

CHAPTER III

Cohomology

In this chapter we define the general notion of cohomology of a sheaf of abelian groups on a topological space, and then study in detail the cohomology of coherent and quasi-coherent sheaves on a noetherian scheme.

Although the end result is usually the same, there are many different ways of introducing cohomology. There are the fine resolutions often used in several complex variables—see Gunning and Rossi [1]; the Čech cohomology used by Serre [3], who first introduced cohomology into abstract algebraic geometry; the canonical flasque resolutions of Godement [1]; and the derived functor approach of Grothendieck [1]. Each is important in its own way.

We will take as our basic definition the derived functors of the global section functor (§1, 2). This definition is the most general, and also best suited for theoretical questions, such as the proof of Serre duality in §7. However, it is practically impossible to calculate, so we introduce Čech cohomology in §4, and use it in §5 to compute explicitly the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space $\mathbf{P}^r$. This calculation is the basis of many later results on projective varieties.

In order to prove that the Čech cohomology agrees with the derived functor cohomology, we need to know that the higher cohomology of a quasi-coherent sheaf on an affine scheme is zero. We prove this in §3 in the noetherian case only, because it is technically much simpler than the case of an arbitrary affine scheme ([EGA III, §1]). Hence we are bound to include noetherian hypotheses in all theorems involving cohomology.

As applications, we show for example that the arithmetic genus of a projective variety X , whose definition in (I, §7) depended on a projective embedding of X , can be computed in terms of the cohomology groups $H^i(X, \mathcal{O}_X)$, and hence is intrinsic (Ex. 5.3). We also show that the arithmetic genus is constant in a family of normal projective varieties (9.13).

Another application is Zariski's main theorem (11.4) which is important in the birational study of varieties.

The latter part of the chapter (§8–12) is devoted to families of schemes, i.e., the study of the fibres of a morphism. In particular, we include a section on flat morphisms and a section on smooth morphisms. While these can be treated without cohomology, it seems to be an appropriate place to include them, because flatness can be understood better using cohomology (9.9).

1 Derived Functors

In this chapter we will assume familiarity with the basic techniques of homological algebra. Since notation and terminology vary from one source to another, we will assemble in this section (without proofs) the basic definitions and results we will need. More details can be found in the following sources: Godement [1, esp. Ch. I, §1.1–1.8, 2.1–2.4, 5.1–5.3], Hilton and Stammbach [1, Ch. II, IV, IX], Grothendieck [1, Ch. II, §1, 2, 3], Cartan and Eilenberg [1, Ch. III, V], Rotman [1, §6].

Definition. An *abelian category* is a category $\mathfrak{A}$, such that: for each $A, B \in \text{Ob } \mathfrak{A}$, $\text{Hom}(A, B)$ has a structure of an abelian group, and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism. (Hilton and Stammbach [1, p. 78].)

The following are all abelian categories.

Example 1.0.1. $\mathfrak{Ab}$, the category of abelian groups.

Example 1.0.2. $\mathfrak{Mod}(A)$, the category of modules over a ring A (commutative with identity as always).

Example 1.0.3. $\mathfrak{Ab}(X)$, the category of sheaves of abelian groups on a topological space X .

Example 1.0.4. $\mathfrak{Mod}(X)$, the category of sheaves of $\mathcal{O}_X$ -modules on a ringed space $(X, \mathcal{O}_X)$.

Example 1.0.5. $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$ -modules on a scheme X (II, 5.7).

Example 1.0.6. $\mathfrak{Coh}(X)$, the category of coherent sheaves of $\mathcal{O}_X$ -modules on a noetherian scheme X (II, 5.7).

Example 1.0.7. $\mathcal{C}oh(\mathfrak{X})$, the category of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$ -modules on a noetherian formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ (II, 9.9).

In the rest of this section, we will be stating some basic results of homological algebra in the context of an arbitrary abelian category. However, in most books, these results are proved only for the category of modules over a ring, and proofs are often done by “diagram-chasing”: you pick an element and chase its images and pre-images through a diagram. Since diagram-chasing doesn’t make sense in an arbitrary abelian category, the conscientious reader may be disturbed. There are at least three ways to handle this difficulty. (1) Provide intrinsic proofs for all the results, starting from the axioms of an abelian category, and without even mentioning an element. This is cumbersome, but can be done—see, e.g., Freyd [1]. Or (2), note that in each of the categories we use (most of which are in the above list of examples), one can in fact carry out proofs by diagram-chasing. Or (3), accept the “full embedding theorem” (Freyd [1, Ch. 7]), which states roughly that any abelian category is equivalent to a subcategory of $\mathfrak{Ab}$. This implies that any category-theoretic statement (e.g., the 5-lemma) which can be proved in $\mathfrak{Ab}$ (e.g., by diagram-chasing) also holds in any abelian category.

Now we begin our review of homological algebra. A *complex* $A^\cdot$ in an abelian category $\mathfrak{A}$ is a collection of objects A^i , $i \in \mathbb{Z}$, and morphisms $d^i: A^i \rightarrow A^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i . If the objects A^i are specified only in a certain range, e.g., $i \geq 0$, then we set $A^i = 0$ for all other i . A *morphism* of complexes, $f: A^\cdot \rightarrow B^\cdot$ is a set of morphisms $f^i: A^i \rightarrow B^i$ for each i , which commute with the coboundary maps d^i .

The i th *cohomology object* $h^i(A^\cdot)$ of the complex $A^\cdot$ is defined to be $\ker d^i / \text{im } d^{i-1}$. If $f: A^\cdot \rightarrow B^\cdot$ is a morphism of complexes, then f induces a natural map $h^i(f): h^i(A^\cdot) \rightarrow h^i(B^\cdot)$. If $0 \rightarrow A^\cdot \rightarrow B^\cdot \rightarrow C^\cdot \rightarrow 0$ is a short exact sequence of complexes, then there are natural maps $\delta^i: h^i(C^\cdot) \rightarrow h^{i+1}(A^\cdot)$ giving rise to a long exact sequence

$$\dots \rightarrow h^i(A^\cdot) \rightarrow h^i(B^\cdot) \rightarrow h^i(C^\cdot) \xrightarrow{\delta^i} h^{i+1}(A^\cdot) \rightarrow \dots$$

Two morphisms of complexes $f, g: A^\cdot \rightarrow B^\cdot$ are *homotopic* (written $f \sim g$) if there is a collection of morphisms $k^i: A^i \rightarrow B^{i-1}$ for each i (which need not commute with the d^i) such that $f - g = dk + kd$. The collection of morphisms, $k = (k^i)$ is called a *homotopy operator*. If $f \sim g$, then f and g induce the *same* morphism $h^i(A^\cdot) \rightarrow h^i(B^\cdot)$ on the cohomology objects, for each i .

A covariant functor $F: \mathfrak{A} \rightarrow \mathfrak{B}$ from one abelian category to another is *additive* if for any two objects A, A' in $\mathfrak{A}$, the induced map $\text{Hom}(A, A') \rightarrow \text{Hom}(FA, FA')$ is a homomorphism of abelian groups. F is *left exact* if it is additive and for every short exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

in $\mathfrak{A}$, the sequence

$$0 \rightarrow FA' \rightarrow FA \rightarrow FA''$$

is exact in $\mathfrak{B}$. If we can write a 0 on the right instead of the left, we say F is *right exact*. If it is both left and right exact, we say it is *exact*. If only the middle part $FA' \rightarrow FA \rightarrow FA''$ is exact, we say F is *exact in the middle*.

For a contravariant functor we make analogous definitions. For example, $F: \mathfrak{A} \rightarrow \mathfrak{B}$ is *left exact* if it is additive, and for every short exact sequence as above, the sequence

$$0 \rightarrow FA'' \rightarrow FA \rightarrow FA'$$

is exact in $\mathfrak{B}$.

Example 1.0.8. If $\mathfrak{A}$ is an abelian category, and A is a fixed object, then the functor $B \rightarrow \text{Hom}(A, B)$, usually denoted $\text{Hom}(A, \cdot)$, is a covariant left exact functor from $\mathfrak{A}$ to $\mathfrak{Ab}$. The functor $\text{Hom}(\cdot, A)$ is a contravariant left exact functor from $\mathfrak{A}$ to $\mathfrak{Ab}$.

Next we come to resolutions and derived functors. An object I of $\mathfrak{A}$ is *injective* if the functor $\text{Hom}(\cdot, I)$ is exact. An *injective resolution* of an object A of $\mathfrak{A}$ is a complex I' , defined in degrees $i \geq 0$, together with a morphism $\varepsilon: A \rightarrow I^0$, such that I^i is an injective object of $\mathfrak{A}$ for each $i \geq 0$, and such that the sequence

$$0 \rightarrow A \xrightarrow{\varepsilon} I^0 \rightarrow I^1 \rightarrow \dots$$

is exact.

If every object of $\mathfrak{A}$ is isomorphic to a subobject of an injective object of $\mathfrak{A}$, then we say $\mathfrak{A}$ *has enough injectives*. If $\mathfrak{A}$ has enough injectives, then every object has an injective resolution. Furthermore, a well-known lemma states that any two injective resolutions are homotopy equivalent.

Now let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F: \mathfrak{A} \rightarrow \mathfrak{B}$ be a covariant left exact functor. Then we construct the *right derived functors* $R^i F$, $i \geq 0$, of F as follows. For each object A of $\mathfrak{A}$, choose once and for all an injective resolution I' of A . Then we define $R^i F(A) = h^i(F(I'))$.

Theorem 1.1A. *Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F: \mathfrak{A} \rightarrow \mathfrak{B}$ be a covariant left exact functor to another abelian category $\mathfrak{B}$. Then*

(a) *For each $i \geq 0$, $R^i F$ as defined above is an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. Furthermore, it is independent (up to natural isomorphism of functors) of the choices of injective resolutions made.*

(b) *There is a natural isomorphism $F \cong R^0 F$.*

(c) *For each short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ and for each $i \geq 0$ there is a natural morphism $\delta^i: R^i F(A'') \rightarrow R^{i+1} F(A')$, such that we obtain a long exact sequence*

$$\dots \rightarrow R^i F(A') \rightarrow R^i F(A) \rightarrow R^i F(A'') \xrightarrow{\delta^i} R^{i+1} F(A') \rightarrow R^{i+1} F(A) \rightarrow \dots$$

(d) Given a morphism of the exact sequence of (c) to another $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$, the δ 's give a commutative diagram

$$\begin{array}{ccc} R^i F(A'') & \xrightarrow{\delta^i} & R^{i+1} F(A') \\ \downarrow & & \downarrow \\ R^i F(B'') & \xrightarrow{\delta^i} & R^{i+1} F(B'). \end{array}$$

(e) For each injective object I of $\mathfrak{A}$, and for each $i > 0$, we have $R^i F(I) = 0$.

Definition. With $F: \mathfrak{A} \rightarrow \mathfrak{B}$ as in the theorem, an object J of $\mathfrak{A}$ is *acyclic* for F if $R^i F(J) = 0$ for all $i > 0$.

Proposition 1.2A. With $F: \mathfrak{A} \rightarrow \mathfrak{B}$ as in (1.1A), suppose there is an exact sequence

$$0 \rightarrow A \rightarrow J^0 \rightarrow J^1 \rightarrow \dots$$

where each J^i is acyclic for F , $i \geq 0$. (We say J is an F -acyclic resolution of A .) Then for each $i \geq 0$ there is a natural isomorphism $R^i F(A) \cong h^i(F(J))$.

We leave to the reader the analogous definitions of projective objects, projective resolutions, an abelian category having enough projectives, and the left derived functors of a covariant right exact functor. Also, the right derived functors of a left exact contravariant functor (use projective resolutions) and the left derived functors of a right exact contravariant functor (use injective resolutions).

Next we will give a universal property of derived functors. For this purpose, we generalize slightly with the following definition.

Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories. A (covariant) δ -functor from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T = (T^i)_{i \geq 0}$, together with a morphism $\delta^i: T^i(A'') \rightarrow T^{i+1}(A')$ for each short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$, and each $i \geq 0$, such that:

(1) For each short exact sequence as above, there is a long exact sequence

$$\begin{aligned} 0 \rightarrow T^0(A') \rightarrow T^0(A) \rightarrow T^0(A'') &\xrightarrow{\delta^0} T^1(A') \rightarrow \dots \\ \dots \rightarrow T^i(A) \rightarrow T^i(A'') &\xrightarrow{\delta^i} T^{i+1}(A') \rightarrow T^{i+1}(A) \rightarrow \dots; \end{aligned}$$

(2) for each morphism of one short exact sequence (as above) into another $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$, the δ 's give a commutative diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B'). \end{array}$$

Definition. The δ -functor $T = (T^i): \mathfrak{A} \rightarrow \mathfrak{B}$ is said to be *universal* if, given any other δ -functor $T' = (T'^i): \mathfrak{A} \rightarrow \mathfrak{B}$, and given any morphism of

functors $f^0: T^0 \rightarrow T'^0$, there exists a unique sequence of morphisms $f^i: T^i \rightarrow T'^i$ for each $i \geq 0$, starting with the given f^0 , which commute with the δ^i for each short exact sequence.

Remark 1.2.1. If $F: \mathfrak{A} \rightarrow \mathfrak{B}$ is a covariant additive functor, then by definition there can exist at most one (up to unique isomorphism) universal δ -functor T with $T^0 = F$. If T exists, the T^i are sometimes called the *right satellite functors* of F .

Definition. An additive functor $F: \mathfrak{A} \rightarrow \mathfrak{B}$ is *effaceable* if for each object A of $\mathfrak{A}$, there is a monomorphism $u: A \rightarrow M$, for some M , such that $F(u) = 0$. It is *coeffaceable* if for each A there exists an epimorphism $u: P \rightarrow A$ such that $F(u) = 0$.

Theorem 1.3A. Let $T = (T^i)_{i \geq 0}$ be a covariant δ -functor from $\mathfrak{A}$ to $\mathfrak{B}$. If T^i is effaceable for each $i > 0$, then T is universal.

PROOF. Grothendieck [1, II, 2.2.1]

Corollary 1.4. Assume that $\mathfrak{A}$ has enough injectives. Then for any left exact functor $F: \mathfrak{A} \rightarrow \mathfrak{B}$, the derived functors $(R^i F)_{i \geq 0}$ form a universal δ -functor with $F \cong R^0 F$. Conversely, if $T = (T^i)_{i \geq 0}$ is any universal δ -functor, then T^0 is left exact, and the T^i are isomorphic to $R^i T^0$ for each $i \geq 0$.

PROOF. If F is a left exact functor, then the $(R^i F)_{i \geq 0}$ form a δ -functor by (1.1A). Furthermore, for any object A , let $u: A \rightarrow I$ be a monomorphism of A into an injective. Then $R^i F(I) = 0$ for $i > 0$ by (1.1A), so $R^i F(u) = 0$. Thus $R^i F$ is effaceable for each $i > 0$. It follows from the theorem that $(R^i F)$ is universal.

On the other hand, given a universal δ -functor T , we have T^0 left exact by the definition of δ -functor. Since $\mathfrak{A}$ has enough injectives, the derived functors $R^i T^0$ exist. We have just seen that $(R^i T^0)$ is another universal δ -functor. Since $R^0 T^0 = T^0$, we find $R^i T^0 \cong T^i$ for each i , by (1.2.1).

2 Cohomology of Sheaves

In this section we define cohomology of sheaves by taking the derived functors of the global section functor. Then as an application of general techniques of cohomology we prove Grothendieck's theorem about the vanishing of cohomology on a noetherian topological space. To begin with, we must verify that the categories we use have enough injectives.

Proposition 2.1A. If A is a ring, then every A -module is isomorphic to a submodule of an injective A -module.

PROOF. Godement [1, I, 1.2.2] or Hilton and Stammach [1, I, 8.3].

Proposition 2.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$ -modules has enough injectives.*

PROOF. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$ -modules. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{x,X}$ -module. Therefore there is an injection $\mathcal{F}_x \rightarrow I_x$, where I_x is an injective $\mathcal{O}_{x,X}$ -module (2.1A). For each point x , let j denote the inclusion of the one-point space $\{x\}$ into X , and consider the sheaf $\mathcal{I} = \prod_{x \in X} j_* (I_x)$. Here we consider I_x as a sheaf on the one-point space $\{x\}$, and j_* is the direct image functor (II, §1).

Now for any sheaf $\mathcal{G}$ of $\mathcal{O}_X$ -modules, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) = \prod \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_* (I_x))$ by definition of the direct product. On the other hand, for each point $x \in X$, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_* (I_x)) \cong \mathrm{Hom}_{\mathcal{O}_{x,X}}(\mathcal{G}_x, I_x)$ as one sees easily. Thus we conclude first that there is a natural morphism of sheaves of $\mathcal{O}_X$ -modules $\mathcal{F} \rightarrow \mathcal{I}$ obtained from the local maps $\mathcal{F}_x \rightarrow I_x$. It is clearly injective. Second, the functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{I})$ is the direct product over all $x \in X$ of the stalk functor $\mathcal{G} \mapsto \mathcal{G}_x$, which is exact, followed by $\mathrm{Hom}_{\mathcal{O}_{x,X}}(\cdot, I_x)$, which is exact, since I_x is an injective $\mathcal{O}_{x,X}$ -module. Hence $\mathrm{Hom}(\cdot, \mathcal{I})$ is an exact functor, and therefore $\mathcal{I}$ is an injective $\mathcal{O}_X$ -module.

Corollary 2.3. *If X is any topological space, then the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

PROOF. Indeed, if we let $\mathcal{O}_X$ be the constant sheaf of rings $\mathbf{Z}$, then $(X, \mathcal{O}_X)$ is a ringed space, and $\mathfrak{Mod}(X) = \mathfrak{Ab}(X)$.

Definition. Let X be a topological space. Let $\Gamma(X, \cdot)$ be the global section functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We define the *cohomology functors* $H^i(X, \cdot)$ to be the right derived functors of $\Gamma(X, \cdot)$. For any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$. Note that even if X and $\mathcal{F}$ have some additional structure, e.g., X a scheme and $\mathcal{F}$ a quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathcal{F}$ simply as a sheaf of abelian groups on the underlying topological space X .

We let the reader write out the long exact sequences which follow from the general properties of derived functors (1.1A).

Recall (II, Ex. 1.16) that a sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective.

Lemma 2.4. *If $(X, \mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$ -module is flasque.*

PROOF. For any open subset $U \subseteq X$, let $\mathcal{O}_U$ denote the sheaf $j_!(\mathcal{O}_X|_U)$, which is the restriction of $\mathcal{O}_X$ to U , extended by zero outside U (II, Ex. 1.19). Now let $\mathcal{I}$ be an injective $\mathcal{O}_X$ -module, and let $V \subseteq U$ be open sets. Then we have an inclusion $0 \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$ -modules. Since $\mathcal{I}$ is injective, we get a surjection $\mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) \rightarrow \mathrm{Hom}(\mathcal{O}_V, \mathcal{I}) \rightarrow 0$. But $\mathrm{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$ and $\mathrm{Hom}(\mathcal{O}_V, \mathcal{I}) = \mathcal{I}(V)$, so $\mathcal{I}$ is flasque.

Proposition 2.5. *If $\mathcal{F}$ is a flasque sheaf on a topological space X , then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

PROOF. Embed $\mathcal{F}$ in an injective object $\mathcal{I}$ of $\mathfrak{Ab}(X)$ and let $\mathcal{G}$ be the quotient:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow 0.$$

Then $\mathcal{F}$ is flasque by hypothesis, $\mathcal{I}$ is flasque by (2.4), and so $\mathcal{G}$ is flasque by (II, Ex. 1.16c). Now since $\mathcal{F}$ is flasque, we have an exact sequence (II, Ex. 1.16b)

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{I}) \rightarrow \Gamma(X, \mathcal{G}) \rightarrow 0.$$

On the other hand, since $\mathcal{I}$ is injective, we have $H^i(X, \mathcal{I}) = 0$ for $i > 0$ (1.1Ae). Thus from the long exact sequence of cohomology, we get $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G})$ for each $i \geq 2$. But $\mathcal{G}$ is also flasque, so by induction on i we get the result.

Remark 2.5.1. This result tells us that flasque sheaves are acyclic for the functor $\Gamma(X, \cdot)$. Hence we can calculate cohomology using flasque resolutions (1.2A). In particular, we have the following result.

Proposition 2.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X, \cdot)$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ coincide with the cohomology functors $H^i(X, \cdot)$.*

PROOF. Considering $\Gamma(X, \cdot)$ as a functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, we calculate its derived functors by taking injective resolutions in the category $\mathfrak{Mod}(X)$. But any injective is flasque (2.4), and flasques are acyclic (2.5) so this resolution gives the usual cohomology functors (1.2A).

Remark 2.6.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $A = \Gamma(X, \mathcal{O}_X)$. Then for any sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$, $\Gamma(X, \mathcal{F})$ has a natural structure of A -module. In particular, since we can calculate cohomology using resolutions in the category $\mathfrak{Mod}(X)$, all the cohomology groups of $\mathcal{F}$ have a natural structure of A -module; the associated exact sequences are sequences of A -modules, and so forth. Thus for example, if X is a scheme over $\text{Spec } B$ for some ring B , the cohomology groups of any $\mathcal{O}_X$ -module $\mathcal{F}$ have a natural structure of B -module.

A Vanishing Theorem of Grothendieck

Theorem 2.7 (Grothendieck [1]). *Let X be a noetherian topological space of dimension n . Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X , we have $H^i(X, \mathcal{F}) = 0$.*

Before proving the theorem, we need some preliminary results, mainly concerning direct limits. If $(\mathcal{F}_\alpha)$ is a direct system of sheaves on X , indexed by a directed set A , then we have defined the direct limit $\varinjlim \mathcal{F}_\alpha$ (II, Ex. 1.10).

Lemma 2.8. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

PROOF. Let $(\mathcal{F}_\alpha)$ be a directed system of flasque sheaves. Then for any inclusion of open sets $V \subseteq U$, and for each α , we have $\mathcal{F}_\alpha(U) \rightarrow \mathcal{F}_\alpha(V)$ is surjective. Since $\varinjlim$ is an exact functor, we get

$$\varinjlim \mathcal{F}_\alpha(U) \rightarrow \varinjlim \mathcal{F}_\alpha(V)$$

is also surjective. But on a noetherian topological space, $\varinjlim \mathcal{F}_\alpha(U) = (\varinjlim \mathcal{F}_\alpha)(U)$ for any open set (II, Ex. 1.11). So we have

$$(\varinjlim \mathcal{F}_\alpha)(U) \rightarrow (\varinjlim \mathcal{F}_\alpha)(V)$$

is surjective, and so $\varinjlim \mathcal{F}_\alpha$ is flasque.

Proposition 2.9. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)$ be a direct system of abelian sheaves. Then there are natural isomorphisms, for each $i \geq 0$*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \rightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

PROOF. For each α we have a natural map $\mathcal{F}_\alpha \rightarrow \varinjlim \mathcal{F}_\alpha$. This induces a map on cohomology, and then we take the direct limit of these maps. For $i = 0$, the result is already known (II, Ex. 1.11). For the general case, we consider the category $\text{ind}_A(\mathfrak{Ab}(X))$ consisting of all directed systems of objects of $\mathfrak{Ab}(X)$, indexed by A . This is an abelian category. Furthermore, since $\varinjlim$ is an exact functor, we have a natural transformation of δ -functors

$$\varinjlim H^i(X, \cdot) \rightarrow H^i(X, \varinjlim \cdot)$$

from $\text{ind}_A(\mathfrak{Ab}(X))$ to $\mathfrak{Ab}$. They agree for $i = 0$, so to prove they are the same, it will be sufficient to show they are both effaceable for $i > 0$. For in that case, they are both universal by (1.3A), and so must be isomorphic.

