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Géométrie algébrique et cohomologie, I

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10 Algèbre homologique

Nous allons introduire et utiliser deux théories cohomologiques différentes : la *cohomologie de Čech* et la *cohomologie des foncteurs dérivés*. La première est plus facile à calculer, mais c'est la seconde qui possède les meilleures propriétés théoriques générales. Les deux sont distinctes en général, mais lorsque les faisceaux de coefficients sont quasi-cohérents elles coïncident au moins pour les schémas séparés, comme nous le démontrerons.

Voici un aperçu très bref de l'histoire sujet. Le développement de la cohomologie de Čech prend ses racines dans les travaux de Poincaré sur l'homologie (1895). Les aspects combinatoires sont explorés par Alexandroff (1925) et Čech (1932) qui introduisent les triangulations et le nerf des recouvrements. Au milieu du vingtième siècle apparaissent les faisceaux, introduits par Leray (1946). Par la suite, la théorie connait une forte algébrisation avec a découverte de la technique des foncteurs dérivés par Cartan et Eilenberg (1956). La cohomologie des faisceaux arrive en géométrie algébrique à partir des années 1950 : c'est d'abord Serre (1955) qui montre comment adapter la cohomologie de Čech, puis Grothendieck (à partir de 1957) qui développe la cohomologie des foncteurs dérivés. L'évolution des théories cohomologiques (catégories dérivées, structures diverses : action de Galois, structures de Hodge...) et la profusion de leurs applications en géométrie algébrique s'est poursuivie sans interruption depuis, et elles restent l'un des outils les plus puissants dont on dispose.

Figure 11 – Samuel Eilenberg (1913-1998)

Comment comprendre intuitivement la cohomologie ? La vérité est que l'égalité $d \circ d = 0$ qui définit un complexe d'objets d'une catégorie abélienne reste un mystère pour les mathématicien.nes, comme en témoignent les propos de Henri Cartan, Jean-Pierre Serre et John Gwyn Griffiths reproduits en épigraphe du livre [GM03], voir figure 12. Il est plus facile de comprendre intuitivement les idées qui motivent l'introduction du complexe de Čech ; la lectrice pourra lire avec profit l'introduction du chapitre 2 sur l'homologie dans le livre de Hatcher [Hat02]. Pour cette raison, nous commencerons par présenter la cohomologie de Čech, plus concrète, avant de passer à la cohomologie des foncteurs dérivés. Aujourd'hui, on considère que la théorie la meilleure pour développer la théorie est celle des foncteurs dérivés, que l'on appelle souvent simplement *cohomologie*, et que la cohomologie de Čech est une sorte d'algorithme qui en permet le calcul dans les bonnes situations.

10.1 Comment représenter un module ?

Toutes les branches des mathématiques, à commencer bien sûr par l'algèbre et la géométrie, sont irriguées par les techniques surpuissantes de l'algèbre linéaire. Lorsque les scalaires vivent dans un anneau commutatif unitaire A qui n'est pas un corps, les objets qui jouent le rôle des espaces vectoriels de dimension finie sont les *modules de type fini*. Leur étude est la généralisation naturelle de l'algèbre linéaire. L'objectif de ce paragraphe est de montrer comment, dans cette théorie, ont émergé les notions de *complexe de modules* et de *résolution d'un module*. Pour cela nous allons énoncer le théorème des syzygies de Hilbert, qui est l'un des points d'ancrage historique de l'algèbre homologique. Pour plus détails, la lectrice peut consulter [Ei95, 1.10].

Les modules de type fini les plus simples sont les modules libres $M \simeq A^n$. Pour eux, une bonne partie de l'algèbre linéaire habituelle s'étend sans grand changement : les morphismes entre modules libres sont représentées par des matrices, les endomorphismes ont un déterminant qui teste leur inversibilité, etc. Pour les modules quelconques, les choses ne sont pas si simples mais on peut essayer de se ramener à des modules libres de la manière suivante. Pour un module M donné, le choix d'une partie génératrice (de cardinal disons r) fournit une surjection depuis le module libre $L_0 = A^r$:

$$L_0 \longrightarrow M \longrightarrow 0.$$

(Nous anticipons un peu en utilisant les notations des suites exactes, voir déf. 10.2.5.) Si ce morphisme est injectif, $M \simeq L_0$ est libre et on a terminé. Sinon, on considère le noyau de $L_0 \to M$ qui (en supposant A noethérien ce que nous supposons pour simplifier) est encore de type fini. Le choix d'une partie génératrice de ce noyau fournit une surjection depuis un module libre L_1. On obtient une suite exacte :

$$L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0.$$

On itère ensuite avec le noyau de $L_1 \to L_0$, etc. Lorsque A est l'anneau des polynômes, David Hilbert a fait une percée décisive que nous illustrons avec un exemple typique de module non libre, à savoir un module quotient de A :

$$A = k[x_1, \dots, x_n] \quad \text{et} \quad M = A/(x_1, \dots, x_n) = k.$$

Dans les cas de dimension $n = 1, 2, 3$, en menant les calculs avec soin pour choisir des générateurs « naturels » des noyaux, on voit que le procédé se termine :

$$0 \longrightarrow A \xrightarrow{(x_1)} A \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} A^2 \xrightarrow{(x_1\ x_2)} A \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}} A^3 \xrightarrow{(x_1\ x_2\ x_3)} A \longrightarrow k \longrightarrow 0$$

10.1.1 Exercice (très instructif). Effectuez ces calculs.

10.1.2 Théorème (des syzygies de Hilbert, 1890) *Soit k un corps. Tout module de type fini M sur l'anneau de polynômes $k[x_1, \dots, x_n]$ peut s'insérer dans une suite exacte de longueur $k \leqslant n$*

$$0 \longrightarrow L_k \longrightarrow L_{k-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0.$$

où les L_i sont libres de type fini. Une telle suite exacte est appelée une résolution libre *de M.*

(La longueur maximale $k = n$ est atteinte pour le module $M = A/(x_1, \dots, x_n)$: on ne peut le mettre dans une suite exacte plus courte.) Une manière moderne de lire ce théorème est qu'il permet de remplacer M par l'objet $L_\bullet = (L_k \to \cdots \to L_0)$, qui est plus agréable à manipuler car c'est un objet d'algèbre linéaire ; nous l'appellerons un *complexe*. La suite exacte de l'énoncé du théorème est une *résolution* de M par $L_\bullet$. Pour les modules sur des anneaux généraux, de telles résolutions libres ne sont pas toujours finies, mais cela importe peu et le développement ultérieur de l'algèbre homologique a montré (et les parties qui viennent vont montrer) le rôle capital joué par les complexes.

10.2 Catégories abéliennes

Pour tout schéma X, notons $\mathrm{Mod}(\mathcal{O}_X)$ la catégorie des faisceaux de $\mathcal{O}_X$-modules sur X. Dans la suite du cours, nous allons étudier la cohomologie des faisceaux, qui fournit un cadre théorique général pour étudier certains foncteurs exacts d'un côté, comme les foncteurs

- de sections globales $\Gamma : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(A)$, où $A = \mathcal{O}_X(X)$,
- plus généralement, d'image directe $f_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$.

Le contexte catégorique adapté est celui des catégories dites *abéliennes*. La lectrice s'assurera facilement que les catégories $\mathrm{Mod}(A)$ et $\mathrm{Mod}(\mathcal{O}_X)$ ainsi que les foncteurs Γ et f_* vérifient les axiomes de la définition suivante.

Avant la définition suivante, il est bon de relire la définition d'un objet zéro ; voir 4.1.14 et 4.1.15. Lorsqu'un tel objet existe, entre deux objets quelconques existe un unique *morphisme nul* et à chaque morphisme on peut associer des notions de *noyau* (voir 4.5.14)) et de *conoyau* (voir 4.5.5).

10.2.1 Définition (Catégorie abélienne) Une catégorie $\mathscr{C}$ est dite *abélienne* si :

1. les ensembles de morphismes $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ sont munis de structures de groupes abéliens et les applications de composition sont bilinéaires ;
2. il existe un objet zéro ;
3. toute paire d'objets possède une somme et un produit ;
4. tout morphisme possède un noyau et un conoyau ;
5. tout monomorphisme est un noyau et tout épimorphisme est un conoyau ;
6. pour tout morphisme u, la flèche $\operatorname{coker} \ker u \to \ker \operatorname{coker} u$ est un isomorphisme.

Si $\mathscr{C}, \mathscr{D}$ sont deux catégories abéliennes, un *foncteur additif* est un foncteur $F : \mathscr{C} \to \mathscr{C}$ tel que $\mathrm{Hom}_{\mathscr{C}}(M, N) \to \mathrm{Hom}_{\mathscr{D}}(FM, FN)$ est un morphisme de groupes, pour tous objets $M, N \in \mathscr{C}$.

10.2.2 Remarques.

1. On peut montrer que cette définition est équivalente à celle de Hartshorne [Har77, III, § 1].
2. Les axiomes donnés ci-dessus sont redondants, mais il nous importe peu de donner une liste minimale. Sous cette forme, la définition permet d'avoir sous la main une liste des propriétés principales des catégories $\mathrm{Mod}(A)$ et $\mathrm{Sh}(X, \mathrm{Mod}(\mathscr{A}))$.

3. Le premier et le dernier axiome sont conséquences des autres. Le premier axiome est spécial car c'est une structure supplémentaire et non une propriété ; mais on peut montrer qu'avec les axiomes 2 à 5, la loi d'addition sur les $\mathrm{Hom}_{\mathscr{C}}(M,N)$ existe et est unique. Il en résulte que le fait d'être abélienne est une *propriété*. Pour des détails, voir [Fr64, chap. 2].
4. On peut montrer que la somme et le produit de deux objets A et B sont isomorphes. On appelle parfois cet objet commun le *biproduit* et on le note $A \oplus B$. Pour plus de détails, voir MacLane [Mac78, chap. VIII, § 2].

10.2.3 Exercice (Catégorie abélienne opposée). Démontrez que si $\mathscr{C}$ est une catégorie abélienne, alors la catégorie opposée $\mathscr{C}^\circ$ est abélienne.

10.2.4 Exercice (Image et coimage).

1. Soit f un morphisme de A-modules. On définit l'*image* $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$ et la *coimage* $\mathrm{coim}(f) = \mathrm{coker}(\ker(f))$. Démontrez qu'il existe un isomorphisme canonique $\mathrm{coim}(f) \xrightarrow{\sim} \mathrm{im}(f)$. Mêmes questions avec un morphisme de faisceaux de $\mathscr{A}$-modules sur un espace topologique X. (En fait l'énoncé est vrai dans n'importe quelle catégorie abélienne.)
2. Soit $\mathscr{C}$ la catégorie des groupes abéliens sans torsion. On rappelle que l'inclusion $i : \mathscr{C} \hookrightarrow \mathrm{Ab}$ possède un adjoint à gauche donné par le quotient sans torsion $F(M) = M/\mathrm{Tors}(M)$, voir l'exercice 4.4.5. Démontrez que les noyaux et conoyaux de morphismes existent dans $\mathscr{C}$ avec les formules $\ker_{\mathscr{C}}(f) = \ker_{\mathrm{Ab}}(f)$ et $\mathrm{coker}_{\mathscr{C}}(f) = F(\mathrm{coker}_{\mathrm{Ab}}(f))$. Démontrez que si $f : \mathbb{Z} \to \mathbb{Z}$ est donné par $f(x) = 2x$, alors $\mathrm{im}(f) = (0)$ alors que $\mathrm{coim}(f) = \mathbb{Z}$.

