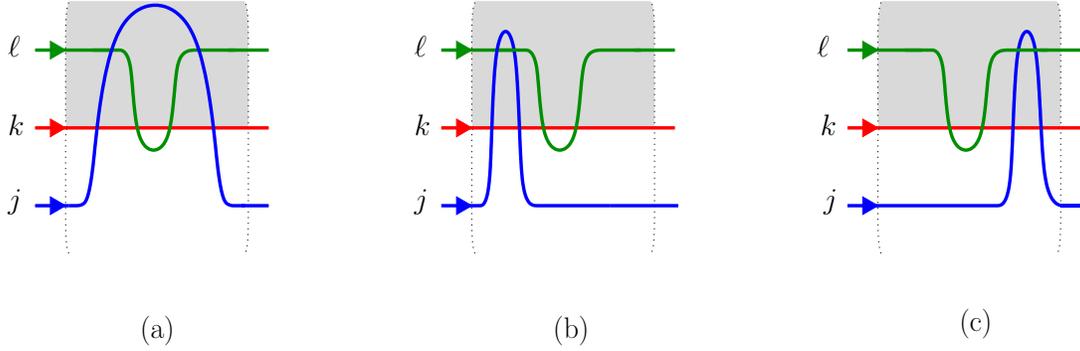


FIGURE 7. Illustration of the proof of Proposition 10.

Regard \mathcal{Q} as a complete 3-uniform hypergraph. We assign to a 3-edge $\{j, k, \ell\}$ with $j < k < \ell$ a colour in $\{1, 2, 3\}$, depending, respectively, on which one of (1), (2), and (3) gives the code of k in $\{j, k, \ell\}$. Since $q = r_3(m, m, m)$, by Ramsey's theorem \mathcal{Q} has a subarrangement $\mathcal{M} = \{1', 2', \dots, m'\}$, where $1' < 2' < \dots < m'$, all of whose 3-edges are of the same colour. If all the 3-edges are of colour 3, then by changing the traversal directions of all pseudocircles and inverting the crossing-sign convention we obtain that all the 3-edges become of colour 2. Thus we may assume that either all the 3-edges are of colour 1, or they are all of colour 2. Again, to avoid unnecessary cluttered notation we relabel the pseudocircles in \mathcal{M} with $1, 2, \dots, m$.

Suppose first that all the 3-edges of \mathcal{M} are of colour 1. Then for each $1 \leq j < k < \ell \leq m$, the situation is as in Figure 7(a). Thus (i) the code of j in $\{j, k, \ell\}$ is $k^- \ell^- \ell^+ k^+$; (ii) the code of k in $\{j, k, \ell\}$ is $j^+ \ell^- \ell^+ j^-$; and (iii) the code of ℓ in $\{j, k, \ell\}$ is $j^+ k^+ k^- j^-$. We first analyze the code of each pseudocircle $i \in \{2, 3, \dots, m-1\}$. By (i), the code of i in \mathcal{M} contains the subpermutation $(i^- : m^-][m^+ : i^+)$; by (iii), this code contains the subpermutation $[1^+ : i^+)(i^- : 1^-]$; and by (ii), it contains the subpermutation $(i-1)^+(i+1)^-(i+1)^+(i-1)^-$. These three conditions hold only if the code of i is $[1^+ : i^+)(i^- : m^-][m^+ : i^+)(i^- : 1^-]$.

We now analyze the codes of pseudocircles 1 and m . By (i), the code of 1 in \mathcal{M} is $(1^- : m^-][m^+ : 1^+) = [1^+ : 1^+)(1^- : m^-][m^+ : 1^+)(1^- : 1^-]$ (since $[1^+ : 1^+)$ and $(1^- : 1^-]$ are empty strings). Finally, by (iii), the code of m in \mathcal{M} is $[1^+ : m^+)(m^- : 1^-] = [1^+ : m^+)(m^- : m^-][m^+ : m^+)(m^- : 1^-]$ (since $(m^- : m^-]$ and $[m^+ : m^+)$ are empty strings). We conclude that the code of every $i \in \mathcal{M}$ is $[1^+ : i^+)(i^- : m^-][m^+ : i^+)(i^- : 1^-]$. By Proposition 5 and Observation 6, then \mathcal{M} is isomorphic to \mathcal{C}_m^2 .

Suppose finally that all the 3-edges of \mathcal{M} have colour 2. Then for each $1 \leq j < k < \ell \leq m$, the situation is as in Figure 7(b). Thus (i) the code of j in $\{j, k, \ell\}$ is $k^- \ell^- \ell^+ k^+$; (ii) the code of k in $\{j, k, \ell\}$ is $j^+ j^- \ell^- \ell^+$; and (iii) the code of ℓ in $\{j, k, \ell\}$ is $j^+ j^- k^+ k^-$. We first analyze the code of each pseudocircle $i \in \{2, 3, \dots, m-1\}$. By (i), the code of i in \mathcal{M} contains the subpermutation $(i^- : m^-][m^+ : i^+)$; by (iii), this code contains the subpermutation $[1^+ 1^- : i^+ i^-)$; and by (ii), it contains the subpermutation $(i-1)^+(i-1)^-(i+1)^-(i+1)^+$. These three conditions hold only if the code of i is $[1^+ 1^- : i^+ i^-)(i^- : m^-][m^+ : i^+)$.

We now analyze the codes of 1 and m . By (i), the code of 1 in \mathcal{M} is $(1^- : m^-][m^+ : 1^+) = [1^+ 1^- : 1^+ 1^-)(1^- : m^-][m^+ : 1^+)$ (since $[1^+ 1^- : 1^+ 1^-)$ is an empty string). Finally, by (iii), the code of m in \mathcal{M} is $[1^+ 1^- : m^+ m^-) = [1^+ 1^- : m^+ m^-)(m^- : m^-][m^+ : m^+)$ (since $(m^- : m^-]$ and $[m^+ : m^+)$ are empty strings). We conclude that the code of every $i \in \mathcal{M}$ is $[1^+ 1^- : i^+ i^-)(i^- : m^-][m^+ : i^+)$. By Proposition 5 and Observation 7, then \mathcal{M} is isomorphic to \mathcal{C}_m^3 . \square

5. OPEN QUESTIONS

To prove Theorem 2 we make repeated use of Ramsey’s theorem, and so an explicit bound for this theorem, derived from our proofs, would be multiply exponential. With additional effort (and considerably more space) we can save several applications of Ramsey’s theorem, and show that for each fixed $m \geq 1$, every arrangement of pseudocircles of size at least $2^{2^{cm^2}}$ contains a subarrangement isomorphic to \mathcal{C}_m^1 , \mathcal{C}_m^2 , or \mathcal{C}_m^3 . This bound is still doubly exponential in m . What is the best explicit bound that can be proved for this theorem?

We now include two questions that were brought to our attention by a reviewer. Let us say that a k -arrangement of pseudocircles is a collection of pseudocircles that pairwise cross at exactly k points. With this terminology, the main result in this paper is that 2-arrangements of pseudocircles have three unavoidable arrangements. Which are the unavoidable k -arrangements of pseudocircles? How does the number of unavoidable k -arrangements of pseudocircles increase with k ?

In this mindset, we finally add a question of our own, which may be regarded as a warm-up to the questions in the previous paragraph. Is it obvious, or (this seems more likely) at least reasonably easy to prove, that for each fixed (even) integer $k \geq 2$ the set of unavoidable k -arrangements of pseudocircles is finite?

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