So let $(\mathcal{F}_\alpha) \in \text{ind}_A(\mathfrak{Ab}(X))$. For each α , let $\mathcal{G}_\alpha$ be the sheaf of discontinuous sections of $\mathcal{F}_\alpha$ (II, Ex. 1.16e). Then $\mathcal{G}_\alpha$ is flasque, and there is a natural inclusion $\mathcal{F}_\alpha \rightarrow \mathcal{G}_\alpha$. Furthermore, the construction of $\mathcal{G}_\alpha$ is functorial, so the $\mathcal{G}_\alpha$ also form a direct system, and we obtain a monomorphism $u: (\mathcal{F}_\alpha) \rightarrow (\mathcal{G}_\alpha)$ in the category $\text{ind}_A(\mathfrak{Ab}(X))$. Now the $\mathcal{G}_\alpha$ are all flasque, so $H^i(X, \mathcal{G}_\alpha) = 0$ for $i > 0$ (2.5). Thus $\varinjlim H^i(X, \mathcal{G}_\alpha) = 0$, and the functor on the left-hand side is effaceable for $i > 0$. On the other hand, $\varinjlim \mathcal{G}_\alpha$ is also flasque by (2.8). So $H^i(X, \varinjlim \mathcal{G}_\alpha) = 0$ for $i > 0$, and we see that the functor on the right-hand side is also effaceable. This completes the proof.

Remark 2.9.1. As a special case we see that cohomology commutes with infinite direct sums.

Lemma 2.10. *Let Y be a closed subset of X , let $\mathcal{F}$ be a sheaf of abelian groups on Y , and let $j: Y \rightarrow X$ be the inclusion. Then $H^i(Y, \mathcal{F}) = H^i(X, j_* \mathcal{F})$, where $j_* \mathcal{F}$ is the extension of $\mathcal{F}$ by zero outside Y (II, Ex. 1.19).*

PROOF. If $\mathcal{J}'$ is a flasque resolution of $\mathcal{F}$ on Y , then $j_*\mathcal{J}'$ is a flasque resolution of $j_*\mathcal{F}$ on X , and for each i , $\Gamma(Y, \mathcal{J}'^i) = \Gamma(X, j_*\mathcal{J}'^i)$. So we get the same cohomology groups.

Remark 2.10.1. Continuing our earlier abuse of notation (II, Ex. 1.19), we often write $\mathcal{F}$ instead of $j_*\mathcal{F}$. This lemma shows there will be no ambiguity about the cohomology groups.

PROOF OF (2.7). First we fix some notation. If Y is a closed subset of X , then for any sheaf $\mathcal{F}$ on X we let $\mathcal{F}_Y = j_*(\mathcal{F}|_Y)$, where $j: Y \rightarrow X$ is the inclusion. If U is an open subset of X , we let $\mathcal{F}_U = i_*(\mathcal{F}|_U)$, where $i: U \rightarrow X$ is the inclusion. In particular, if $U = X - Y$, we have an exact sequence (II, Ex. 1.19)

$$0 \rightarrow \mathcal{F}_U \rightarrow \mathcal{F} \rightarrow \mathcal{F}_Y \rightarrow 0.$$

We will prove the theorem by induction on $n = \dim X$, in several steps.

Step 1. Reduction to the case X irreducible. If X is reducible, let Y be one of its irreducible components, and let $U = X - Y$. Then for any $\mathcal{F}$ we have an exact sequence

$$0 \rightarrow \mathcal{F}_U \rightarrow \mathcal{F} \rightarrow \mathcal{F}_Y \rightarrow 0.$$

From the long exact sequence of cohomology, it will be sufficient to prove that $H^i(X, \mathcal{F}_Y) = 0$ and $H^i(X, \mathcal{F}_U) = 0$ for $i > n$. But Y is closed and irreducible, and $\mathcal{F}_U$ can be regarded as a sheaf on the closed subset $\bar{U}$, which has one fewer irreducible components than X . Thus using (2.10) and induction on the number of irreducible components, we reduce to the case X irreducible.

Step 2. Suppose X is irreducible of dimension 0. Then the only open subsets of X are X and the empty set. For otherwise, X would have a proper irreducible closed subset, and $\dim X$ would be ≥ 1 . Thus $\Gamma(X, \cdot)$ induces an equivalence of categories $\mathfrak{Ab}(X) \rightarrow \mathfrak{Ab}$. In particular, $\Gamma(X, \cdot)$ is an exact functor, so $H^i(X, \mathcal{F}) = 0$ for $i > 0$, and for all $\mathcal{F}$.

Step 3. Now let X be irreducible of dimension n , and let $\mathcal{F} \in \mathfrak{Ab}(X)$. Let $B = \bigcup_{U \subseteq X} \mathcal{F}(U)$, and let A be the set of all finite subsets of B . For each $\alpha \in A$, let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the sections in α (over various open sets). Then A is a directed set, and $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$. So by (2.9), it will be sufficient to prove vanishing of cohomology for each $\mathcal{F}_\alpha$. If α' is a subset of α , then we have an exact sequence

$$0 \rightarrow \mathcal{F}_{\alpha'} \rightarrow \mathcal{F}_\alpha \rightarrow \mathcal{G} \rightarrow 0,$$

where $\mathcal{G}$ is a sheaf generated by $\#(\alpha - \alpha')$ sections over suitable open sets. Thus, using the long exact sequence of cohomology, and induction on $\#(\alpha)$, we reduce to the case that $\mathcal{F}$ is generated by a single section over some open set U . In that case $\mathcal{F}$ is a quotient of the sheaf $\mathbf{Z}_U$ (where $\mathbf{Z}$ denotes the constant sheaf $\mathbf{Z}$ on X). Letting $\mathcal{R}$ be the kernel, we have an exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathbf{Z}_U \rightarrow \mathcal{F} \rightarrow 0.$$

Again using the long exact sequence of cohomology, it will be sufficient to prove vanishing for $\mathcal{R}$ and for $\mathbf{Z}_U$.

Step 4. Let U be an open subset of X and let $\mathcal{R}$ be a subsheaf of $\mathbf{Z}_U$. For each $x \in U$, the stalk $\mathcal{R}_x$ is a subgroup of $\mathbf{Z}$. If $\mathcal{R} = 0$, skip to Step 5. If not, let d be the least positive integer which occurs in any of the groups $\mathcal{R}_x$. Then there is a nonempty open subset $V \subseteq U$ such that $\mathcal{R}|_V \cong d \cdot \mathbf{Z}|_V$ as a subsheaf of $\mathbf{Z}|_V$. Thus $\mathcal{R}_V \cong \mathbf{Z}_V$ and we have an exact sequence

$$0 \rightarrow \mathbf{Z}_V \rightarrow \mathcal{R} \rightarrow \mathcal{R}/\mathbf{Z}_V \rightarrow 0.$$

Now the sheaf $\mathcal{R}/\mathbf{Z}_V$ is supported on the closed subset $(U - V)^-$ of X , which has dimension $< n$, since X is irreducible. So using (2.10) and the induction hypothesis, we know $H^i(X, \mathcal{R}/\mathbf{Z}_V) = 0$ for $i \geq n$. So by the long exact sequence of cohomology, we need only show vanishing for $\mathbf{Z}_V$.

Step 5. To complete the proof, we need only show that for any open subset $U \subseteq X$, we have $H^i(X, \mathbf{Z}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \rightarrow \mathbf{Z}_U \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}_Y \rightarrow 0.$$

Now $\dim Y < \dim X$ since X is irreducible, so using (2.10) and the induction hypothesis, we have $H^i(X, \mathbf{Z}_Y) = 0$ for $i \geq n$. On the other hand, $\mathbf{Z}$ is flasque, since it is a constant sheaf on an irreducible space (II, Ex. 1.16a). Hence $H^i(X, \mathbf{Z}) = 0$ for $i > 0$ by (2.5). So from the long exact sequence of cohomology we have $H^i(X, \mathbf{Z}_U) = 0$ for $i > n$. q.e.d.

Historical Note: The derived functor cohomology which we defined in this section was introduced by Grothendieck [1]. It is the theory which is used in [EGA]. The use of sheaf cohomology in algebraic geometry started with Serre [3]. In that paper, and in the later paper [4], Serre used Čech cohomology for coherent sheaves on an algebraic variety with its Zariski topology. The equivalence of this theory with the derived functor theory follows from the “theorem of Leray” (Ex. 4.11). The same argument, using Cartan’s “Theorem B” shows that the Čech cohomology of a coherent analytic sheaf on a complex analytic space is equal to the derived functor cohomology. Gunning and Rossi [1] use a cohomology theory computed by fine resolutions of a sheaf on a paracompact Hausdorff space. The equivalence of this theory with ours is shown by Godement [1, Thm. 4.7.1, p. 181 and Ex. 7.2.1, p. 263], who shows at the same time that both theories coincide with his theory which is defined by a canonical flasque resolution. Godement also shows [1, Thm. 5.10.1, p. 228] that on a paracompact Hausdorff space, his theory coincides with Čech cohomology. This provides a bridge to the standard topological theories with constant coefficients, as developed in the book of Spanier [1]. He shows that on a paracompact Hausdorff space, Čech cohomology and Alexander cohomology and singular cohomology all agree (see Spanier [1, pp. 314, 327, 334]).

The vanishing theorem (2.7) was proved by Serre [3] for coherent sheaves on algebraic curves and projective algebraic varieties, and later [5] for abstract algebraic varieties. It is analogous to the theorem that singular cohomology on a (real) manifold of dimension n vanishes in degrees $i > n$.

EXERCISES

2.1. (a) Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k . Let P, Q be distinct closed points of X , and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathcal{Z}_U) \neq 0$.

***(b)** More generally, let $Y \subseteq X = \mathbf{A}_k^n$ be the union of $n + 1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathcal{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

2.2. Let $X = \mathbf{P}_k^1$ be the projective line over an algebraically closed field k . Show that the exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{K} \rightarrow \mathcal{K}/\mathcal{O} \rightarrow 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X, \cdot)$ by $H_Y^i(X, \cdot)$. They are the *cohomology groups of X with supports in Y* , and coefficients in a given sheaf.

(b) If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \rightarrow \Gamma_Y(X, \mathcal{F}') \rightarrow \Gamma_Y(X, \mathcal{F}) \rightarrow \Gamma_Y(X, \mathcal{F}'') \rightarrow 0$$

is exact.

(c) Show that if $\mathcal{F}$ is flasque, then $H_Y^i(X, \mathcal{F}) = 0$ for all $i > 0$.

(d) If $\mathcal{F}$ is flasque, show that the sequence

$$0 \rightarrow \Gamma_Y(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X - Y, \mathcal{F}) \rightarrow 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 \rightarrow H_Y^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(U, \mathcal{F}|_U) \rightarrow \\ \rightarrow H_Y^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(U, \mathcal{F}|_U) \rightarrow \\ \rightarrow H_Y^2(X, \mathcal{F}) \rightarrow \dots \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y . Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H_Y^i(X, \mathcal{F}) \cong H_Y^i(V, \mathcal{F}|_V).$$

2.4. *Mayer-Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X . Then there is a long exact sequence of cohomology with supports

$$\begin{aligned} \dots \rightarrow H_{Y_1 \cap Y_2}^i(X, \mathcal{F}) \rightarrow H_{Y_1}^i(X, \mathcal{F}) \oplus H_{Y_2}^i(X, \mathcal{F}) \rightarrow H_{Y_1 \cup Y_2}^i(X, \mathcal{F}) \rightarrow \\ \rightarrow H_{Y_1 \cap Y_2}^{i+1}(X, \mathcal{F}) \rightarrow \dots \end{aligned}$$

- 2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the *local space* of X at P , and give it the induced topology. Let $j: X_P \rightarrow X$ be the inclusion, and for any sheaf $\mathcal{F}$ on X , let $\mathcal{F}_P = j^*\mathcal{F}$. Show that for all i , $\mathcal{F}$, we have

$$H_P^i(X, \mathcal{F}) = H_P^i(X_P, \mathcal{F}_P).$$

- 2.6. Let X be a noetherian topological space, and let $\{\mathcal{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X . Then $\varinjlim \mathcal{I}_\alpha$ is also injective. [Hints: First show that a sheaf $\mathcal{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathcal{R} \subseteq \mathcal{Z}_U$, and for every map $f: \mathcal{R} \rightarrow \mathcal{I}$, there exists an extension of f to a map of $\mathcal{Z}_U \rightarrow \mathcal{I}$. Secondly, show that any such sheaf $\mathcal{R}$ is finitely generated, so any map $\mathcal{R} \rightarrow \varinjlim \mathcal{I}_\alpha$ factors through one of the $\mathcal{I}_\alpha$.]
- 2.7. Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.
 (a) Show that $H^1(S^1, \mathbf{Z}) \cong \mathbf{Z}$, using our definition of cohomology.
 (b) Now let $\mathcal{R}$ be the sheaf of germs of continuous real-valued functions on S^1 . Show that $H^1(S^1, \mathcal{R}) = 0$.