Le théorème des syzygies 10.1.2 a montré l'importance de la notion de suite exacte, qui a un sens dans toute catégorie abélienne et que nous rappelons maintenant.

10.2.5 Définition (Suites exactes) Soit $\mathscr{C}$ une catégorie abélienne. On appelle :

- *suite exacte* dans $\mathscr{C}$ une suite d'objets et de morphismes

$$\cdots \xrightarrow{f_{i-2}} X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} \cdots$$

  telle que $\mathrm{im}(f_{i-1}) = \ker(f_i)$ pour tout $i \in \mathbb{Z}$,
- *suite exacte courte* une suite exacte de la forme $0 \to X \to Y \to Z \to 0$.

De nombreux foncteurs naturels préservent partiellement les suites exactes ; voici les termes qui leur sont attachés.

10.2.6 Définition (Foncteurs exacts) Soient $\mathscr{C} \to \mathscr{D}$ deux catégories abéliennes et $F : \mathscr{C} \to \mathscr{D}$ un foncteur additif. On dit que F est :

- *exact à gauche* s'il préserve les noyaux :

$$0 \to X \to Y \to Z \text{ exacte} \implies 0 \to FX \to FY \to FZ \text{ exacte.}$$

- *exact à droite* s'il préserve les conoyaux :

$$X \to Y \to Z \to 0 \text{ exacte} \implies FX \to FY \to FZ \to 0 \text{ exacte.}$$

- *exact* s'il est exact à gauche et à droite :

$$0 \to X \to Y \to Z \to 0 \text{ exacte} \implies 0 \to FX \to FY \to FZ \to 0 \text{ exacte.}$$

10.2.7 Remarque. Cette notion d'exactitude coïncide, dans le contexte des foncteurs additifs entre catégories abéliennes, avec la notion générale donnée en 10.5.6. Ces faits sont énoncés dans [Mac78], chapitre VIII, fin de la section 3. Dans la suite, nous utiliserons exclusivement la définition 10.2.6.

10.2.8 Exercice (Décomposer les suites exactes en suites exactes courtes). On considère une suite exacte $\cdots \to X_{i-1} \to X_i \to X_{i+1} \to \cdots$ dont on note $f_i : X_i \to X_{i+1}$ les morphismes. En utilisant les suites exactes courtes $0 \to \ker(f_i) \to X_i \to \mathrm{im}(f_i) \to 0$, démontrez les faits suivants.

1. Un foncteur exact préserve les suites exactes de longueur arbitraire (éventuellement infinie).
2. Dans une suite exacte $1 \to G_1 \to G_2 \to \cdots \to G_n \to 1$ de groupes finis, le produit alterné de cardinaux $\prod_{i=1}^n |G_i|^{(-1)^i}$ est égal à 1. (Ceci est vrai même si les G_i sont non abéliens.)
3. Dans une suite exacte $0 \to E_1 \to E_2 \to \cdots \to E_n \to 0$ d'espaces vectoriels de dimension finie sur un corps k, la somme alternée de dimensions $\sum_{i=1}^n (-1)^i \dim(E_i)$ est égale à 0.

10.2.9 $\boxed{\text{Exercice}}$ (Exemples de suites exactes à quatre termes).

1. Soiet $a, b \geqslant 1$ entiers, d leur pgcd et m leur ppcm. On considère la suite de morphismes :

$$0 \longrightarrow \mathbb{Z}/d\mathbb{Z} \xrightarrow{i} (\mathbb{Z}/ab\mathbb{Z})^* \xrightarrow{\Delta} (\mathbb{Z}/a\mathbb{Z})^* \times (\mathbb{Z}/b\mathbb{Z})^* \xrightarrow{p} (\mathbb{Z}/d\mathbb{Z})^* \longrightarrow 1$$

   avec $i(r) = 1 + mr$, $\Delta(s) = (s,s)$ et $p(t,u) = tu^{-1}$. Montrez qu'il s'agit d'une suite exacte de groupes finis. Déduisez-en une formule pour $\varphi(ab)$ où φ est la fonction indicatrice d'Euler.

2. Soit A un anneau intègre et K son corps de fractions. Montrez que pour tout A-module M, de sous-module de torsion noté M_{tor}, on a une suite exacte :

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \longrightarrow M \otimes K \longrightarrow M \otimes K/A \longrightarrow 0.$$

3. Soit $\mathbb{C}((z)) = \mathrm{Frac}(\mathbb{C}[\![z]\!]) = \{f(z) = \sum_{n=-N}^{\infty} f_n z^n \text{ pour un entier } N\}$ le corps de séries formelles de Laurent. Soient $D : \mathbb{C}((z)) \to \mathbb{C}((z))$ la dérivation et $\mathrm{Res} : \mathbb{C}((z)) \to \mathbb{C}$ l'application résidu, définie par $\mathrm{Res}(f) = f_{-1}$. Démontrez qu'on a une suite exacte :

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}((z)) \xrightarrow{D} \mathbb{C}((z)) \xrightarrow{\mathrm{Res}} \mathbb{C} \longrightarrow 0.$$

4. Soit (M, d) un module différentiel, voir définition 10.3.3. Démontrez qu'on a une suite exacte :

$$0 \longrightarrow \mathrm{H}(M) \longrightarrow M/\mathrm{B}(M) \xrightarrow{d} \mathrm{Z}(M) \longrightarrow \mathrm{H}(M) \longrightarrow 0.$$

10.2.10 Exercice (Hom est exact à gauche). Soit $\mathscr{C} = \mathrm{Mod}(A)$. Démontrez les fais suivants :

1. Pour tout $P \in \mathscr{C}$, le foncteur $F = \mathrm{Hom}(P, -) : \mathscr{C} \to \mathscr{C}$ est exact à gauche.
2. Pour tout $I \in \mathscr{C}$, le foncteur $F = \mathrm{Hom}(-, I) : \mathscr{C}^\circ \to \mathscr{C}$ est exact à gauche. (*Comme ici F est contravariant, il est important de se rappeler des conventions de 4.2.3. Précisément, il faut montrer : $X \to Y \to Z \to 0$ exacte $\Rightarrow 0 \to \mathrm{Hom}(Z, I) \to \mathrm{Hom}(Y, I) \to \mathrm{Hom}(X, I)$ exacte.*)
3. Il existe $P \in \mathscr{C}$ tel que $\mathrm{Hom}(P, -)$ n'est pas exact à droite et il existe $I \in \mathscr{C}$ tel que $\mathrm{Hom}(-, I)$ n'est pas exact à droite.
4. Mêmes questions dans une catégorie abélienne $\mathscr{C}$ arbitraire.

10.2.11 $\boxed{\text{Exercice}}$ (Le foncteur des points fixes est exact à gauche). Soient A un anneau et G un groupe. On note $\mathscr{C}$ la catégorie des G-A-modules, c'est-à-dire les A-modules munis d'une action A-linéaire de G. (Lorsque A est un corps, il s'agit de représentations linéaires au sens usuel.)

1. Montrez que le foncteur $F : \mathscr{C} \to \mathrm{Mod}(A)$ défini par $F(M) = M^G$, le sous-A-module des points fixes de G, est exact à gauche.
2. On suppose que A est un corps k et on prend le groupe $G = (k, +)$. On considère le G-k-module $M = k^2$ où G agit par $\lambda \cdot (a, b) = (a + \lambda b, b)$ (c'est l'action naturelle de la matrice $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ sur k^2). Montrez que $N := \{(a, b) \in M;\ b = 0\}$ est une droite G-stable. Montrez que F n'est pas exact à droite en regardant son action sur la suite exacte $0 \to N \to M \to M/N \to 0$ de $\mathscr{C}$.

10.2.12 $\boxed{\text{Exercice}}$ (Un foncteur exact à droite sans adjoint à droite). Soit Vect la catégorie des espaces vectoriels sur un corps k. On considère l'espace vectoriel $E = k^{(\mathbb{N})} := \oplus_{n \geqslant 0} k$, sous-espace de $k^{\mathbb{N}} := \prod_{n \geqslant 0} k$ composé des familles $(x_n)_{n \geqslant 0}$ telles que tous les x_n sont nuls sauf un nombre fini. Enfin on note $F : \mathrm{Vect} \to \mathrm{Vect}$ le foncteur défini par $F(V) = \mathrm{Hom}(E, V)$.

1. Démontrez que F est exact à droite.
2. Démontrez que F ne préserve pas les sommes directes infinies dénombrables, c'est-à-dire que l'application canonique $c : \oplus_{n \geqslant 0} \mathrm{Hom}(E, k) \to \mathrm{Hom}(E, \oplus_{n \geqslant 0} k)$ n'est pas un isomorphisme. Pour cela, démontrez que l'image de c est composée des applications linéaires $E \to \oplus_{n \geqslant 0} k$ de rang fini ; en particulier $\mathrm{id}_E \notin \mathrm{im}(c)$. (*Indication : notez $H_n = \mathrm{Hom}(E, k)$ le n-ième terme de la source de c. Quelle est l'image par c d'une forme linéaire $\varphi \in H_n$?*)

10.2.13 $\boxed{\text{Exercice}}$ (Un foncteur exact à gauche sans adjoint à gauche, 1). On garde les notations de l'exercice précédent et on note $F : \mathrm{Vect} \to \mathrm{Vect}$ le foncteur défini par $F(V) = V \otimes_k E$.

1. Démontrez que F est exact à gauche.
2. Démontrez que F ne préserve pas les produits infinis dénombrables, c'est-à-dire que l'application canonique $d : (\prod_{n \geqslant 0} k) \otimes_k E \to \prod_{n \geqslant 0} (k \otimes_k E)$ n'est pas un isomorphisme. Pour cela, montrez que l'image de d est composée des familles $(v_n)_{n \geqslant 0}$ de vecteurs $v_n \in k \otimes_k E = E$ qui engendrent un sous-espace de dimension finie. (*Indication : les vecteurs de la source de d sont des sommes finies de tenseurs élémentaires $a_i \otimes v_i$.*)

10.2.14 $\boxed{\text{Exercice}}$ (Un foncteur exact à gauche sans adjoint à gauche, 2). Soit $X = \mathbb{A}^1_{\mathbb{C}}$ la droite affine complexe et $x = 0$ son origine. On considère le foncteur fibre $F : \mathrm{Sh}(X, \mathrm{Ab}) \to \mathrm{Ab}$, $\mathscr{F} \mapsto \mathscr{F}_x = \mathscr{F}_0$.