3 Cohomology of a Noetherian Affine Scheme

In this section we will prove that if $X = \text{Spec } A$ is a noetherian affine scheme, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves $\mathcal{F}$ of $\mathcal{O}_X$ -modules. The key point is to show that if I is an injective A -module, then the sheaf $\tilde{I}$ on $\text{Spec } A$ is flasque. We begin with some algebraic preliminaries.

Proposition 3.1A (Krull's Theorem). *Let A be a noetherian ring, let $M \subseteq N$ be finitely generated A -modules, and let $\mathfrak{a}$ be an ideal of A . Then the $\mathfrak{a}$ -adic topology on M is induced by the $\mathfrak{a}$ -adic topology on N . In particular, for any $n > 0$, there exists an $n' \geq n$ such that $\mathfrak{a}^{n'}M \cong M \cap \mathfrak{a}^{n'}N$.*

PROOF. Atiyah–Macdonald [1, 10.11] or Zariski–Samuel [1, vol. II, Ch. VIII, Th. 4].

Recall (II, Ex. 5.6) that for any ring A , and any ideal $\mathfrak{a} \subseteq A$, and any A -module M , we have defined the submodule $\Gamma_{\mathfrak{a}}(M)$ to be $\{m \in M \mid \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$.

Lemma 3.2. *Let A be a noetherian ring, let $\mathfrak{a}$ be an ideal of A , and let I be an injective A -module. Then the submodule $J = \Gamma_{\mathfrak{a}}(I)$ is also an injective A -module.*

PROOF. To show that J is injective, it will be sufficient to show that for any ideal $\mathfrak{b} \subseteq A$, and for any homomorphism $\varphi: \mathfrak{b} \rightarrow J$, there exists a homomorphism $\psi: A \rightarrow J$ extending φ . (This is a well-known criterion for an injective module—Godement [1, I, 1.4.1]). Since A is noetherian, $\mathfrak{b}$ is finitely generated. On the other hand, every element of J is annihilated by some

power of $\mathfrak{a}$, so there exists an $n > 0$ such that $\mathfrak{a}^n \varphi(b) = 0$, or equivalently, $\varphi(\mathfrak{a}^n b) = 0$. Now applying (3.1A) to the inclusion $b \subseteq A$, we find that there is an $n' \geq n$ such that $\mathfrak{a}^{n'} b \subseteq b \cap \mathfrak{a}^{n'}$. Hence $\varphi(b \cap \mathfrak{a}^{n'}) = 0$, and so the map $\varphi: b \rightarrow J$ factors through $b/(b \cap \mathfrak{a}^{n'})$. Now we consider the following diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & A/\mathfrak{a}^{n'} & & \\
 \uparrow & & \uparrow & \searrow \psi' & \\
 b & \xrightarrow{\quad} & b/(b \cap \mathfrak{a}^{n'}) & \xrightarrow{\quad} & J \xrightarrow{\quad} I \\
 & \searrow \varphi & & & \\
 & & & &
 \end{array}$$

Since I is injective, the composed map of $b/(b \cap \mathfrak{a}^{n'})$ to I extends to a map $\psi': A/\mathfrak{a}^{n'} \rightarrow I$. But the image of ψ' is annihilated by $\mathfrak{a}^{n'}$, so it is contained in J . Composing with the natural map $A \rightarrow A/\mathfrak{a}^{n'}$, we obtain the required map $\psi: A \rightarrow J$ extending φ .

Lemma 3.3. *Let I be an injective module over a noetherian ring A . Then for any $f \in A$, the natural map of I to its localization I_f is surjective.*

PROOF. For each $i > 0$, let b_i be the annihilator of f^i in A . Then $b_1 \subseteq b_2 \subseteq \dots$, and since A is noetherian, there is an r such that $b_r = b_{r+1} = \dots$. Now let $\theta: I \rightarrow I_f$ be the natural map, and let $x \in I_f$ be any element. Then by definition of localization, there is a $y \in I$ and an $n \geq 0$ such that $x = \theta(y)/f^n$. We define a map φ from the ideal (f^{n+r}) of A to I by sending f^{n+r} to $f^r y$. This is possible, because the annihilator of f^{n+r} is $b_{n+r} = b_r$, and b_r annihilates $f^r y$. Since I is injective, φ extends to a map $\psi: A \rightarrow I$. Let $\psi(1) = z$. Then $f^{n+r} z = f^r y$. But this implies that $\theta(z) = \theta(y)/f^n = x$. Hence θ is surjective.

Proposition 3.4. *Let I be an injective module over a noetherian ring A . Then the sheaf $\tilde{I}$ on $X = \text{Spec } A$ is flasque.*

PROOF. We will use noetherian induction on $Y = (\text{Supp } \tilde{I})^-$. See (II, Ex. 1.14) for the notion of support. If Y consists of a single closed point of X , then $\tilde{I}$ is a skyscraper sheaf (II, Ex. 1.17) which is obviously flasque.

In the general case, to show that $\tilde{I}$ is flasque, it will be sufficient to show, for any open set $U \subseteq X$, that $\Gamma(X, \tilde{I}) \rightarrow \Gamma(U, \tilde{I})$ is surjective. If $Y \cap U = \emptyset$, there is nothing to prove. If $Y \cap U \neq \emptyset$, we can find an $f \in A$ such that the open set $X_f = D(f)$ (II, §2) is contained in U and $X_f \cap Y \neq \emptyset$. Let $Z = X - X_f$, and consider the following diagram:

$$\begin{array}{ccccc}
 \Gamma(X, \tilde{I}) & \rightarrow & \Gamma(U, \tilde{I}) & \rightarrow & \Gamma(X_f, \tilde{I}) \\
 \uparrow & & \uparrow & & \\
 \Gamma_Z(X, \tilde{I}) & \rightarrow & \Gamma_Z(U, \tilde{I}), & &
 \end{array}$$

where Γ_Z denotes sections with support in Z (II, Ex. 1.20). Now given a section $s \in \Gamma(U, \tilde{I})$, we consider its image s' in $\Gamma(X_f, \tilde{I})$. But $\Gamma(X_f, \tilde{I}) = I_f$ (II, 5.1), so by (3.3), there is a $t \in I = \Gamma(X, \tilde{I})$ restricting to s' . Let t' be the restriction of t to $\Gamma(U, \tilde{I})$. Then $s - t'$ goes to 0 in $\Gamma(X_f, \tilde{I})$, so it has support in Z . Thus to complete the proof, it will be sufficient to show that $\Gamma_Z(X, \tilde{I}) \rightarrow \Gamma_Z(U, \tilde{I})$ is surjective.

Let $J = \Gamma_Z(X, \tilde{I})$. If α is the ideal generated by f , then $J = \Gamma_\alpha(I)$ (II, Ex. 5.6), so by (3.2), J is also an injective A -module. Furthermore, the support of $\tilde{J}$ is contained in $Y \cap Z$, which is strictly smaller than Y . Hence by our induction hypothesis, $\tilde{J}$ is flasque. Since $\Gamma(U, \tilde{J}) = \Gamma_Z(U, \tilde{I})$ (II, Ex. 5.6), we conclude that $\Gamma_Z(X, \tilde{I}) \rightarrow \Gamma_Z(U, \tilde{I})$ is surjective, as required.

Theorem 3.5. *Let $X = \text{Spec } A$ be the spectrum of a noetherian ring A . Then for all quasi-coherent sheaves $\mathcal{F}$ on X , and for all $i > 0$, we have $H^i(X, \mathcal{F}) = 0$.*

PROOF. Given $\mathcal{F}$, let $M = \Gamma(X, \mathcal{F})$, and take an injective resolution $0 \rightarrow M \rightarrow I'$ of M in the category of A -modules. Then we obtain an exact sequence of sheaves $0 \rightarrow \tilde{M} \rightarrow \tilde{I}'$ on X . Now $\mathcal{F} = \tilde{M}$ (II, 5.5) and each $\tilde{I}'^i$ is flasque by (3.4), so we can use this resolution of $\mathcal{F}$ to calculate cohomology (2.5.1). Applying the functor Γ , we recover the exact sequence of A -modules $0 \rightarrow M \rightarrow I'$. Hence $H^0(X, \mathcal{F}) = M$, and $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Remark 3.5.1. This result is also true without the noetherian hypothesis, but the proof is more difficult [EGA III, 1.3.1].

Corollary 3.6. *Let X be a noetherian scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X . Then $\mathcal{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathcal{G}$.*

PROOF. Cover X with a finite number of open affines $U_i = \text{Spec } A_i$, and let $\mathcal{F}|_{U_i} = \tilde{M}_i$ for each i . Embed M_i in an injective A_i -module I_i . For each i , let $f_i: U_i \rightarrow X$ be the inclusion, and let $\mathcal{G} = \bigoplus f_{i*}(\tilde{I}_i)$. For each i we have an injective map of sheaves $\mathcal{F}|_{U_i} \rightarrow \tilde{I}_i$. Hence we obtain a map $\mathcal{F} \rightarrow \mathcal{G}$. Taking the direct sum over i gives a map $\mathcal{F} \rightarrow \mathcal{G}$ which is clearly injective. On the other hand, for each i , $\tilde{I}_i$ is flasque (3.4) and quasi-coherent on U_i . Hence $f_{i*}(\tilde{I}_i)$ is also flasque (II, Ex. 1.16d) and quasi-coherent (II, 5.8). Taking the direct sum of these, we see that $\mathcal{G}$ is flasque and quasi-coherent.

Theorem 3.7 (Serre [5]). *Let X be a noetherian scheme. Then the following conditions are equivalent:*

- (i) X is affine;
- (ii) $H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and all $i > 0$;
- (iii) $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

PROOF. (i) $\Rightarrow$ (ii) is (3.5). (ii) $\Rightarrow$ (iii) is trivial, so we have only to prove (iii) $\Rightarrow$ (i). We use the criterion of (II, Ex. 2.17). First we show that X can

be covered by open affine subsets of the form X_f , with $f \in A = \Gamma(X, \mathcal{O}_X)$. Let P be a closed point of X , let U be an open affine neighborhood of P , and let $Y = X - U$. Then we have an exact sequence

$$0 \rightarrow \mathcal{I}_{Y \cup \{P\}} \rightarrow \mathcal{I}_Y \rightarrow k(P) \rightarrow 0,$$

where $\mathcal{I}_Y$ and $\mathcal{I}_{Y \cup \{P\}}$ are the ideal sheaves of the closed sets Y and $Y \cup \{P\}$, respectively. The quotient is the skyscraper sheaf $k(P) = \mathcal{O}_P/\mathfrak{m}_P$ at P . Now from the exact sequence of cohomology, and hypothesis (iii), we get an exact sequence

$$\Gamma(X, \mathcal{I}_Y) \rightarrow \Gamma(X, k(P)) \rightarrow H^1(X, \mathcal{I}_{Y \cup \{P\}}) = 0.$$

So there is an element $f \in \Gamma(X, \mathcal{I}_Y)$ which goes to 1 in $k(P)$, i.e., $f_P \equiv 1 \pmod{\mathfrak{m}_P}$. Since $\mathcal{I}_Y \subseteq \mathcal{O}_X$, we can consider f as an element of A . Then by construction, we have $P \in X_f \subseteq U$. Furthermore, $X_f = U_{\bar{f}}$, where $\bar{f}$ is the image of f in $\Gamma(U, \mathcal{O}_U)$, so X_f is affine.