1. Démontrez que F préserve les limites finies.
2. Démontrez que F ne préserve pas les produits infinis dénombrables, en considérant les faisceaux $\mathscr{F}_n = \mathcal{O}_X^\times$ pour tout $n \geqslant 1$ et en démontrant que le morphisme $(\prod_{n \geqslant 1} \mathscr{F}_n)_x \to \prod_{n \geqslant 1} \mathscr{F}_{n,x}$ n'est pas surjectif. (*Indication : notant $f_{n,x} \in \mathscr{F}_{n,x}$ le ferme de la fonction $f_n = 1 - nx$, démontrez que la collection $(f_{n,x})$ ne provient pas d'un germe du faisceau $\mathscr{F} = \prod_{n \geqslant 1} \mathscr{F}_n$.*)

10.3 Complexes

L'application d'un foncteur additif à une suite exacte fait perdre l'exactitude en général ; elle crée naturellement des objets plus généraux : les *complexes*.

10.3.1 Définition (Complexes) Soit $\mathscr{C}$ une catégorie abélienne.

- Un *complexe (de cochaînes)* est une suite $M^\bullet = (M^\bullet, d)$ d'objets et de morphismes
$$\cdots \longrightarrow M^{i-1} \xrightarrow{d_{i-1}} M^i \xrightarrow{d_i} M^{i+1} \longrightarrow \cdots$$
telle que $d_{i+1} \circ d_i = 0$ pour tout $i \in \mathbb{Z}$. Les d_i sont appelées *différentielles*. Un complexe est dit *positif* lorsque $M^i = 0$ pour $i < 0$ et *négatif* lorsque M^i pour $i > 0$.

- Un *morphisme de complexes* $f : M^\bullet \to N^\bullet$ est une collection de morphismes $f_i : M^i \to N^i$
$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"] & M^i \arrow[r, "d_i"] \arrow[d, "f_i"] & M^{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & N^{i-1} \arrow[r, "d_{i-1}"] & N^i \arrow[r, "d_i"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$
tels que $f_{i+1}d_i = d_i f_i$ pour tout $i \in \mathbb{Z}$.

10.3.2 Remarque. La donnée de $M^\bullet$ est équivalente à la donnée de l'objet $\mathbb{Z}$-gradué $\oplus_{i\in\mathbb{Z}} M^i$ muni de l'application graduée $d : \oplus_{i\in\mathbb{Z}} M^i \to \oplus_{i\in\mathbb{Z}} M^i$ définie par $d_{|M^i} := d_i$, vérifiant $d^2 = 0$.

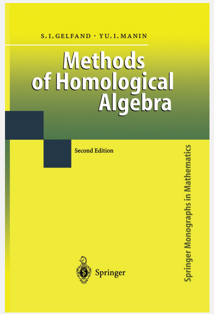
- Lorsque $\deg(d) = +1$, comme dans la définition 10.3.1, on parle de *complexe de cochaînes*. Un exemple est le complexe de de Rham $0 \to \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n) \to \ldots$.

- Lorsque $\deg(d) = -1$, on parle de *complexe de chaînes*. Un exemple est fourni par les résolutions libres d'un module apparues dans le théorème des syzygies 10.1.2.

La convention habituelle est d'utiliser l'indexation en haut $M^\bullet = \oplus M^i$ pour les complexes de cochaînes et l'indexation en bas $M_\bullet = \oplus M_i$ pour les complexes de chaînes.

Le défaut d'exactitude d'un complexe est mesuré à l'aide de sa *cohomologie*, qui est associée à la différentielle et à son intrigante propriété $d^2 = 0$ (voir figure 12). Pour présenter cette notion, la graduation ne joue pas de rôle et nous la laissons donc de côté jusqu'à la fin de cette sous-section.

10.3.3 Définition (Cohomologie des complexes) Soit (M, d) un *objet différentiel* de $\mathscr{C}$, c'est-à-dire un objet M muni d'un endomorphisme $d : M \to M$ tel que $d^2 = 0$. On note :



... utinam intelligere possim rationacinationes pulcherrimas quae e propositione concisa DE QUADRATUM NIHILO EXAEQUARI fluunt.

(... if I could only understand the beautiful consequence following from the concise proposition $d^2 = 0$.)

From Henri Cartan Laudatio on receiving the degree of Doctor Honoris Causa, Oxford University, 1980.

FIGURE 12 – Épigraphe du livre Gelfand et Manin, *Methods of homological algebra* [GM03]

- $\mathrm{Z}(M) = \ker(d)$ l'*objet des cocycles*,
- $\mathrm{B}(M) = \mathrm{im}(d)$ l'*objet des cobords*,
- $\mathrm{H}(M) = \mathrm{Z}(M)/\mathrm{B}(M)$ l'*objet de cohomologie de* M.

La notation emprunte les initiales des mots allemands Zyklus, Begrenzung, Homologie. Par ailleurs, très souvent on notera avec la même lettre d les différentielles de tous les objets en présence, lorsque cela n'engendre pas de confusion.

Un *morphisme d'objets différentiels* $f : (M, d) \to (M', d')$ est un morphisme $f : M \to M'$ tel que $d' \circ f = f \circ d$. Un morphisme d'objets différentiels envoie $\mathrm{Z}(M)$ dans $\mathrm{Z}(M')$ et $\mathrm{B}(M)$ dans $\mathrm{B}(M')$ et induit donc un morphisme en cohomologie
$$\mathrm{H}(f) : \mathrm{H}(M) \longrightarrow \mathrm{H}(M').$$

Une des propriétés fondamentales de la cohomologie est son comportement par rapport aux suites exactes, que voici :

10.3.4 Lemme (du serpent). *Considérons le diagramme en traits pleins ci-dessous, qui représente un morphisme (α, β, γ) de complexes. On suppose que les deux lignes sont exactes. Alors, il existe une suite exacte indiquée en pointillés, où le morphisme ∂ est défini par $\partial(c) = (f')^{-1}\beta g^{-1}(c)$.*

$$\begin{tikzcd}
 & & \ker f \arrow[r, dashed] & \ker\alpha \arrow[r, dashed] \arrow[d] & \ker\beta \arrow[r, dashed] \arrow[d] & \ker\gamma \arrow[d] \arrow[ddll, dashed, out=0, in=180] & \\
0 \arrow[urr, dashed] & & & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
 & & 0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & \\
 & & & \operatorname{coker}\alpha \arrow[r, dashed] & \operatorname{coker}\beta \arrow[r, dashed] & \operatorname{coker}\gamma \arrow[r, dashed] & \operatorname{coker} g' \arrow[r, dashed] & 0
\end{tikzcd}$$

Démonstration :

□

10.3.5 Proposition. *Soit* $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ *une suite exacte d'objets différentiels d'une catégorie abélienne. Alors il existe un morphisme* ∂ : $\mathrm{H}(M'') \to \mathrm{H}(M')$ *appelé* morphisme connectant *(*connecting homomorphism *en anglais) tel que le triangle suivant soit exact :*

$$\begin{tikzcd}
 & \mathrm{H}(M) \arrow[dr, "\mathrm{H}(p)"] & \\
\mathrm{H}(M') \arrow[ur, "\mathrm{H}(i)"] & & \mathrm{H}(M'') \arrow[ll, "\partial"']
\end{tikzcd}$$

c'est-à-dire qu'en chaque sommet, le noyau de la flèche qui part est égal à l'image de celle qui arrive.

Démonstration : Comme $\mathrm{B}(M) = \mathrm{im}(d) \subset \ker(d) = \mathrm{Z}(M)$, la différentielle de M induit un morphisme dont le noyau et le conoyau sont la cohomologie (voir exercice 10.2.9.4) :

$$0 \longrightarrow \mathrm{H}(M) \longrightarrow M/\mathrm{B}(M) \xrightarrow{d} \mathrm{Z}(M) \longrightarrow \mathrm{H}(M) \longrightarrow 0.$$

La même chose vaut pour M' et M''. En appliquant le lemme du serpent au diagramme

$$\begin{tikzcd}
 & M'/\mathrm{B}(M') \arrow[r] \arrow[d, "d"] & M/\mathrm{B}(M) \arrow[r] \arrow[d, "d"] & M''/\mathrm{B}(M'') \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \mathrm{Z}(M') \arrow[r] & \mathrm{Z}(M) \arrow[r] & \mathrm{Z}(M'') &
\end{tikzcd}$$

on obtient le morphisme connectant et l'exactitude annoncée. □

Notons que pour un complexe de cochaînes $M^\bullet = \oplus M^i$, les modules $\mathrm{Z}(M^\bullet)$, $\mathrm{B}(M^\bullet)$ et $\mathrm{H}(M^\bullet)$ sont gradués avec

$$\mathrm{Z}^i(M^\bullet) := \mathrm{Z}^i(M^i), \quad \mathrm{B}^i(M^\bullet) := \mathrm{B}(M^i), \quad \mathrm{H}^i(M^\bullet) := \mathrm{H}(M^i).$$

Si la suite de la proposition 10.3.5 est une suite exacte de complexes de cochaînes, la formule explicite pour le connectant (voir l'énoncé du lemme du serpent) montre que $\partial(\mathrm{H}^i(M'')) \subset \mathrm{H}^{i+1}(M')$. Le triangle exact prend donc la forme d'une suite exacte longue :

$$\cdots \longrightarrow \mathrm{H}^i(M') \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^i(M'') \xrightarrow{\partial} \mathrm{H}^{i+1}(M') \longrightarrow \cdots$$

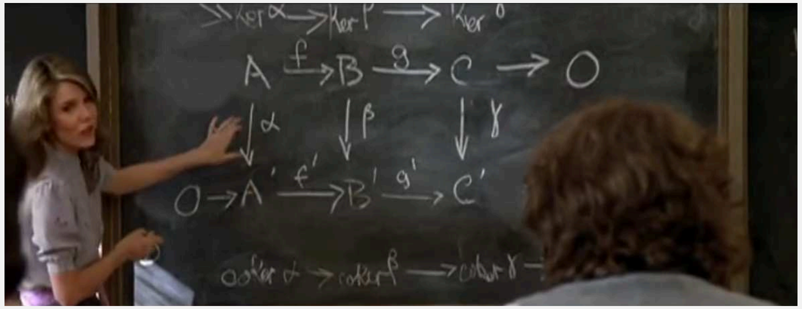


FIGURE 13 – Le lemme du serpent démontré par Jill Clayburgh dans le film *It's my turn* (1980)
Pour voir la démonstration : `https://youtu.be/aXBNPjrvx-I`

10.3.6 Exercice. Une *homotopie* $h : f \to g$ entre deux morphismes $f, g : M^\bullet \to N^\bullet$ est une collection de morphismes $h_i : M^i \to N^{i-1}$

$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r, "d_{i-1}"] \arrow[d, shift left] \arrow[d, shift right] & M^i \arrow[r, "d_i"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h_i"'] & M^{i+1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, "h_{i+1}"'] & \cdots \\
\cdots \arrow[r] & N^{i-1} \arrow[r, "d_{i-1}"'] & N^i \arrow[r, "d_i"'] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

tels que $f_i - g_i = d_{i-1}h_i + h_{i+1}d_i$ pour tout $i \in \mathbb{Z}$.

On dit que f et g sont homotopes s'il existe une homotopie entre eux. Dans ce cas ils induisent le même morphisme en cohomologie.

10.4 Résolutions

Nous introduisons maintenant les objets injectifs et projectifs, qui vont servir à « résoudre » les objets quelconques d'une catégorie abélienne.

10.4.1 Définition (Injectifs, projectifs) Soient I, P deux objets d'une catégorie abélienne $\mathscr{C}$. On dit que :

- I est *injectif* si $\mathrm{Hom}(-, I)$ est exact.
- P est *projectif* si $\mathrm{Hom}(P, -)$ est exact.
- *$\mathscr{C}$ possède assez d'injectifs* si pour tout $X \in \mathscr{C}$, il existe un injectif I et un mono $X \hookrightarrow I$.
- *$\mathscr{C}$ possède assez de projectifs* si pour tout $X \in \mathscr{C}$, il existe un projectif P et un épi $P \twoheadrightarrow X$.

10.4.2 Exercice (Cf figure 14). Compte tenu de l'exactitude à gauche (exercice 10.2.10), les objets I resp. P sont injectif resp. projectif si et seulement si $\mathrm{Hom}(-, I)$ resp. $\mathrm{Hom}(P, -)$ est exact à droite. Vérifiez que cela équivaut aux énoncés en apparence plus forts suivants :

1. L'objet I est injectif si et seulement si pour toute suite exacte $0 \to X \to Y$, tout morphisme $X \to I$ s'étend en un morphisme $Y \to I$.
2. L'objet P est projectif si et seulement si pour toute suite exacte $Y \to Z \to 0$, tout morphisme $P \to Z$ se relève en un morphisme $P \to Y$.

10.4.3 Exercice (Projectif *ssi* facteur direct d'un libre). Soit A un anneau et P un A-module. Démontrez que P est projectif si et seulement s'il est facteur direct d'un module libre, c'est-à-dire qu'il existe un module libre L et un module C tels que $L \simeq P \oplus C$.

La présence d'assez d'injectifs et de projectifs dans une catégorie abélienne est un fait crucial pour développer l'algèbre homologique. Pour l'heure, nous allons vérifier que c'est bien le cas dans la catégorie des A-modules.

10.4.4 Théorème. *Soit A un anneau. Alors la catégorie $\mathscr{C} = \mathrm{Mod}(A)$ possède assez de projectifs.*

Démonstration : D'abord observons que tout module libre P est projectif. En effet, on peut écrire $P = \oplus_{j \in J} A e_j$ où $(e_j)_{j \in J}$ est une base. Soit une suite exacte $Y \to Z \to 0$ et un morphisme $u : P \to Z$. Par propriété des modules libres, la donnée de u équivaut à la donnée des images $z_j = u(e_j)$. Soit $y_j \in Y$ un élément qui relève z_j, alors le morphisme $v : P \to Y$ défini par $v(e_j) = y_j$ relève u, comme désiré. Or tout module est quotient d'un module libre : en effet, pour un module M on peut prendre le module libre de base M, noté $P = \oplus_{m \in M} A e_m$ et considérer l'application surjective $P \to M$ qui envoie e_m sur m. On en déduit que $\mathscr{C}$ possède assez de projectifs. ☐

$$\begin{tikzcd} & & I \\ 0 \arrow[r] & X \arrow[r] \arrow[ur] & Y \arrow[u, dashed] \end{tikzcd}$$

Tout morphisme depuis un sous-objet s'étend

$$\begin{tikzcd} P \arrow[d, dashed] \arrow[dr] & & \\ Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

Tout morphisme vers un quotient se relève

FIGURE 14 – Propriétés de définition des objets injectifs et projectifs

Pour démontrer que $\mathrm{Mod}(A)$ possède assez d'injectifs, nous utiliserons un important théorème dû à Baer qui permet de simplifier la vérification du fait qu'un module I est injectif en se limitant aux suites exactes $0 \to X \to Y$ dans lequelles $Y = A$ (et donc X est un idéal). Voici l'énoncé :

10.4.5 Théorème (Critère de Baer) *Soit A un anneau. Un A-module I est injectif si et seulement si pour toute suite exacte $0 \to \mathfrak{a} \to A$, tout morphisme $\mathfrak{a} \to I$ s'étend en un morphisme $A \to I$.*

Démonstration : La démonstration est un simple argument d'extension basé sur le lemme de Zorn. Nous renvoyons à Rotman [Rot09, th. 3.30] ou Weibel [Wei94, 2.3.1] pour les détails. ☐

10.4.6 Théorème. *Soit A un anneau. Alors la catégorie $\mathscr{C} = \mathrm{Mod}(A)$ possède assez d'injectifs.*

Démonstration : Supposons d'abord que $A = \mathbb{Z}$. Démontrons que dans ce cas, tout $\mathbb{Z}$-module divisible D est injectif. Pour cela, utilisons le critère de Baer 10.4.5. Considérons une suite exacte $0 \to (n) \to \mathbb{Z}$ et un morphisme $u : (n) \to D$. Comme D est divisible, il existe $d \in D$ tel que $u(n) = nd$; alors le morphisme $v : \mathbb{Z} \to D$ défini par $v(x) = dx$ étend u, ce qui conclut. Maintenant montrons que tout $\mathbb{Z}$-module M se plonge dans un injectif. Pour cela exprimons M comme quotient d'un module libre : $M = (\oplus_{j\in J}\mathbb{Z}e_j)/N$. Celui-ci se plonge dans le module $D = (\oplus_{j\in J}\mathbb{Q}e_j)/N$ qui est divisible car quotient d'un module divisible, voir 4.1.12. Comme un divisible est injectif, ceci conclut.

Passons au cas A quelconque. Dans l'exercice 4.4.11, à tout $\mathbb{Z}$-module D on a associé un A-module $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ où la structure de A-module est donnée par $(af)(x) = f(ax)$ pour tous $a \in A$ et $f : A \to D$, et la lectrice a démontré une adjonction :

$$\mathrm{Hom}_A(X, \mathrm{Hom}_{\mathbb{Z}}(A, D)) = \mathrm{Hom}_{\mathbb{Z}}(X, D).$$

Ceci implique que si D est un $\mathbb{Z}$-module injectif, alors $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ est un A-module injectif : en effet, étant donnés une suite exacte $0 \to X \to Y$ et un morphisme A-linéaire $u : X \to \mathrm{Hom}_{\mathbb{Z}}(A, D)$, par adjonction il lui correspond un morphisme $\mathbb{Z}$-linéaire $u' : X \to D$, qui s'étend en un morphisme $v' : Y \to D$ par $\mathbb{Z}$-injectivité de D, et par adjonction de nouveau fournit une extension $v : Y \to \mathrm{Hom}_{\mathbb{Z}}(A, D)$ pour u. Nous allons utiliser ce fait pour montrer que tout A-module M se plonge dans un injectif. Pour chaque m on définit $\varphi_m : A \to M$, $\varphi_m(a) = am$. L'application $\varphi : M \to \mathrm{Hom}_{\mathbb{Z}}(A, M)$, $m \mapsto \varphi_m$ est A-linéaire et injective car $m = \varphi_m(1)$. D'après la première partie de la démonstration (cas $A = \mathbb{Z}$) on peut plonger M dans un $\mathbb{Z}$-module injectif D, d'où un morphisme composé :

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, D).$$

La dernière flèche est injective par exactitude à gauche de $\mathrm{Hom}_{\mathbb{Z}}(A, -)$, et le but est un A-module injectif d'après ce qui précède, d'où le plongement cherché. $\square$

10.4.7 Exercice (Les adjoints à droite préservent les injectifs). Revisitez la deuxième partie de la démonstration de 10.4.6 pour montrer plus généralement que l'adjoint à droite $G : \mathscr{D} \to \mathscr{C}$ d'un foncteur *exact* $F : \mathscr{C} \to \mathscr{D}$ préserve les objects injectifs : $I \in \mathscr{C}$ injectif $\Rightarrow G(I) \in \mathscr{D}$ injectif.

10.4.8 Définition (Résolutions) Soit M un objet d'une catégorie abélienne $\mathscr{C}$.

- Une *résolution à gauche* de M est un complexe $P_\bullet$ tel que $P_k = 0$ lorsque $k < 0$, avec un morphisme $P_0 \to M$ tel que la suite suivante est exacte :

  $$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{d} M \longrightarrow 0.$$

  C'est une *résolution projective* si tous les P_k sont projectifs.
- Une *résolution à droite* de M est un complexe $I^\bullet$ tel que $I^k = 0$ lorsque $i < 0$, avec un morphisme $M \to I^0$ tel que la suite suivante est exacte :

  $$0 \xrightarrow{d} M \xrightarrow{d} I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \longrightarrow \cdots$$

  C'est une *résolution injective* si tous les I^k sont injectifs.

10.4.9 Corollaire. *Tout A-module M possède une résolution projective*

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

et une résolution injective

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

— objet projectif, objet injectif. Montrer que $\mathrm{Mod}(A)$ en a assez.

— résolution. Homotopies. Foncteurs dérivés.

— δ-foncteur.

10.5 Exactitude (cas non abélien)

Cette sous-section est indépendante du reste du texte et sa lecture n'est pas strictement nécessaire pour le comprendre. Les seules notions dont nous aurons besoin concernant l'exactitude des foncteurs seront développées dans le contexte des catégories abéliennes, dans la sous-section 10.2 qui fournira également les exemples principaux pour nos applications.

Allons-y, pour le bénéfice de la lectrice intéressée.

10.5.1 Préservation des (co)limites. Les produits, produits fibrés, égalisateurs, noyaux apparaissent constamment en algèbre et ailleurs, et c'est donc une information très intéressante de savoir si un foncteur les préserve. Par exemple, dans la catégorie des A-modules, un morphisme $f : N_1 \to N_2$ est un noyau si et seulement s'il est injectif (il est alors le noyau de la projection $N_2 \to N_2/f(N_1)$). Pour un A-module M, demander si le foncteur de produit tensoriel par M préserve les noyaux revient à demander : si un morphisme $N_1 \to N_2$ est injectif, le morphisme induit $N_1 \otimes_A M \to N_1 \otimes_A M$ est-il injectif ? C'est vrai pour certains M et faux pour d'autres. Pour un autre exemple, si une application continue d'espaces topologiques pointés $(X,x) \to (Y,y)$ est injective, le morphisme de groupes fondamentaux $\pi_1(X,x) \to \pi_1(Y,y)$ est-il injectif d'image un sous-groupe distingué ? Dualement, il est très intéressant de savoir si un foncteur préserve les sommes, coégalisateurs, et autres colimites.

La proposition qui suit énonce un fait remarquable : les foncteurs adjoints jouissent de telles propriétés de préservation pour les limites arbitraires ou les colimites arbitraires.