Thus every closed point of X has an open affine neighborhood of the form X_f . By quasi-compactness, we can cover X with a finite number of these, corresponding to $f_1, \dots, f_r \in A$.

Now by (II, Ex. 2.17), to show that X is affine, we need only verify that $f_1, \dots, f_r$ generate the unit ideal in A . We use $f_1, \dots, f_r$ to define a map $\alpha: \mathcal{O}_X^r \rightarrow \mathcal{O}_X$ by sending $\langle a_1, \dots, a_r \rangle$ to $\sum f_i a_i$. Since the X_{f_i} cover X , this is a surjective map of sheaves. Let $\mathcal{F}$ be the kernel:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X^r \xrightarrow{\alpha} \mathcal{O}_X \rightarrow 0.$$

We filter $\mathcal{F}$ as follows:

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}_X^r \supseteq \mathcal{F} \cap \mathcal{O}_X^{r-1} \supseteq \dots \supseteq \mathcal{F} \cap \mathcal{O}_X$$

for a suitable ordering of the factors of $\mathcal{O}_X^r$. Each of the quotients of this filtration is a coherent sheaf of ideals in $\mathcal{O}_X$. Thus using our hypothesis (iii) and the long exact sequence of cohomology, we climb up the filtration and deduce that $H^1(X, \mathcal{F}) = 0$. But then $\Gamma(X, \mathcal{O}_X^r) \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X)$ is surjective, which tells us that $f_1, \dots, f_r$ generate the unit ideal in A . q.e.d.

Remark 3.7.1. This result is analogous to another theorem of Serre in complex analytic geometry, which characterizes Stein spaces by the vanishing of coherent analytic sheaf cohomology.

EXERCISES

3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine. [Hint: Use (3.7), and for any coherent sheaf $\mathcal{F}$ on X , consider the filtration $\mathcal{F} \supseteq \mathcal{N} \cdot \mathcal{F} \supseteq \mathcal{N}^2 \cdot \mathcal{F} \supseteq \dots$, where $\mathcal{N}$ is the sheaf of nilpotent elements on X .]

3.2. Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A .

- (a) Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A -modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H_{\mathfrak{a}}^i(\cdot)$.
- (b) Now let $X = \text{Spec } A$, $Y = V(\mathfrak{a})$. Show that for any A -module M ,

$$H_{\mathfrak{a}}^i(M) = H_Y^i(X, \tilde{M}),$$

where $H_Y^i(X, \cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

- (c) For any i , show that $\Gamma_{\mathfrak{a}}(H_{\mathfrak{a}}^i(M)) = H_{\mathfrak{a}}^i(M)$.

3.4. *Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A -module, then $\text{depth}_{\mathfrak{a}} M$ is the maximum length of an M -regular sequence $x_1, \dots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II, §8).

- (a) Assume that A is noetherian. Show that if $\text{depth}_{\mathfrak{a}} M \geq 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated. [*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M .]
- (b) Show inductively, for M finitely generated, that for any $n \geq 0$, the following conditions are equivalent:

- (i) $\text{depth}_{\mathfrak{a}} M \geq n$;
- (ii) $H_{\mathfrak{a}}^i(M) = 0$ for all $i < n$.

For more details, and related results, see Grothendieck [7].

3.5. Let X be a noetherian scheme, and let P be a closed point of X . Show that the following conditions are equivalent:

- (i) $\text{depth } \mathcal{O}_P \geq 2$;
- (ii) if U is any open neighborhood of P , then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U .

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

3.6. Let X be a noetherian scheme.

- (a) Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X . Thus $\mathfrak{Qco}(X)$ has enough injectives.

*** (b)** Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will *not* work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathcal{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathcal{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines . . .]

- (c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

3.7. Let A be a noetherian ring, let $X = \text{Spec } A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

- (a) For any A -module M , establish the following formula of Deligne:

$$\Gamma(U, \tilde{M}) \cong \varinjlim_n \text{Hom}_A(\mathfrak{a}^n, M).$$

- (b) Apply this in the case of an injective A -module I , to give another proof of (3.4).

3.8. Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0, x_1, x_2, \dots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \dots$. Let I be an injective A -module containing A . Show that $I \rightarrow I_{x_0}$ is not surjective.

4 Čech Cohomology

In this section we construct the Čech cohomology groups for a sheaf of abelian groups on a topological space X , with respect to a given open covering of X . We will prove that if X is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups defined in §2. The value of this result is that it gives a practical method for computing cohomology of quasi-coherent sheaves on a scheme.

Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of X . Fix, once and for all, a well-ordering of the index set I . For any finite set of indices $i_0, \dots, i_p \in I$ we denote the intersection $U_{i_0} \cap \dots \cap U_{i_p}$ by $U_{i_0, \dots, i_p}$.

Now let $\mathcal{F}$ be a sheaf of abelian groups on X . We define a complex $C(\mathcal{U}, \mathcal{F})$ of abelian groups as follows. For each $p \geq 0$, let

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0, \dots, i_p}).$$

Thus an element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is determined by giving an element

$$\alpha_{i_0, \dots, i_p} \in \mathcal{F}(U_{i_0, \dots, i_p}),$$

for each $(p+1)$ -tuple $i_0 < \dots < i_p$ of elements of I . We define the co-boundary map $d: C^p \rightarrow C^{p+1}$ by setting

$$(d\alpha)_{i_0, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}}|_{U_{i_0, \dots, i_{p+1}}}.$$

Here the notation $\hat{i}_k$ means omit i_k . Then since $\alpha_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}}$ is an element of $\mathcal{F}(U_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}})$, we restrict to $U_{i_0, \dots, i_{p+1}}$ to get an element of $\mathcal{F}(U_{i_0, \dots, i_{p+1}})$. One checks easily that $d^2 = 0$, so we have indeed defined a complex of abelian groups.

Remark 4.0.1. If $\alpha \in C^p(\mathcal{U}, \mathcal{F})$, it is sometimes convenient to have the symbol $\alpha_{i_0, \dots, i_p}$ defined for *all* $(p+1)$ -tuples of elements of I . If there is a repeated index in the set $\{i_0, \dots, i_p\}$, we define $\alpha_{i_0, \dots, i_p} = 0$. If the indices are all distinct, we define $\alpha_{i_0, \dots, i_p} = (-1)^\sigma \alpha_{\sigma i_0, \dots, \sigma i_p}$, where σ is the permutation for which $\sigma i_0 < \dots < \sigma i_p$. With these conventions, one can check that the formula given above for $d\alpha$ remains correct for any $(p+2)$ -tuple $i_0, \dots, i_{p+1}$ of elements of I .

Definition. Let X be a topological space and let $\mathfrak{U}$ be an open covering of X . For any sheaf of abelian groups $\mathcal{F}$ on X , we define the p th Čech cohomology group of $\mathcal{F}$, with respect to the covering $\mathfrak{U}$, to be

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = h^p(C^*(\mathfrak{U}, \mathcal{F})).$$

Caution 4.0.2. Keeping X and $\mathfrak{U}$ fixed, if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence of sheaves of abelian groups on X , we do *not* in general get a long exact sequence of Čech cohomology groups. In other words, the functors $\check{H}^p(\mathfrak{U}, \cdot)$ do not form a δ -functor (§1). For example, if $\mathfrak{U}$ consists of the single open set X , then this results from the fact that the global section functor $\Gamma(X, \cdot)$ is not exact.

Example 4.0.3. To illustrate how well suited Čech cohomology is for computations, we will compute some examples. Let $X = \mathbf{P}_k^1$, let $\mathcal{F}$ be the sheaf of differentials Ω (II, §8), and let $\mathfrak{U}$ be the open covering by the two open sets $U = \mathbf{A}^1$ with affine coordinate x , and $V = \mathbf{A}^1$ with affine coordinate $y = 1/x$. Then the Čech complex has only two terms:

$$C^0 = \Gamma(U, \Omega) \times \Gamma(V, \Omega)$$

$$C^1 = \Gamma(U \cap V, \Omega).$$

Now

$$\Gamma(U, \Omega) = k[x] dx$$

$$\Gamma(V, \Omega) = k[y] dy$$

$$\Gamma(U \cap V, \Omega) = k \left[x, \frac{1}{x} \right] dx,$$

and the map $d: C^0 \rightarrow C^1$ is given by

$$x \mapsto x$$

$$y \mapsto \frac{1}{x}$$

$$dy \mapsto -\frac{1}{x^2} dx.$$

So $\ker d$ is the set of pairs $\langle f(x) dx, g(y) dy \rangle$ such that

$$f(x) = -\frac{1}{x^2} g\left(\frac{1}{x}\right).$$

This can happen only if $f = g = 0$, since one side is a polynomial in x and the other side is a polynomial in $1/x$ with no constant term. So $\check{H}^0(\mathfrak{U}, \Omega) = 0$.

To compute H^1 , note that the image of d is the set of all expressions

$$\left(f(x) + \frac{1}{x^2} g\left(\frac{1}{x}\right) \right) dx,$$

where f and g are polynomials. This gives the subvector space of $k[x, 1/x] dx$ generated by all $x^n dx$, $n \in \mathbf{Z}$, $n \neq -1$. Therefore $\check{H}^1(\mathfrak{U}, \Omega) \cong k$, generated by the image of $x^{-1} dx$.

Example 4.0.4. Let S^1 be the circle (in its usual topology), let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$, and let $\mathfrak{U}$ be the open covering by two connected open semi-circles U, V , which overlap at each end, so that $U \cap V$ consists of two small intervals. Then

$$\begin{aligned} C^0 &= \Gamma(U, \mathbf{Z}) \times \Gamma(V, \mathbf{Z}) = \mathbf{Z} \times \mathbf{Z} \\ C^1 &= \Gamma(U \cap V, \mathbf{Z}) = \mathbf{Z} \times \mathbf{Z} \end{aligned}$$

and the map $d: C^0 \rightarrow C^1$ takes $\langle a, b \rangle$ to $\langle b - a, b - a \rangle$. Thus $\check{H}^0(\mathfrak{U}, \mathbf{Z}) = \mathbf{Z}$ and $\check{H}^1(\mathfrak{U}, \mathbf{Z}) = \mathbf{Z}$. Since we know this is the right answer (Ex. 2.7), this illustrates the general principle that Čech cohomology agrees with the usual cohomology provided the open covering is taken fine enough so that there is no cohomology on any of the open sets (Ex. 4.11).

Now we will study some properties of the Čech cohomology groups.

Lemma 4.1. *For any $X, \mathfrak{U}, \mathcal{F}$ as above, we have $\check{H}^0(\mathfrak{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.*

PROOF. $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \ker(d: C^0(\mathfrak{U}, \mathcal{F}) \rightarrow C^1(\mathfrak{U}, \mathcal{F}))$. If $\alpha \in C^0$ is given by $\{\alpha_i \in \mathcal{F}(U_i)\}$, then for each $i < j$, $(d\alpha)_{ij} = \alpha_j - \alpha_i$. So $d\alpha = 0$ says the sections α_i and α_j agree on $U_i \cap U_j$. Thus it follows from the sheaf axioms that $\ker d = \Gamma(X, \mathcal{F})$.