10.5.2 Proposition ([Le14, Th. 6.3.1]). Soit $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ une adjonction. Alors :

1. *Le foncteur F préserve les colimites* : pour tout foncteur $X : I \to \mathscr{C}$ tel que X et $F \circ X$ possèdent des colimites, l'application canonique $\operatorname{colim} FX \to F(\operatorname{colim} X)$ est un isomorphisme.
2. *Le foncteur G préserve les limites* : pour tout foncteur $X : I \to \mathscr{C}$ tel que X et $G \circ X$ possèdent des limites, l'application canonique $G(\lim X) \to \lim GX$ est un isomorphisme.

(On dit aussi que F *commute avec les colimites* et G *commute avec les limites.*)

Si l'application canonique $\operatorname{colim} FX \to F(\operatorname{colim} X)$ de l'énoncé n'est pas claire pour la lectrice, elle pourra lire le détail de sa construction dans la remarque suivante avant de passer à la démonstration.

10.5.3 Remarque. Avec les notations de la proposition, soit $u : X \to \Delta_C$ la colimite de X. Par composition on obtient une transformation naturelle :

$$Fu : FX \longrightarrow F\Delta_C = \Delta_{F(C)}$$

entre foncteurs $I \to \mathscr{D}$. Par propriété universelle de la colimite $v : FX \to \Delta_{\operatorname{colim} FX}$, il existe un unique morphisme $f : \operatorname{colim} FX \to F(C) = F(\operatorname{colim} X)$ tel que $Fu = \Delta_f \circ v$.

Démonstration : Par dualité, il suffit de démontrer 1. Pour tout objet $D \in \mathscr{D}$, on a :

$$\begin{aligned}
\operatorname{Hom}(\operatorname{colim} F(X_i), D) &= \lim \operatorname{Hom}(F(X_i), D) \text{ par déf. 4.5.13(i) de la colimite,}\\
&= \lim \operatorname{Hom}(X_i, G(D)) \text{ par adjonction,}\\
&= \operatorname{Hom}(\operatorname{colim} X_i, G(D)) \text{ par déf. 4.5.13(i) de la colimite,}\\
&= \operatorname{Hom}(F(\operatorname{colim} X_i), D) \text{ par adjonction.}
\end{aligned}$$

Ces isomorphismes étant fonctoriels en D, par le lemme de Yoneda ils proviennent d'un unique isomorphisme $f : \operatorname{colim} F(X_i) \to F(\operatorname{colim} X_i)$. Un petit exercice de déroulement des définitions, fastidieux mais sans difficulté, permet de voir qu'il s'agit du morphisme décrit dans 10.5.3. □

10.5.4 Exercice (Les limites commutent avec les limites (et idem avec co-)). Soient I, J deux petites catégories. Supposons que dans $\mathscr{C}$, les limites de tous les foncteurs indicés par I ou J existent. Démontrez qu'alors les limites de tous les foncteurs $X : I \times J \to \mathscr{C}$ existent, et que l'on a des isomorphismes canoniques :

$$\lim_{i\in I}\lim_{j\in J} X_{i,j} \xrightarrow{\sim} \lim_{(i,j)\in I\times J} X_{i,j} \xleftarrow{\sim} \lim_{j\in J}\lim_{i\in I} X_{i,j}.$$

(*Indication : procédez soit par calcul direct, soit par application de la proposition 10.5.2.*)

10.5.5 Préservation des (co)limites finies. Les propriétés de 10.5.2 sont d'autant plus utiles que les adjoints abondent, comme nous l'avons vu dans la section 4.4. Il existe cependant de nombreux exemples de foncteurs qui ont un comportement intermédiaire : ils ne préservent pas les colimites arbitraires (resp. les limites arbitraires), mais préservent celles qui sont indicées par des catégories qui sont *finies* c'est-à-dire avec un nombre d'objets et de morphismes *finis*. Pour des exemples, nous renvoyons la lectrice aux exercices 10.2.12, 10.2.13, 10.2.14.

La préservation des limites ou colimites *finies* est de fait assez courante en algèbre et en géométrie, car les objets qu'on y étudie sont la plupart du temps définis par un nombre fini de données (par exemple : un ensemble muni de certaines lois internes...), satisfaisant un nombre fini d'axiomes (associativité, commutativité...), ou possédant des propriétés de finitude intrinsèques (être une algèbre de type fini sur un corps, être défini comme lieu de zéros d'un nombre fini de polynômes...). Non contente d'être assez courante, cette propriété est assez utile, pour les mêmes raisons que précédemment : le plus souvent les (co)limites qui apparaissent en situation portent sur un nombre fini d'objets, vérifiant un nombre fini de relations entre eux ; bref, ce sont des (co)limites finies. En résumé, l'omniprésence de la finitude en géométrie algébrique amène naturellement la définition que voici.

10.5.6 Définition. Soit $F : \mathscr{C} \to \mathscr{D}$ un foncteur. On dit que F est :

- *exact à gauche* s'il préserve les limites finies,
- *exact à droite* s'il préserve les colimites finies.

Il n'est pas apparent à ce stade que les foncteurs exacts à droite ou à gauche ont des propriétés particulières et qu'il est possible de développer un arsenal de techniques très puissantes pour les étudier. Nous verrons ceci dans la deuxième partie du cours, dans le cas de foncteurs entre certaines catégories dites *abéliennes* (voir définition 10.2.1), où ces techniques portent le nom de *cohomologie*.

10.5.7 Des réciproques. Nous avons découvert trois propriétés importantes, ainsi reliées :

F possède un adjoint à gauche $\overset{\text{(i)}}{\Longrightarrow}$ F préserve les limites $\overset{\text{(ii)}}{\Longrightarrow}$ F est exact à gauche.

(On dispose bien sûr d'implications en version « à droite »). Pour terminer cette courte discussion de l'exactitude, voici quelques indications sur les implications réciproques.

(i) Sous certaines hypothèses qui affirment que les catégories en jeu ne sont pas « trop grosses », des théorèmes connus sous les noms de GAFT et SAFT (General / Special Adjoint Functor Theorem) établissent qu'un foncteur qui préserve les limites est un adjoint à gauche. Ces hypothèses sont peu contraignantes, de sorte qu'il est sage de garder en mémoire que dans la pratique le résultat est toujours vrai. Il est à noter toutefois que la construction des adjoints fournie par la démonstration de GAFT et SAFT n'a pas d'intérêt autre que pour établir l'existence ; même dans les cas où l'adjoint est connu, on peine à le reconnaître dans cette construction. On renvoie à [Le14, Th. 6.3.10 et 6.3.13] où ceci est discuté.

(ii) Ici aussi, on dispose d'un résultat permettant de comprendre l'implication réciproque :

10.5.8 Proposition. *Soit $F : \mathscr{C} \to \mathscr{D}$ un foncteur exact à gauche (c'est-à-dire qui préserve les limites finies). Si F préserve les produits, alors F préserve toutes les limites.*

Démonstration : [Le14, Prop. 5.1.26]. $\square$

Exemple, le foncteur sections globales, est exact à gauche mais pas à droite, c'est cette obstruction à relever globalement des choses qui se relèvent localement qui est riche géométriquement. C'est le point de départ de la cohomologie, qu'on verra dans la 2ème partie du cours.

Dans cet exemple, la catégorie en jeu est particulière, on dit *abélienne*, comme le sont les catégories de groupes abéliens ou modules, et faisceaux de groupes abéliens ou modules. Dans ces catégories, un foncteur additif est exact à gauche ssi il préserve les noyaux i.e. les injections (cf MacLane page 201). C'est la définition habituelle de foncteur exact à gauche.

11 Cohomologie de Čech

Soit X un espace topologique. Soit $\mathcal{U} = (U_i)_{i\in I}$ un recouvrement ouvert de X. Pour chaque ensemble d'indices $\{i_0, \dots, i_p\}$ de I on notera $U_{i_0,\dots,i_p} := U_{i_0} \cap \dots \cap U_{i_p}$.

11.0.1 Définition. Soit $\mathscr{F} \in \mathrm{Sh}(X, \mathrm{Ab})$ un faisceau de groupes abéliens sur X. Fixons un ordre total sur I. On définit un complexe de groupes abéliens $C^\bullet(\mathcal{U}, \mathscr{F})$ en posant, pour tout $p \geqslant 0$:

$$C^p(\mathcal{U}, \mathscr{F}) = \prod_{i_0<\dots<i_p} \mathscr{F}(U_{i_0,\dots,i_p}).$$

avec les différentielles $d = d^p : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ définies pour $\alpha = (\alpha_{i_0,\dots,i_p}) \in C^p(\mathcal{U}, \mathscr{F})$ par :

$$(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}.$$

11.0.2 Remarques.

1. Deux précisions sur la formule dans le membre de droite :
   (a) pour simplifier nous avons noté $\alpha_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}$ au lieu de sa restriction à l'ouvert $U_{i_0,\dots,i_{p+1}}$,
   (b) le chapeau signifie que l'indice situé dessous est omis.
2. Par exemple : $(d\alpha)_{i,j} = \alpha_j - \alpha_i$ si $\alpha \in C^0$ et $(d\alpha)_{i,j,k} = \alpha_{j,k} - \alpha_{i,k} + \alpha_{i,j}$ si $\alpha \in C^1$.
3. Il existe une variante utile, le complexe de Čech « total » $C^\bullet_{\mathrm{tot}}(\mathcal{U}, \mathscr{F})$ dans lequel C^p est indicé par *tous* les uplets $(i_0, \dots, i_p) \in I^{p+1}$. Les deux approches sont essentiellement équivalentes ; pour la comparaison entre les deux, voir [Har77, III, Rem. 4.0.1] ou [SP23, Tag 01ED] et [SP23, Tag 01FG]. Avantages respectifs des deux complexes :
   (a) le complexe ordonné est plus petit et donc plus agréable pour les calculs explicites ; dans les exemples I sera un ensemble fini $\{1, \dots, N\}$ muni de son ordre naturel ;
   (b) le complexe total est plus intrinsèque, donc plus facile à utiliser pour certaines questions théoriques, par exemple lorsqu'on considère des raffinements de recouvrements.

11.0.3 Lemme. *$C^\bullet(\mathcal{U}, \mathscr{F})$ est un complexe, i.e. $d \circ d = 0$.*

Démonstration : Soit $\beta = d\alpha = (\beta_{i_0,\dots,i_{p+1}})$ avec $\beta_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k\alpha_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}$. On calcule :

$$\begin{aligned}
(d\beta)_{i_0,\dots,i_{p+2}} &= \sum_{k=0}^{p+2}(-1)^k\beta_{i_0,\dots,\widehat{i_k},\dots,i_{p+2}} \\
&= \sum_{k=0}^{p+2}(-1)^k\left(\sum_{l<k}(-1)^l\alpha_{i_0,\dots,\widehat{i_l},\dots,\widehat{i_k},\dots,i_{p+2}} + \sum_{l>k}(-1)^{l-1}\alpha_{i_0,\dots,\widehat{i_k},\dots,\widehat{i_l},\dots,i_{p+2}}\right) \\
&= \sum_{k<l}((-1)^{k+l}+(-1)^{k+l-1})\,\alpha_{i_0,\dots,\widehat{i_k},\dots,\widehat{i_l},\dots,i_{p+2}} \\
&= 0.
\end{aligned}$$

Ceci montre que $(d\circ d)(\alpha)=0$. □

Par exemple, pour $d^1\circ d^0$ nous avons $\beta = d\alpha$, $\beta_{i,j} = \alpha_j - \alpha_i$ puis

$$(d\beta)_{i,j,k} = \beta_{j,k} - \beta_{i,k} + \beta_{i,j} = (\alpha_k-\alpha_j) - (\alpha_k-\alpha_i) + (\alpha_j-\alpha_i) = 0.$$

11.0.4 Définition. Avec les notations précédentes, on appelle *p-ième groupe de cohomologie de Čech de X à coefficients dans $\mathscr{F}$ relatif au recouvrement $\mathcal{U}$* le groupe

$$\check{\mathrm{H}}^p(\mathcal{U},\mathscr{F}) = h^p(C^\bullet(\mathcal{U},\mathscr{F})).$$

11.0.5 Exemple. Soient A un anneau et $n\in\mathbb{Z}$ un entier. Calculons la cohomologie de la droite projective $X=\mathbb{P}^1_A$ à coefficients dans le $\mathcal{O}_X$-module $\mathscr{F}=\mathcal{O}(n)$, relativement au recouvrement standard $\mathcal{U}=\{U_0,U_1\}$. Rappelons que celui-ci est composé des deux droites affines $U_0=\mathrm{Spec}(A[x])$ et $U_1=\mathrm{Spec}(A[y])$ avec $y=1/x$ (dans les notations de 6.3.7, on a $x=t_1/t_0$ et $y=t_0/t_1$). On a :

- $C^0 = \Gamma(U_0,\mathscr{F})\times\Gamma(U_1,\mathscr{F}) = (t_0)^nA[t_1/t_0]\times(t_1)^nA[t_0/t_1]$,
- $C^1 = \Gamma(U_0\cap U_1,\mathscr{F}) = (t_0)^nA[t_1/t_0,t_0/t_1]$,
- $d: C^0\to C^1$, $((t_0)^nf(t_1/t_0),(t_1)^ng(t_0/t_1))\mapsto (t_1)^ng(t_0/t_1)-(t_0)^nf(t_1/t_0)$.

Calculons $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{O}(n))=\ker(d)$. Si $n\geqslant 0$, l'égalité $(t_1)^ng(t_0/t_1)=(t_0)^nf(t_1/t_0)$ implique que $(t_1)^ng(t_0/t_1)$ est un polynôme en t_1 donc $\deg(g)\leqslant n$, puis que $f(x)=x^ng(1/x)$. Ceci montre que $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{O}(n)) = A[t_0,t_1]_{\deg=n}$ qui est un A-module libre de rang $n+1$. Si $n=-m<0$, cette même égalité fournit $(t_0)^mg(t_0/t_1)=(t_1)^mf(t_1/t_0)$. On voit qu'il ne peut exister de monôme $\mu\neq 0$ dans cette écriture, car en regardant le membre de gauche (resp. de droite) on aurait $\deg_{t_0}(\mu)>0$, resp $\deg_{t_0}(\mu)\leqslant 0$. Dans ce cas $f=g=0$ et $\check{\mathrm{H}}^0(\mathcal{U},\mathcal{O}(n))=0$.

Calculons $\check{\mathrm{H}}^1(\mathcal{U},\mathcal{O}(n))=\mathrm{coker}(d)$. Le A-module libre C^1 a pour base l'ensemble des monômes $(t_0)^i(t_1)^j$ avec $i,j\in\mathbb{Z}$ et $i+j=n$. L'image de d est le sous-module libre de base le sous-ensemble des monômes tels que $i\geqslant 0$ ou $j\geqslant 0$ (puisque f,g sont des *polynômes*). Si $n\geqslant 0$, le conoyau de d est donc nul. Si $n=-m<0$, le conoyau est égal au module libre de base $\{(t_0)^{-1}(t_1)^{-m+1},(t_0)^{-2}(t_1)^{-m+2},\dots,(t_0)^{-m+1}(t_1)^{-1}\}$ qui est de rang $m-1=-n-1$. Voir figure 15.

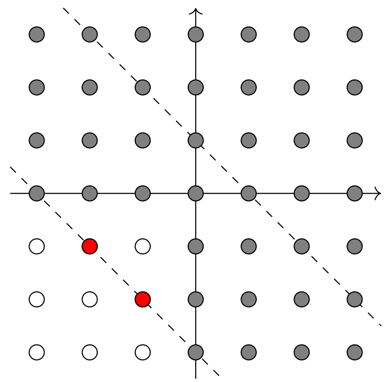


FIGURE 15 – Base du module $\check{\mathrm{H}}^1(\mathcal{U},\mathcal{O}(n))=\mathrm{coker}(d:C^0\to C^1)$

En degré 0, la cohomologie de Čech est explicite et fournit le « bon » groupe de cohomologie :

11.0.6 Lemme. *Pour tout faisceau abélien $\mathscr{F}$ sur X et tout recouvrement ouvert $\mathcal{U}$, on a :*

$$\check{\mathrm{H}}^0(\mathcal{U},\mathscr{F}) = \Gamma(X,\mathscr{F}).$$

Démonstration : Par définition $\check{\mathrm{H}}^0(\mathcal{U},\mathscr{F})=\ker(d:C^0\to C^1)$. Un élément de $\ker(d)$ est une famille $(\alpha_i)_{i\in I}$, $\alpha_i\in\mathscr{F}(U_i)$, telle que $\alpha_j-\alpha_i=0$ pour tous i,j. D'après la condition de faisceau, ceci implique que les α_i se recollent en une unique section $\alpha\in\mathscr{F}(X)$. Ceci conclut. □

En degré 1, la cohomologie de Čech d'un recouvrement fixé peut être pathologique, mais la situation s'améliore si l'on prend en compte tous les recouvrements ouverts possibles de X. Pour ceci, on appelle *raffinement* de $\mathcal{U}$ un recouvrement ouvert $\mathcal{V} = (V_j)_{j \in J}$ muni d'une application $\lambda : J \to I$ telle que $V_j \subset U_{\lambda(j)}$ pour tout $j \in J$. (Si on travaille avec les complexes ordonnés, I et J sont munis d'ordres totaux et on doit supposer λ croissante.) On note alors $\mathcal{V} \prec \mathcal{U}$ ce qui définit une relation d'ordre sur l'ensemble des recouvrements ouverts de X. On définit alors le *p-ième groupe de cohomologie de Čech de X à coefficients dans $\mathscr{F}$* par :

$$\check{\mathrm{H}}^p(X, \mathscr{F}) := \operatorname*{colim}_{\mathcal{U}} \check{\mathrm{H}}^p(\mathcal{U}, \mathscr{F}).$$

Dans la sous-section suivante, on constuira un morphisme canonique

$$\mathrm{can} : \check{\mathrm{H}}^p(X, \mathscr{F}) \longrightarrow \mathrm{H}^p(X, \mathscr{F})$$

vers le p-ième groupe de cohomologie au sens des foncteurs dérivés. Dans certains cas favorables, ce morphisme sera un isomorphisme, par exemple si $p = 1$ (voir [Har77, II, Exercice 4.4] ou [SP23, Tag 09V0]) ou si X est un schéma séparé, voir ? ?.

Comment construire le morphisme can ? Le point de départ pour faire le lien avec la cohomologie des foncteurs dérivés sera la *résolution de Čech* obtenue en faisceautisant le complexe de Čech.

11.0.7 Définition. On note $\mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$ le complexe de faisceaux abéliens défini par :

$$\mathscr{C}^p(\mathcal{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} f_*(\mathscr{F}_{|U_{i_0,\dots,i_p}})$$

où $f : U_{i_0,\dots,i_p} \hookrightarrow X$ est l'inclusion ouverte, avec différentielle $d : \mathscr{C}^p \to \mathscr{C}^{p+1}$ définie par la même formule que précédemment. On dispose d'un morphisme $\varepsilon : \mathscr{F} \to \mathscr{C}^0(\mathcal{U}, \mathscr{F})$ obtenu en prenant le produit des morphismes d'adjonction $\mathscr{F} \to f_*(\mathscr{F}_{|U_i})$.

11.0.8 Lemme. *Le complexe $\mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$ est une résolution de $\mathscr{F}$, c'est-à-dire qu'on a une suite exacte :*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathcal{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathcal{U}, \mathscr{F}) \longrightarrow \mathscr{C}^2(\mathcal{U}, \mathscr{F}) \longrightarrow \dots$$

Démonstration :

□

11.1 Cohomologie de Čech des schémas affines

11.2 Cohomologie de Čech de l'espace projectif

Références

[AM04] Éric Amar, Étienne Matheron, *Analyse complexe*, Cassini, 2004.

[Aw06] Steve Awodey, *Category Theory*, Oxford University Press, 2006.

[CL21] Antoine Chambert-Loir, *(Mostly) Commutative Algebra*, Universitext, Springer, 2021.

[DPP10] Apóstolos Doxiádis, Christos Papadimitriou, Alecos Papadatos, *Logicomix*, Vuibert, 2010.

[DW82] Richard Dedekind, Heinrich Weber, *Theorie der algebraischen Functionen einer Veränderlichen*, J. Reine Angew. Math. 92 (1882), 181–290. Édition en langue anglaise : *Theory of algebraic functions of one variable*, Hist. Math., 39. American Mathematical Society, 2012.

[EH00] David Eisenbud, Joe Harris, *The geometry of schemes*, Graduate Texts in Math., 197. Springer-Verlag, 2000.

[Ei95] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Math., Springer, 1995.

[Fr64] Peter Freyd, *Abelian categories*, Harper & Row, 1964.

[Gl81] Georges Glaeser, *Epistémologie des nombres relatifs*, Recherches en didactique des mathématiques 2(3), 303-346, 1981.

[GM03] S. Gelfand, Y. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics, Springer, 2003. Une explication plus complète sur l'épigraphe de ce livre se trouve ici : `https://mathoverflow.net/questions/453656/who-wrote-if-only-i-could-understand-the-equation-d2-0`.

[GW20] Ulrich Görtz, Torsten Wedhorn, *Algebraic Geometry 1, Schemes with examples and exercises*, Springer Studium Mathematik, Springer Spektrum, 2020.

[Hal67] Paul Halmos, *Introduction à la théorie des ensembles*, Gauthier-Villars, 1967.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Math. 52, Springer-Verlag, 1977.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002. Accessible à l'adresse `https://pi.math.cornell.edu/~hatcher/`.

[Hi93] Marc Hindry, *Géométrie arithmétique*, Chabert, Jean-Luc (ed.) et al., Analyse diophantienne et géométrie algébrique. Exposés du Séminaire d'Histoire des Mathématiques de l'Institut Henri Poincaré, Cah. Sémin. Hist. Math., 2. Sér. 3, 79-84, 1993.