Next we define a “sheafified” version of the Čech complex. For any open set $V \subseteq X$, let $f: V \rightarrow X$ denote the inclusion map. Now given $X, \mathfrak{U}, \mathcal{F}$ as above, we construct a complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ of sheaves on X as follows. For each $p \geq 0$, let

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} f_*(\mathcal{F}|_{U_{i_0, \dots, i_p}}),$$

and define

$$d: \mathcal{C}^p \rightarrow \mathcal{C}^{p+1}$$

by the same formula as above. Note by construction that for each p we have $\Gamma(X, \mathcal{C}^p(\mathfrak{U}, \mathcal{F})) = C^p(\mathfrak{U}, \mathcal{F})$.

Lemma 4.2. *For any sheaf of abelian groups $\mathcal{F}$ on X , the complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, i.e., there is a natural map $\varepsilon: \mathcal{F} \rightarrow \mathcal{C}^0$ such that the sequence of sheaves*

$$0 \rightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \rightarrow \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \rightarrow \dots$$

is exact.

PROOF. We define $\varepsilon: \mathcal{F} \rightarrow \mathcal{C}^0$ by taking the product of the natural maps $\mathcal{F} \rightarrow f_*(\mathcal{F}|_{U_i})$ for $i \in I$. Then the exactness at the first step follows from the sheaf axioms for $\mathcal{F}$.

To show the exactness of the complex $\mathcal{C}'$ for $p \geq 1$, it is enough to check exactness on the stalks. So let $x \in X$, and suppose $x \in U_j$. For each $p \geq 1$, we define a map

$$k: \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \rightarrow \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

as follows. Given $\alpha_x \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$, it is represented by a section $\alpha \in \Gamma(V, \mathcal{C}^p(\mathcal{U}, \mathcal{F}))$ over a neighborhood V of x , which we may choose so small that $V \subseteq U_j$. Now for any p -tuple $i_0 < \dots < i_{p-1}$, we set

$$(k\alpha)_{i_0, \dots, i_{p-1}} = \alpha_{j, i_0, \dots, i_{p-1}},$$

using the notational convention of (4.0.1). This makes sense because $V \cap U_{i_0, \dots, i_{p-1}} = V \cap U_{j, i_0, \dots, i_{p-1}}$. Then take the stalk of $k\alpha$ at x to get the required map k . Now one checks that for any $p \geq 1$, $\alpha \in \mathcal{C}_x^p$,

$$(dk + kd)(\alpha) = \alpha.$$

Thus k is a homotopy operator for the complex $\mathcal{C}'_x$, showing that the identity map is homotopic to the zero map. It follows (§1) that the cohomology groups $h^p(\mathcal{C}'_x)$ of this complex are 0 for $p \geq 1$.

Proposition 4.3. *Let X be a topological space, let $\mathcal{U}$ be an open covering, and let $\mathcal{F}$ be a flasque sheaf of abelian groups on X . Then for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$.*

PROOF. Consider the resolution $0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}'(\mathcal{U}, \mathcal{F})$ given by (4.2). Since $\mathcal{F}$ is flasque, the sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are flasque for each $p \geq 0$. Indeed, for any $i_0, \dots, i_p$, $\mathcal{F}|_{U_{i_0, \dots, i_p}}$ is a flasque sheaf on $U_{i_0, \dots, i_p}$; f_* preserves flasque sheaves (II, Ex. 1.16d), and a product of flasque sheaves is flasque. So by (2.5.1) we can use this resolution to compute the usual cohomology groups of $\mathcal{F}$. But $\mathcal{F}$ is flasque, so $H^p(X, \mathcal{F}) = 0$ for $p > 0$ by (2.5). On the other hand, the answer given by this resolution is

$$h^p(\Gamma(X, \mathcal{C}'(\mathcal{U}, \mathcal{F}))) = \check{H}^p(\mathcal{U}, \mathcal{F}).$$

So we conclude that $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.

Lemma 4.4. *Let X be a topological space, and $\mathcal{U}$ an open covering. Then for each $p \geq 0$ there is a natural map, functorial in $\mathcal{F}$,*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}).$$

PROOF. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}'$ be an injective resolution of $\mathcal{F}$ in $\mathcal{U}\mathbf{b}(X)$. Comparing with the resolution $0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}'(\mathcal{U}, \mathcal{F})$ of (4.2), it follows from a general result on complexes (Hilton and Stammbach [1, IV, 4.4]) that there is a morphism of complexes $\mathcal{C}'(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{I}'$, inducing the identity map on $\mathcal{F}$, and unique up to homotopy. Applying the functors $\Gamma(X, \cdot)$ and h^p , we get the required map.

Theorem 4.5. *Let X be a noetherian separated scheme, let $\mathcal{U}$ be an open affine cover of X , and let $\mathcal{F}$ be a quasi-coherent sheaf on X . Then for all $p \geq 0$, the natural maps of (4.4) give isomorphisms*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}).$$

PROOF. For $p = 0$ we have an isomorphism by (4.1). For the general case, embed $\mathcal{F}$ in a flasque, quasi-coherent sheaf $\mathcal{G}$ (3.6), and let $\mathcal{R}$ be the quotient:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{R} \rightarrow 0.$$

For each $i_0 < \dots < i_p$, the open set $U_{i_0, \dots, i_p}$ is affine, since it is an intersection of affine open subsets of a separated scheme (II, Ex. 4.3). Since $\mathcal{F}$ is quasi-coherent, we therefore have an exact sequence

$$0 \rightarrow \mathcal{F}(U_{i_0, \dots, i_p}) \rightarrow \mathcal{G}(U_{i_0, \dots, i_p}) \rightarrow \mathcal{R}(U_{i_0, \dots, i_p}) \rightarrow 0$$

of abelian groups, by (3.5) or (II, 5.6). Taking products, we find that the corresponding sequence of Čech complexes

$$0 \rightarrow C^*(\mathcal{U}, \mathcal{F}) \rightarrow C^*(\mathcal{U}, \mathcal{G}) \rightarrow C^*(\mathcal{U}, \mathcal{R}) \rightarrow 0$$

is exact. Therefore we get a long exact sequence of Čech cohomology groups. Since $\mathcal{G}$ is flasque, its Čech cohomology vanishes for $p > 0$ by (4.3), so we have an exact sequence

$$0 \rightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^0(\mathcal{U}, \mathcal{G}) \rightarrow \check{H}^0(\mathcal{U}, \mathcal{R}) \rightarrow \check{H}^1(\mathcal{U}, \mathcal{F}) \rightarrow 0$$

and isomorphisms

$$\check{H}^p(\mathcal{U}, \mathcal{R}) \xrightarrow{\sim} \check{H}^{p+1}(\mathcal{U}, \mathcal{F})$$

for each $p \geq 1$. Now comparing with the long exact sequence of usual cohomology for the above short exact sequence, using the case $p = 0$, and (2.5), we conclude that the natural map

$$\check{H}^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(X, \mathcal{F})$$

is an isomorphism. But $\mathcal{R}$ is also quasi-coherent (II, 5.7), so we obtain the result for all p by induction.

EXERCISES

- 4.1.** Let $f: X \rightarrow Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathcal{F}$ on X , there are natural isomorphisms for all $i \geq 0$,

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_* \mathcal{F}).$$

[Hint: Use (II, 5.8).]

- 4.2.** Prove Chevalley's theorem: Let $f: X \rightarrow Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

- (a) Let $f: X \rightarrow Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathcal{M}$ on X , and a morphism of sheaves $\alpha: \mathcal{O}_Y^r \rightarrow f_* \mathcal{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y .

- (b) For any coherent sheaf $\mathcal{F}$ on Y , show that there is a coherent sheaf $\mathcal{G}$ on X , and a morphism $\beta: f_*\mathcal{G} \rightarrow \mathcal{F}$ which is an isomorphism at the generic point of Y . [Hint: Apply $\mathcal{H}om(\cdot, \mathcal{F})$ to α and use (II, Ex. 5.17e).]
- (c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider $\ker \beta$ and $\operatorname{coker} \beta$, and use noetherian induction on Y .

4.3. Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k -vector space spanned by $\{x^i y^j | i, j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—cf. (I, Ex. 3.6).)

4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathcal{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1 , there is an isomorphism if one takes the limit over all coverings.

- (a) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of the topological space X . A refinement of $\mathcal{U}$ is a covering $\mathcal{V} = (V_j)_{j \in J}$, together with a map $\lambda: J \rightarrow I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathcal{F}$, and for each i ,

$$\lambda^i: \check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^i(\mathcal{V}, \mathcal{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F}).$$

- (b) For any abelian sheaf $\mathcal{F}$ on X , show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$$

are compatible with the refinement maps above.

- (c) Now prove the following theorem. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(X, \mathcal{F})$$

is an isomorphism. [Hint: Embed $\mathcal{F}$ in a flasque sheaf $\mathcal{G}$, and let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so that we have an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{R} \rightarrow 0.$$

Define a complex $D(\mathcal{U})$ by

$$0 \rightarrow C^0(\mathcal{U}, \mathcal{F}) \rightarrow C^0(\mathcal{U}, \mathcal{G}) \rightarrow D(\mathcal{U}) \rightarrow 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D(\mathcal{U}) \rightarrow C^0(\mathcal{U}, \mathcal{R}),$$

and see what happens under refinement.]

- 4.5. For any ringed space $(X, \mathcal{O}_X)$, let $\text{Pic } X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\text{Pic } X \cong H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation. [Hint: For any invertible sheaf $\mathcal{L}$ on X , cover X by open sets U_i on which $\mathcal{L}$ is free, and fix isomorphisms $\varphi_i: \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$. Now use (Ex. 4.4).]
- 4.6. Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{I}$ be a sheaf of ideals with $\mathcal{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathcal{I})$. Show that there is an exact sequence of sheaves of abelian groups on X ,

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{O}_{X_0}^* \rightarrow 0,$$

where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$); the map $\mathcal{I} \rightarrow \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathcal{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\dots \rightarrow H^1(X, \mathcal{I}) \rightarrow \text{Pic } X \rightarrow \text{Pic } X_0 \rightarrow H^2(X, \mathcal{I}) \rightarrow \dots$$

- 4.7. Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d . (Do not assume f is irreducible.) Assume that $(1, 0, 0)$ is not on X . Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \rightarrow \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1,$$

$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

- 4.8. *Cohomological Dimension* (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X , denoted $\text{cd}(X)$, to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\text{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\text{cd}(X) \leq \dim X$.
- (a) In the definition of $\text{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X . Use (II, Ex. 5.15) and (2.9).
- (b) If X is quasi-projective over a field k , then it is even sufficient to consider only locally free coherent sheaves on X . Use (II, 5.18).
- (c) Suppose X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\text{cd}(X) \leq r$.
- * (d) If X is a quasi-projective scheme of dimension r over a field k , then X can be covered by $r + 1$ open affine subsets. Conclude (independently of (2.7)) that $\text{cd}(X) \leq \dim X$.
- (e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\text{cd}(X - Y) \leq r - 1$.
- 4.9. Let $X = \text{Spec } k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k . Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X . Therefore the projective closure

$\bar{Y}$ in $\mathbf{P}_k^4$ is also not a set-theoretic complete intersection. [Hints: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P, \mathcal{O}_X) \neq 0$.]

***4.10.** Let X be a nonsingular variety over an algebraically closed field k , and let $\mathcal{F}$ be a coherent sheaf on X . Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X (II, §8). [Hint: Use (II, Ex. 8.6) and (4.5).]

4.11. This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups, and $\mathcal{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \dots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathcal{F}|_V) = 0$. Then prove that for all $p \geq 0$, the natural maps

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

5 The Cohomology of Projective Space

In this section we make explicit calculations of the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space, by using Čech cohomology for a suitable open affine covering. These explicit calculations form the basis for various general results about cohomology of coherent sheaves on projective varieties.

Let A be a noetherian ring, let $S = A[x_0, \dots, x_r]$, and let $X = \text{Proj } S$ be the projective space $\mathbf{P}_A^r$ over A . Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre (II, §5). For any sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$, we denote by $\Gamma_*(\mathcal{F})$ the graded S -module $\bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$ (see II, §5).

Theorem 5.1. *Let A be a noetherian ring, and let $X = \mathbf{P}_A^r$, with $r \geq 1$. Then:*

- (a) *the natural map $S \rightarrow \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S -modules;*
- (b) *$H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbf{Z}$;*
- (c) *$H^r(X, \mathcal{O}_X(-r-1)) \cong A$;*
- (d) *The natural map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \rightarrow H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A -modules, for each $n \in \mathbf{Z}$.

PROOF. Let $\mathcal{F}$ be the quasi-coherent sheaf $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)$. Since cohomology commutes with arbitrary direct sums on a noetherian topological space (2.9.1), the cohomology of $\mathcal{F}$ will be the direct sum of the cohomology of the sheaves $\mathcal{O}(n)$. So we will compute the cohomology of $\mathcal{F}$, and keep track

of the grading by n , so that we can sort out the pieces at the end. Note that all the cohomology groups in question have a natural structure of A -module (2.6.1).

For each $i = 0, \dots, r$, let U_i be the open set $D_+(x_i)$. Then each U_i is an open affine subset of X , and the U_i cover X , so we can compute the cohomology of $\mathcal{F}$ by using Čech cohomology for the covering $\mathfrak{U} = (U_i)$, by (4.5). For any set of indices $i_0, \dots, i_p$, the open set $U_{i_0, \dots, i_p}$ is just $D_+(x_{i_0} \cdots x_{i_p})$, so by (II, 5.11) we have

$$\mathcal{F}(U_{i_0, \dots, i_p}) \cong S_{x_{i_0} \cdots x_{i_p}},$$

the localization of S with respect to the element $x_{i_0} \cdots x_{i_p}$. Furthermore, the grading on $\mathcal{F}$ corresponds to the natural grading of $S_{x_{i_0} \cdots x_{i_p}}$ under this isomorphism. Thus the Čech complex of $\mathcal{F}$ is given by

$$C^*(\mathfrak{U}, \mathcal{F}): \prod S_{x_{i_0}} \rightarrow \prod S_{x_{i_0} x_{i_1}} \rightarrow \dots \rightarrow S_{x_0 \cdots x_r},$$

and the modules all have a natural grading compatible with the grading on $\mathcal{F}$.

Now $H^0(X, \mathcal{F})$ is the kernel of the first map, which is just S , as we have seen earlier (II, 5.13). This proves (a).

Next we consider $H^r(X, \mathcal{F})$. It is the cokernel of the last map in the Čech complex, which is

$$d^{r-1}: \prod_k S_{x_0 \cdots \hat{x}_k \cdots x_r} \rightarrow S_{x_0 \cdots x_r}.$$

We think of $S_{x_0 \cdots x_r}$ as a free A -module with basis $x_0^{l_0} \cdots x_r^{l_r}$, with $l_i \in \mathbb{Z}$. The image of d^{r-1} is the free submodule generated by those basis elements for which at least one $l_i \geq 0$. Thus $H^r(X, \mathcal{F})$ is a free A -module with basis consisting of the “negative” monomials

$$\{x_0^{l_0} \cdots x_r^{l_r} | l_i < 0 \text{ for each } i\}.$$

Furthermore the grading is given by $\sum l_i$. There is only one such monomial of degree $-r-1$, namely $x_0^{-1} \cdots x_r^{-1}$, so we see that $H^r(X, \mathcal{O}_X(-r-1))$ is a free A -module of rank 1. This proves (c).

To prove (d), first note that if $n < 0$, then $H^0(X, \mathcal{O}_X(n)) = 0$ by (a), and $H^r(X, \mathcal{O}_X(-n-r-1)) = 0$ by what we have just seen, since in that case $-n-r-1 > -r-1$, and there are no negative monomials of that degree. So the statement is trivial for $n < 0$. For $n \geq 0$, $H^0(X, \mathcal{O}_X(n))$ has a basis consisting of the usual monomials of degree n , i.e., $\{x_0^{m_0} \cdots x_r^{m_r} | m_i \geq 0 \text{ and } \sum m_i = n\}$. The natural pairing with $H^r(X, \mathcal{O}_X(-n-r-1))$ into $H^r(X, \mathcal{O}_X(-r-1))$ is determined by

$$(x_0^{m_0} \cdots x_r^{m_r}) \cdot (x_0^{l_0} \cdots x_r^{l_r}) = x_0^{m_0+l_0} \cdots x_r^{m_r+l_r},$$

where $\sum l_i = -n-r-1$, and the object on the right becomes 0 if any $m_i + l_i \geq 0$. So it is clear that we have a perfect pairing, under which $x_0^{-m_0-1} \cdots x_r^{-m_r-1}$ is the dual basis element of $x_0^{m_0} \cdots x_r^{m_r}$.

It remains to prove statement (b), which we will do by induction on r . If $r = 1$ there is nothing to prove, so let $r > 1$. If we localize the complex $C(\mathcal{U}, \mathcal{F})$ with respect to x_r , as graded S -modules, we get the Čech complex for the sheaf $\mathcal{F}|_{U_r}$ on the space U_r , with respect to the open affine covering $\{U_i \cap U_r \mid i = 0, \dots, r\}$. By (4.5), this complex gives the cohomology of $\mathcal{F}|_{U_r}$ on U_r , which is 0 for $i > 0$ by (3.5). Since localization is an exact functor, we conclude that $H^i(X, \mathcal{F})_{x_r} = 0$ for $i > 0$. In other words, every element of $H^i(X, \mathcal{F})$, for $i > 0$, is annihilated by some power of x_r .

To complete the proof of (b), we will show that for $0 < i < r$, multiplication by x_r induces a bijective map of $H^i(X, \mathcal{F})$ into itself. Then it will follow that this module is 0.

Consider the exact sequence of graded S -modules

$$0 \rightarrow S(-1) \xrightarrow{x_r} S \rightarrow S/(x_r) \rightarrow 0.$$

This gives the exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0$$

on X , where H is the hyperplane $x_r = 0$. Twisting by all $n \in \mathbb{Z}$ and taking the direct sum, we have

$$0 \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_H \rightarrow 0,$$

where $\mathcal{F}_H = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_H(n)$. Taking cohomology, we get a long exact sequence

$$\dots \rightarrow H^i(X, \mathcal{F}(-1)) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(X, \mathcal{F}_H) \rightarrow \dots$$

Considered as graded S -modules, $H^i(X, \mathcal{F}(-1))$ is just $H^i(X, \mathcal{F})$ shifted one place, and the map $H^i(X, \mathcal{F}(-1)) \rightarrow H^i(X, \mathcal{F})$ of the exact sequence is multiplication by x_r .

Now H is isomorphic to $\mathbf{P}_A^{r-1}$, and $H^i(X, \mathcal{F}_H) = H^i(H, \bigoplus \mathcal{O}_H(n))$ by (2.10). So we can apply our induction hypothesis to $\mathcal{F}_H$, and find that $H^i(X, \mathcal{F}_H) = 0$ for $0 < i < r - 1$. Furthermore, for $i = 0$ we have an exact sequence

$$0 \rightarrow H^0(X, \mathcal{F}(-1)) \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{F}_H) \rightarrow 0,$$

by (a), since $H^0(X, \mathcal{F}_H)$ is just $S/(x_r)$. At the other end of the exact sequence we have

$$0 \rightarrow H^{r-1}(X, \mathcal{F}_H) \xrightarrow{\delta} H^r(X, \mathcal{F}(-1)) \xrightarrow{x_r} H^r(X, \mathcal{F}) \rightarrow 0.$$

Indeed, we have described $H^r(X, \mathcal{F})$ above as the free A -module with basis formed by the negative monomials in $x_0, \dots, x_r$. So it is clear that x_r is surjective. On the other hand, the kernel of x_r is the free submodule generated by those negative monomials $x_0^{l_0} \dots x_r^{l_r}$ with $l_r = -1$. Since $H^{r-1}(X, \mathcal{F}_H)$ is the free A -module with basis consisting of the negative monomials in $x_0, \dots, x_{r-1}$, and δ is division by x_r , the sequence is exact. In particular, δ is injective.

Putting these results all together, the long exact sequence of cohomology shows that the map multiplication by $x_r: H^i(X, \mathcal{F}(-1)) \rightarrow H^i(X, \mathcal{F})$ is bijective for $0 < i < r$, as required. q.e.d.

Theorem 5.2 (Serre [3]). *Let X be a projective scheme over a noetherian ring A , and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over $\text{Spec } A$. Let $\mathcal{F}$ be a coherent sheaf on X . Then:*

- (a) *for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A -module;*
- (b) *there is an integer n_0 , depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.*

PROOF. Since $\mathcal{O}_X(1)$ is a very ample sheaf on X over $\text{Spec } A$, there is a closed immersion $i: X \rightarrow \mathbf{P}_A^r$ of schemes over A , for some r , such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbf{P}^r}(1)$ —cf. (II, 5.16.1). If $\mathcal{F}$ is coherent on X , then $i_*\mathcal{F}$ is coherent on $\mathbf{P}_A^r$ (II, Ex. 5.5), and the cohomology is the same (2.10). Thus we reduce to the case $X = \mathbf{P}_A^r$.

For $X = \mathbf{P}_A^r$, we observe that (a) and (b) are true for any sheaf of the form $\mathcal{O}_X(q)$, $q \in \mathbb{Z}$. This follows immediately from the explicit calculations (5.1). Hence the same is true for any finite direct sum of such sheaves.

To prove the theorem for arbitrary coherent sheaves, we use descending induction on i . For $i > r$, we have $H^i(X, \mathcal{F}) = 0$, since X can be covered by $r + 1$ open affines (Ex. 4.8), so the result is trivial in this case.

In general, given a coherent sheaf $\mathcal{F}$ on X , we can write $\mathcal{F}$ as a quotient of a sheaf $\mathcal{E}$, which is a finite direct sum of sheaves $\mathcal{O}(q_i)$, for various integers q_i (II, 5.18). Let $\mathcal{R}$ be the kernel,

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0.$$

Then $\mathcal{R}$ is also coherent. We get an exact sequence of A -modules

$$\dots \rightarrow H^i(X, \mathcal{E}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^{i+1}(X, \mathcal{R}) \rightarrow \dots$$

Now the module on the left is finitely generated because $\mathcal{E}$ is a sum of $\mathcal{O}(q_i)$, as remarked above. The module on the right is finitely generated by the induction hypothesis. Since A is a noetherian ring, we conclude that the one in the middle is also finitely generated. This proves (a).

To prove (b), we twist and again write down a piece of the long exact sequence

$$\dots \rightarrow H^i(X, \mathcal{E}(n)) \rightarrow H^i(X, \mathcal{F}(n)) \rightarrow H^{i+1}(X, \mathcal{R}(n)) \rightarrow \dots$$

Now for $n \gg 0$, the module on the left vanishes because $\mathcal{E}$ is a sum of $\mathcal{O}(q_i)$. The module on the right also vanishes for $n \gg 0$ because of the induction hypothesis. Hence $H^i(X, \mathcal{F}(n)) = 0$ for $n \gg 0$. Note since there are only finitely many i involved in statement (b), namely $0 < i \leq r$, it is sufficient to determine n_0 separately for each i . This proves (b).

Remark 5.2.1. As a special case of (a), we see that for any coherent sheaf $\mathcal{F}$ on X , $\Gamma(X, \mathcal{F})$ is a finitely generated A -module. This generalizes, and gives another proof of (II, 5.19).

As an application of these results, we give a cohomological criterion for an invertible sheaf to be ample (II, §7).

Proposition 5.3. *Let A be a noetherian ring, and let X be a proper scheme over $\text{Spec } A$. Let $\mathcal{L}$ be an invertible sheaf on X . Then the following conditions are equivalent:*

- (i) $\mathcal{L}$ is ample;
- (ii) For each coherent sheaf $\mathcal{F}$ on X , there is an integer n_0 , depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.