[Kr82] Leopold Kronecker, *Grundzüge einer arithmetischen Theorie der algebraische Grössen*, J. Reine Angew. Math. 92 (1882), 1–122.

[La99] Tsit Yuen Lam, *Lectures on modules and rings*, Graduate texts in mathematics 189, Springer, 1999.

[Le14] Tom Leinster, *Basic category theory*, Cambridge Studies in Advanced Math. 143, Cambridge University Press, 2014.

[Liu06] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Math. 6, Oxford University Press, 2006.

[LS22] Bernard Le Stum, *Algèbre commutative et géométrie algébrique*, cours de master 1, Université de Rennes, 2002. Polycopié accessible à l'adresse `https://perso.univ-rennes1.fr/bernard.le-stum/bernard.le-stum/Enseignement_files/ACGA.pdf`.

[Mac78] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Math. no. 5, Springer-Verlag, 1978.

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Math. 8, Cambridge University Press, 1989.

[Mu99] David Mumford, *The red book of varieties and schemes*, Second edition, Lecture Notes in Mathematics 1358, Springer-Verlag, 1999.

[Na62] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics 13, Interscience Publishers, John Wiley & Sons, 1962.

[Pe01] Daniel Perrin, *Géométrie Algébrique, une introduction*, EDP Sciences - CNRS éditions, 2001. Édition en anglais : *Algebraic geometry. An introduction*, Universitext, Springer, EDP Sciences, 2008.

[Ra22] Lorenzo Ramero, *Grimoire d'algèbre commutative*, Les Presses insousmises, 2022. Accessible à l'adresse `https://pro.univ-lille.fr/fileadmin/user_upload/pages_pros/lorenzo_ramero/The-Grimoire-Project.html`.

[Rot09] J. Rotman, *An introduction to homological algebra*, 2nd ed, Universitext, Springer, 2009.

[SP23] The Stacks Project Authors, *Stacks Project*, located at `http://www.math.columbia.edu/algebraic_geometry/stacks-git`.

[Wed16] Torsten Wedhorn, *Manifolds, Sheaves, and Cohomology*, Springer Studium Mathematik, Springer Spektrum, 2016.

[We50] André Weil, *Number-theory and algebraic geometry*, Proc. Internat. Congr. Math. (Cambridge, Mass.), 1950, vol. 2, 90-100 (1952).

[Wei94] Ch. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.

CHAPTER III

Cohomology

In this chapter we define the general notion of cohomology of a sheaf of abelian groups on a topological space, and then study in detail the cohomology of coherent and quasi-coherent sheaves on a noetherian scheme.

Although the end result is usually the same, there are many different ways of introducing cohomology. There are the fine resolutions often used in several complex variables—see Gunning and Rossi [1]; the Čech cohomology used by Serre [3], who first introduced cohomology into abstract algebraic geometry; the canonical flasque resolutions of Godement [1]; and the derived functor approach of Grothendieck [1]. Each is important in its own way.

We will take as our basic definition the derived functors of the global section functor (§1, 2). This definition is the most general, and also best suited for theoretical questions, such as the proof of Serre duality in §7. However, it is practically impossible to calculate, so we introduce Čech cohomology in §4, and use it in §5 to compute explicitly the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space $\mathbf{P}^r$. This calculation is the basis of many later results on projective varieties.

In order to prove that the Čech cohomology agrees with the derived functor cohomology, we need to know that the higher cohomology of a quasi-coherent sheaf on an affine scheme is zero. We prove this in §3 in the noetherian case only, because it is technically much simpler than the case of an arbitrary affine scheme ([EGA III, §1]). Hence we are bound to include noetherian hypotheses in all theorems involving cohomology.

As applications, we show for example that the arithmetic genus of a projective variety X, whose definition in (I, §7) depended on a projective embedding of X, can be computed in terms of the cohomology groups $H^i(X,\mathcal{O}_X)$, and hence is intrinsic (Ex. 5.3). We also show that the arithmetic genus is constant in a family of normal projective varieties (9.13).

Another application is Zariski's main theorem (11.4) which is important in the birational study of varieties.

The latter part of the chapter (§8–12) is devoted to families of schemes, i.e., the study of the fibres of a morphism. In particular, we include a section on flat morphisms and a section on smooth morphisms. While these can be treated without cohomology, it seems to be an appropriate place to include them, because flatness can be understood better using cohomology (9.9).

1 Derived Functors

In this chapter we will assume familiarity with the basic techniques of homological algebra. Since notation and terminology vary from one source to another, we will assemble in this section (without proofs) the basic definitions and results we will need. More details can be found in the following sources: Godement [1, esp. Ch. I, §1.1–1.8, 2.1–2.4, 5.1–5.3], Hilton and Stammbach [1, Ch. II,IV,IX], Grothendieck [1, Ch. II, §1,2,3], Cartan and Eilenberg [1, Ch. III,V], Rotman [1, §6].

Definition. An *abelian category* is a category $\mathfrak{A}$, such that: for each $A,B \in \operatorname{Ob} \mathfrak{A}$, $\operatorname{Hom}(A,B)$ has a structure of an abelian group, and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism. (Hilton and Stammbach [1, p. 78].)

The following are all abelian categories.

Example 1.0.1. $\mathfrak{Ab}$, the category of abelian groups.

Example 1.0.2. $\mathfrak{Mod}(A)$, the category of modules over a ring A (commutative with identity as always).

Example 1.0.3. $\mathfrak{Ab}(X)$, the category of sheaves of abelian groups on a topological space X.

Example 1.0.4. $\mathfrak{Mod}(X)$, the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X,\mathcal{O}_X)$.

Example 1.0.5. $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X (II, 5.7).

Example 1.0.6. $\mathfrak{Coh}(X)$, the category of coherent sheaves of $\mathcal{O}_X$-modules on a noetherian scheme X (II, 5.7).

Example 1.0.7. $\mathfrak{Coh}(\mathfrak{X})$, the category of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ (II, 9.9).

In the rest of this section, we will be stating some basic results of homological algebra in the context of an arbitrary abelian category. However, in most books, these results are proved only for the category of modules over a ring, and proofs are often done by "diagram-chasing": you pick an element and chase its images and pre-images through a diagram. Since diagram-chasing doesn't make sense in an arbitrary abelian category, the conscientious reader may be disturbed. There are at least three ways to handle this difficulty. (1) Provide intrinsic proofs for all the results, starting from the axioms of an abelian category, and without even mentioning an element. This is cumbersome, but can be done—see, e.g., Freyd [1]. Or (2), note that in each of the categories we use (most of which are in the above list of examples), one can in fact carry out proofs by diagram-chasing. Or (3), accept the "full embedding theorem" (Freyd [1, Ch. 7]), which states roughly that any abelian category is equivalent to a subcategory of $\mathfrak{Ab}$. This implies that any category-theoretic statement (e.g., the 5-lemma) which can be proved in $\mathfrak{Ab}$ (e.g., by diagram-chasing) also holds in any abelian category.

Now we begin our review of homological algebra. A *complex* $A^\cdot$ in an abelian category $\mathfrak{A}$ is a collection of objects A^i, $i \in \mathbf{Z}$, and morphisms $d^i : A^i \to A^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i. If the objects A^i are specified only in a certain range, e.g., $i \geqslant 0$, then we set $A^i = 0$ for all other i. A *morphism* of complexes, $f : A^\cdot \to B^\cdot$ is a set of morphisms $f^i : A^i \to B^i$ for each i, which commute with the coboundary maps d^i.

The ith *cohomology object* $h^i(A^\cdot)$ of the complex $A^\cdot$ is defined to be $\ker d^i/\operatorname{im} d^{i-1}$. If $f : A^\cdot \to B^\cdot$ is a morphism of complexes, then f induces a natural map $h^i(f) : h^i(A^\cdot) \to h^i(B^\cdot)$. If $0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$ is a short exact sequence of complexes, then there are natural maps $\delta^i : h^i(C^\cdot) \to h^{i+1}(A^\cdot)$ giving rise to a long exact sequence

$$\dots \to h^i(A^\cdot) \to h^i(B^\cdot) \to h^i(C^\cdot) \xrightarrow{\delta^i} h^{i+1}(A^\cdot) \to \dots$$

Two morphisms of complexes $f,g : A^\cdot \to B^\cdot$ are *homotopic* (written $f \sim g$) if there is a collection of morphisms $k^i : A^i \to B^{i-1}$ for each i (which need not commute with the d^i) such that $f - g = dk + kd$. The collection of morphisms, $k = (k^i)$ is called a *homotopy operator.* If $f \sim g$, then f and g induce the *same* morphism $h^i(A^\cdot) \to h^i(B^\cdot)$ on the cohomology objects, for each i.

A covariant functor $F : \mathfrak{A} \to \mathfrak{B}$ from one abelian category to another is *additive* if for any two objects A,A' in $\mathfrak{A}$, the induced map $\operatorname{Hom}(A,A') \to \operatorname{Hom}(FA,FA')$ is a homomorphism of abelian groups. F is *left exact* if it is additive and for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathfrak{A}$, the sequence

$$0 \to FA' \to FA \to FA''$$

is exact in $\mathfrak{B}$. If we can write a 0 on the right instead of the left, we say F is *right exact.* If it is both left and right exact, we say it is *exact.* If only the middle part $FA' \to FA \to FA''$ is exact, we say F is *exact in the middle.*

For a contravariant functor we make analogous definitions. For example, $F : \mathfrak{A} \to \mathfrak{B}$ is *left exact* if it is additive, and for every short exact sequence as above, the sequence

$$0 \to FA'' \to FA \to FA'$$

is exact in $\mathfrak{B}$.

Example 1.0.8. If $\mathfrak{A}$ is an abelian category, and A is a fixed object, then the functor $B \to \operatorname{Hom}(A,B)$, usually denoted $\operatorname{Hom}(A,\cdot)$, is a covariant left exact functor from $\mathfrak{A}$ to $\mathfrak{Ab}$. The functor $\operatorname{Hom}(\cdot,A)$ is a contravariant left exact functor from $\mathfrak{A}$ to $\mathfrak{Ab}$.

Next we come to resolutions and derived functors. An object I of $\mathfrak{A}$ is *injective* if the functor $\operatorname{Hom}(\cdot,I)$ is exact. An *injective resolution* of an object A of $\mathfrak{A}$ is a complex $I^\cdot$, defined in degrees $i \geqslant 0$, together with a morphism $\varepsilon : A \to I^0$, such that I^i is an injective object of $\mathfrak{A}$ for each $i \geqslant 0$, and such that the sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \dots$$

is exact.

If every object of $\mathfrak{A}$ is isomorphic to a subobject of an injective object of $\mathfrak{A}$, then we say $\mathfrak{A}$ *has enough injectives.* If $\mathfrak{A}$ has enough injectives, then every object has an injective resolution. Furthermore, a well-known lemma states that any two injective resolutions are homotopy equivalent.