PROOF. (i) $\Rightarrow$ (ii). If $\mathcal{L}$ is ample on X , then for some $m > 0$, $\mathcal{L}^m$ is very ample on X over $\text{Spec } A$, by (II, 7.6). Since X is proper over $\text{Spec } A$, it is necessarily projective (II, 5.16.1). Now applying (5.2) to each of the sheaves $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^2, \dots, \mathcal{F} \otimes \mathcal{L}^{m-1}$ gives (ii). Cf. (II, 7.5) for a similar technique of proof.

(ii) $\Rightarrow$ (i). To show that $\mathcal{L}$ is ample, we will show that for any coherent sheaf $\mathcal{F}$ on X , there is an integer n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for all $n \geq n_0$. This is the definition of ampleness (II, §7).

Given $\mathcal{F}$, let P be a closed point of X , and let $\mathcal{I}_P$ be the ideal sheaf of the closed subset $\{P\}$. Then there is an exact sequence

$$0 \rightarrow \mathcal{I}_P \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes k(P) \rightarrow 0,$$

where $k(P)$ is the skyscraper sheaf $\mathcal{O}_X/\mathcal{I}_P$. Tensoring with $\mathcal{L}^n$, we get

$$0 \rightarrow \mathcal{I}_P \mathcal{F} \otimes \mathcal{L}^n \rightarrow \mathcal{F} \otimes \mathcal{L}^n \rightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes k(P) \rightarrow 0.$$

Now by our hypothesis (ii), there is an n_0 such that $H^1(X, \mathcal{I}_P \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $n \geq n_0$. Therefore

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n) \rightarrow \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n \otimes k(P))$$

is surjective for all $n \geq n_0$. It follows from Nakayama's lemma over the local ring $\mathcal{O}_P$, that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ at P is generated by global sections. Since it is a coherent sheaf, we conclude that for each $n \geq n_0$, there is an open neighborhood U of P , depending on n , such that the global sections of $\mathcal{F} \otimes \mathcal{L}^n$ generate the sheaf at every point of U .

In particular, taking $\mathcal{F} = \mathcal{O}_X$, we find there is an integer $n_1 > 0$ and an open neighborhood V of P such that $\mathcal{L}^{n_1}$ is generated by global sections over V . On the other hand, for each $r = 0, 1, \dots, n_1 - 1$, the above argument gives a neighborhood U_r of P such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r . Now let

$$U_P = V \cap U_0 \cap \dots \cap U_{n_1-1}.$$

Then over U_P , all of the sheaves $\mathcal{F} \otimes \mathcal{L}^n$, for $n \geq n_0$, are generated by global sections. Indeed, any such sheaf can be written as a tensor product

$$(\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m$$

for suitable $0 \leq r < n_1$ and $m \geq 0$.

Now cover X by a finite number of the open sets U_P , for various closed points P , and let the new n_0 be the maximum of the n_0 corresponding to those points P . Then $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections over all of X , for all $n \geq n_0$. q.e.d.

EXERCISES

- 5.1. Let X be a projective scheme over a field k , and let $\mathcal{F}$ be a coherent sheaf on X . We define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

If

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

is a short exact sequence of coherent sheaves on X , show that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

- 5.2. (a) Let X be a projective scheme over a field k , let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k , and let $\mathcal{F}$ be a coherent sheaf on X . Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the *Hilbert polynomial* of $\mathcal{F}$ with respect to the sheaf $\mathcal{O}_X(1)$. [Hints: Use induction on $\dim \text{Supp } \mathcal{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0.]$$

- (b) Now let $X = \mathbb{P}_k^r$, and let $M = \Gamma_*(\mathcal{F})$, considered as a graded $S = k[x_0, \dots, x_r]$ -module. Use (5.2) to show that the Hilbert polynomial of $\mathcal{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

- 5.3. *Arithmetic Genus*. Let X be a projective scheme of dimension r over a field k . We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r (\chi(\mathcal{O}_X) - 1).$$

Note that it depends only on X , not on any projective embedding.

- (a) If X is integral, and k algebraically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X).$$

[Hint: Use (I, 3.4).]

- (b) If X is a closed subvariety of $\mathbb{P}_k^r$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.
 (c) If X is a nonsingular projective curve over an algebraically closed field k , show that $p_a(X)$ is in fact a *birational invariant*. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

- 5.4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X .

- (a) Let X be a projective scheme over a field k , and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X . Show that there is a (unique) additive homomorphism

$$P: K(X) \rightarrow \mathbf{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X , $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

- (b) Now let $X = \mathbf{P}^r_k$. For each $i = 0, 1, \dots, r$, let L_i be a linear space of dimension i in X . Then show that

- (1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) \mid i = 0, \dots, r\}$, and
- (2) the map $P: K(X) \rightarrow \mathbf{Q}[z]$ is injective.

[Hint: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r , using (II, Ex. 6.10c).]

- 5.5.** Let k be a field, let $X = \mathbf{P}^r_k$, and let Y be a closed subscheme of dimension $q \geq 1$, which is a complete intersection (II, Ex. 8.4). Then:

- (a) for all $n \in \mathbf{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \rightarrow H^0(Y, \mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

- (b) Y is connected;
- (c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbf{Z}$;
- (d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

[Hint: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

- 5.6. Curves on a Nonsingular Quadric Surface.** Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}^3_k$ over a field k . We will consider locally principal closed subschemes Y of Q . These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\text{Pic } Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the type (a, b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$.

- (a) Use the special cases $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q , to show:

- (1) if $|a - b| \leq 1$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$;
- (2) if $a, b < 0$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$;
- (3) If $a \leq -2$, then $H^1(Q, \mathcal{O}_Q(a, 0)) \neq 0$.

- (b) Now use these results to show:

- (1) if Y is a locally principal closed subscheme of type (a, b) , with $a, b > 0$, then Y is connected;
- (2) now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b) . Use (II, 7.6.2) and (II, 8.18).
- (3) an irreducible nonsingular curve Y of type (a, b) , $a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type $(1, 3)$, which is just the rational quartic curve (I, Ex. 3.18).

- (c) If Y is a locally principal subscheme of type (a, b) in Q , show that $p_a(Y) = ab - a - b + 1$. [Hint: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q, 0)$ which is a disjoint union of q copies of $\mathbf{P}^1$. See (V, 1.5.2) for another method.]

5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A . We denote by $\mathcal{L}$ an invertible sheaf.

- (a) If $\mathcal{L}$ is ample on X , and Y is any closed subscheme of X , then $i^*\mathcal{L}$ is ample on Y , where $i: Y \rightarrow X$ is the inclusion.
 (b) $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{\text{red}} = \mathcal{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red} .
 (c) Suppose X is reduced. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i , for each irreducible component X_i of X .
 (d) Let $f: X \rightarrow Y$ be a finite surjective morphism, and let $\mathcal{L}$ be an invertible sheaf on Y . Then $\mathcal{L}$ is ample on Y if and only if $f^*\mathcal{L}$ is ample on X . [Hints: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

- (a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).
 (b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f: \tilde{X} \rightarrow X$ be the projection. Let $\mathcal{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathcal{L}(D) \cong \mathcal{L}$, and such that $f(P_i)$ is a nonsingular point of X , for each i . Conclude that there is an invertible sheaf $\mathcal{L}_0$ on X with $f^*\mathcal{L}_0 \cong \mathcal{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.
 (c) If X is reduced, but not necessarily irreducible, let $X_1, \dots, X_r$ be the irreducible components of X . Use (Ex. 4.5) to show $\text{Pic } X \rightarrow \bigoplus \text{Pic } X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.
 (d) Finally, if X is any one-dimensional proper scheme over k , use (2.7) and (Ex. 4.6) to show that $\text{Pic } X \rightarrow \text{Pic } X_{\text{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

5.9. A Nonprojective Scheme. We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbf{P}_k^2$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathcal{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X , let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1 , and since $\dim X = 2$, we have $\omega \otimes \mathcal{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\dots \rightarrow H^1(X, \omega) \rightarrow \text{Pic } X' \rightarrow \text{Pic } X \xrightarrow{\delta} H^2(X, \omega) \rightarrow \dots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X, \omega) \cong k$. Since $\text{char } k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X, \omega) = 0$, we see that $\text{Pic } X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω , such that X' is not projective over k .

Indeed, let D be an ample divisor on X . Then D determines an element $c_1(D) \in H^1(X, \Omega^1)$ which we use to define X' , as above. Then for any divisor E on X one can show that $\delta(\mathcal{L}(E)) = (D, E)$, where (D, E) is the intersection number (Chapter V), considered as an element of k . Hence if E is ample, $\delta(\mathcal{L}(E)) \neq 0$. Therefore X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

- 5.10.** Let X be a projective scheme over a noetherian ring A , and let $\mathcal{F}^1 \rightarrow \mathcal{F}^2 \rightarrow \dots \rightarrow \mathcal{F}^r$ be an exact sequence of coherent sheaves on X . Show that there is an integer n_0 , such that for all $n \geq n_0$, the sequence of global sections

$$\Gamma(X, \mathcal{F}^1(n)) \rightarrow \Gamma(X, \mathcal{F}^2(n)) \rightarrow \dots \rightarrow \Gamma(X, \mathcal{F}^r(n))$$

is exact.

6 Ext Groups and Sheaves

In this section we develop the properties of Ext groups and sheaves, which we will need for the duality theorem. We work on a ringed space $(X, \mathcal{O}_X)$, and all sheaves will be sheaves of $\mathcal{O}_X$ -modules.

If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$ -modules, we denote by $\text{Hom}(\mathcal{F}, \mathcal{G})$ the group of $\mathcal{O}_X$ -module homomorphisms, and by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the sheaf Hom (II, §5). If necessary, we put a subscript X to indicate which space we are on: $\text{Hom}_X(\mathcal{F}, \mathcal{G})$. For fixed $\mathcal{F}$, $\text{Hom}(\mathcal{F}, \cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, and $\mathcal{H}om(\mathcal{F}, \cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. Since $\mathfrak{Mod}(X)$ has enough injectives (2.2) we can make the following definition.

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$ -module. We define the functors $\text{Ext}^i(\mathcal{F}, \cdot)$ as the right derived functors of $\text{Hom}(\mathcal{F}, \cdot)$, and $\mathcal{E}xt^i(\mathcal{F}, \cdot)$ as the right derived functors of $\mathcal{H}om(\mathcal{F}, \cdot)$.

Consequently, according to the general properties of derived functors (1.1A) we have $\text{Ext}^0 = \text{Hom}$, a long exact sequence for a short exact sequence in the second variable, $\text{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for $i > 0$, $\mathcal{G}$ injective in $\mathfrak{Mod}(X)$, and ditto for the $\mathcal{E}xt$ sheaves.

Lemma 6.1. *If $\mathcal{I}$ is an injective object of $\mathfrak{Mod}(X)$, then for any open subset $U \subseteq X$, $\mathcal{I}|_U$ is an injective object of $\mathfrak{Mod}(U)$.*

PROOF. Let $j: U \rightarrow X$ be the inclusion map. Then given an inclusion $\mathcal{F} \subseteq \mathcal{G}$ in $\mathfrak{Mod}(U)$, and given a map $\mathcal{F} \rightarrow \mathcal{I}|_U$, we get an inclusion $j_!\mathcal{F} \subseteq j_!\mathcal{G}$, and a map $j_!\mathcal{F} \rightarrow j_!(\mathcal{I}|_U)$, where $j_!$ is extension by zero (II, Ex. 1.19). But $j_!(\mathcal{I}|_U)$ is a subsheaf of $\mathcal{I}$, so we have a map $j_!\mathcal{F} \rightarrow \mathcal{I}$. Since $\mathcal{I}$ is injective in $\mathfrak{Mod}(X)$, this extends to a map of $j_!\mathcal{G} \hookrightarrow \mathcal{I}$. Restricting to U gives the required map of $\mathcal{G}$ to $\mathcal{I}|_U$.