Now let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor. Then we construct the *right derived functors* R^iF, $i \geqslant 0$, of F as follows. For each object A of $\mathfrak{A}$, choose once and for all an injective resolution $I^\cdot$ of A. Then we define $R^iF(A) = h^i(F(I^\cdot))$.

Theorem 1.1A. *Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor to another abelian category $\mathfrak{B}$. Then*

(a) *For each $i \geqslant 0$, R^iF as defined above is an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. Furthermore, it is independent (up to natural isomorphism of functors) of the choices of injective resolutions made.*

(b) *There is a natural isomorphism $F \cong R^0F$.*

(c) *For each short exact sequence $0 \to A' \to A \to A'' \to 0$ and for each $i \geqslant 0$ there is a natural morphism $\delta^i : R^iF(A'') \to R^{i+1}F(A')$, such that we obtain a long exact sequence*

$$\dots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \dots.$$

(d) *Given a morphism of the exact sequence of* (c) *to another* $0 \to B' \to B \to B'' \to 0$, *the* δ*'s give a commutative diagram*

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^{i+1}F(B'). \end{array}$$

(e) *For each injective object* I *of* $\mathfrak{A}$, *and for each* $i > 0$, *we have* $R^iF(I) = 0$.

Definition. With $F:\mathfrak{A} \to \mathfrak{B}$ as in the theorem, an object J of $\mathfrak{A}$ is *acyclic* for F if $R^iF(J) = 0$ for all $i > 0$.

Proposition 1.2A. *With* $F:\mathfrak{A} \to \mathfrak{B}$ *as in* (1.1A), *suppose there is an exact sequence*

$$0 \to A \to J^0 \to J^1 \to \dots$$

where each J^i *is acyclic for* F, $i \geqslant 0$. (*We say* $J^{\cdot}$ *is an* F-acyclic resolution *of* A.) *Then for each* $i \geqslant 0$ *there is a natural isomorphism* $R^iF(A) \cong h^i(F(J^{\cdot}))$.

We leave to the reader the analogous definitions of projective objects, projective resolutions, an abelian category having enough projectives, and the left derived functors of a covariant right exact functor. Also, the right derived functors of a left exact contravariant functor (use projective resolutions) and the left derived functors of a right exact contravariant functor (use injective resolutions).

Next we will give a universal property of derived functors. For this purpose, we generalize slightly with the following definition.

Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories. A (*covariant*) δ*-functor* from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T = (T^i)_{i \geqslant 0}$, together with a morphism $\delta^i: T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$, and each $i \geqslant 0$, such that:

(1) For each short exact sequence as above, there is a long exact sequence

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to \dots$$

$$\dots \to T^i(A) \to T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \to T^{i+1}(A) \to \dots;$$

(2) for each morphism of one short exact sequence (as above) into another $0 \to B' \to B \to B'' \to 0$, the δ's give a commutative diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B'). \end{array}$$

Definition. The δ-functor $T = (T^i):\mathfrak{A} \to \mathfrak{B}$ is said to be *universal* if, given any other δ-functor $T' = (T'^i):\mathfrak{A} \to \mathfrak{B}$, and given any morphism of functors $f^0: T^0 \to T'^0$, there exists a unique sequence of morphisms $f^i: T^i \to T'^i$ for each $i \geqslant 0$, starting with the given f^0, which commute with the δ^i for each short exact sequence.

Remark 1.2.1. If $F:\mathfrak{A} \to \mathfrak{B}$ is a covariant additive functor, then by definition there can exist at most one (up to unique isomorphism) universal δ-functor T with $T^0 = F$. If T exists, the T^i are sometimes called the *right satellite functors* of F.

Definition. An additive functor $F:\mathfrak{A} \to \mathfrak{B}$ is *effaceable* if for each object A of $\mathfrak{A}$, there is a monomorphism $u:A \to M$, for some M, such that $F(u) = 0$. It is *coeffaceable* if for each A there exists an epimorphism $u:P \to A$ such that $F(u) = 0$.

Theorem 1.3A. *Let* $T = (T^i)_{i \geqslant 0}$ *be a covariant* δ*-functor from* $\mathfrak{A}$ *to* $\mathfrak{B}$. *If* T^i *is effaceable for each* $i > 0$, *then* T *is universal.*

PROOF. Grothendieck [1, II, 2.2.1]

Corollary 1.4. *Assume that* $\mathfrak{A}$ *has enough injectives. Then for any left exact functor* $F:\mathfrak{A} \to \mathfrak{B}$, *the derived functors* $(R^iF)_{i \geqslant 0}$ *form a universal* δ*-functor with* $F \cong R^0F$. *Conversely, if* $T = (T^i)_{i \geqslant 0}$ *is any universal* δ*-functor, then* T^0 *is left exact, and the* T^i *are isomorphic to* R^iT^0 *for each* $i \geqslant 0$.

PROOF. If F is a left exact functor, then the $(R^iF)_{i \geqslant 0}$ form a δ-functor by (1.1A). Furthermore, for any object A, let $u:A \to I$ be a monomorphism of A into an injective. Then $R^iF(I) = 0$ for $i > 0$ by (1.1A), so $R^iF(u) = 0$. Thus R^iF is effaceable for each $i > 0$. It follows from the theorem that (R^iF) is universal.

On the other hand, given a universal δ-functor T, we have T^0 left exact by the definition of δ-functor. Since $\mathfrak{A}$ has enough injectives, the derived functors R^iT^0 exist. We have just seen that (R^iT^0) is another universal δ-functor. Since $R^0T^0 = T^0$, we find $R^iT^0 \cong T^i$ for each i, by (1.2.1).

2 Cohomology of Sheaves

In this section we define cohomology of sheaves by taking the derived functors of the global section functor. Then as an application of general techniques of cohomology we prove Grothendieck's theorem about the vanishing of cohomology on a noetherian topological space. To begin with, we must verify that the categories we use have enough injectives.

Proposition 2.1A. *If* A *is a ring, then every* A*-module is isomorphic to a submodule of an injective* A*-module.*

PROOF. Godement [1, I, 1.2.2] or Hilton and Stammbach [1, I, 8.3].

Proposition 2.2. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

Proof. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{x,X}$-module. Therefore there is an injection $\mathcal{F}_x \to I_x$, where I_x is an injective $\mathcal{O}_{x,X}$-module (2.1A). For each point x, let j denote the inclusion of the one-point space $\{x\}$ into X, and consider the sheaf $\mathcal{I} = \prod_{x \in X} j_*(I_x)$. Here we consider I_x as a sheaf on the one-point space $\{x\}$, and j_* is the direct image functor (II, §1).

Now for any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{I}) = \prod \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G},j_*(I_x))$ by definition of the direct product. On the other hand, for each point $x \in X$, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G},j_*(I_x)) \cong \operatorname{Hom}_{\mathcal{O}_{x,X}}(\mathcal{G}_x,I_x)$ as one sees easily. Thus we conclude first that there is a natural morphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{I}$ obtained from the local maps $\mathcal{F}_x \to I_x$. It is clearly injective. Second, the functor $\operatorname{Hom}_{\mathcal{O}_X}(\cdot,\mathcal{I})$ is the direct product over all $x \in X$ of the stalk functor $\mathcal{G} \mapsto \mathcal{G}_x$, which is exact, followed by $\operatorname{Hom}_{\mathcal{O}_{x,X}}(\cdot,I_x)$, which is exact, since I_x is an injective $\mathcal{O}_{x,X}$-module. Hence $\operatorname{Hom}(\cdot,\mathcal{I})$ is an exact functor, and therefore $\mathcal{I}$ is an injective $\mathcal{O}_X$-module.

Corollary 2.3. *If X is any topological space, then the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

Proof. Indeed, if we let $\mathcal{O}_X$ be the constant sheaf of rings $\mathbf{Z}$, then $(X,\mathcal{O}_X)$ is a ringed space, and $\mathfrak{Mod}(X) = \mathfrak{Ab}(X)$.

Definition. Let X be a topological space. Let $\Gamma(X,\cdot)$ be the global section functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We define the *cohomology functors* $H^i(X,\cdot)$ to be the right derived functors of $\Gamma(X,\cdot)$. For any sheaf $\mathcal{F}$, the groups $H^i(X,\mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$. Note that even if X and $\mathcal{F}$ have some additional structure, e.g., X a scheme and $\mathcal{F}$ a quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathcal{F}$ simply as a sheaf of abelian groups on the underlying topological space X.

We let the reader write out the long exact sequences which follow from the general properties of derived functors (1.1A).

Recall (II, Ex. 1.16) that a sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Lemma 2.4. *If $(X,\mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$-module is flasque.*

Proof. For any open subset $U \subseteq X$, let $\mathcal{O}_U$ denote the sheaf $j_!(\mathcal{O}_X|_U)$, which is the restriction of $\mathcal{O}_X$ to U, extended by zero outside U (II, Ex. 1.19). Now let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module, and let $V \subseteq U$ be open sets. Then we have an inclusion $0 \to \mathcal{O}_V \to \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{I}$ is injective, we get a surjection $\operatorname{Hom}(\mathcal{O}_U,\mathcal{I}) \to \operatorname{Hom}(\mathcal{O}_V,\mathcal{I}) \to 0$. But $\operatorname{Hom}(\mathcal{O}_U,\mathcal{I}) = \mathcal{I}(U)$ and $\operatorname{Hom}(\mathcal{O}_V,\mathcal{I}) = \mathcal{I}(V)$, so $\mathcal{I}$ is flasque.

Proposition 2.5. *If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.*

Proof. Embed $\mathcal{F}$ in an injective object $\mathcal{I}$ of $\mathfrak{Ab}(X)$ and let $\mathcal{G}$ be the quotient:

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

Then $\mathcal{F}$ is flasque by hypothesis, $\mathcal{I}$ is flasque by (2.4), and so $\mathcal{G}$ is flasque by (II, Ex. 1.16c). Now since $\mathcal{F}$ is flasque, we have an exact sequence (II, Ex. 1.16b)

$$0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{I}) \to \Gamma(X,\mathcal{G}) \to 0.$$

On the other hand, since $\mathcal{I}$ is injective, we have $H^i(X,\mathcal{I}) = 0$ for $i > 0$ (1.1Ae). Thus from the long exact sequence of cohomology, we get $H^1(X,\mathcal{F}) = 0$ and $H^i(X,\mathcal{F}) \cong H^{i-1}(X,\mathcal{G})$ for each $i \geqslant 2$. But $\mathcal{G}$ is also flasque, so by induction on i we get the result.

Remark 2.5.1. This result tells us that flasque sheaves are acyclic for the functor $\Gamma(X,\cdot)$. Hence we can calculate cohomology using flasque resolutions (1.2A). In particular, we have the following result.

Proposition 2.6. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X,\cdot)$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ coincide with the cohomology functors $H^i(X,\cdot)$.*

Proof. Considering $\Gamma(X,\cdot)$ as a functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, we calculate its derived functors by taking injective resolutions in the category $\mathfrak{Mod}(X)$. But any injective is flasque (2.4), and flasques are acyclic (2.5) so this resolution gives the usual cohomology functors (1.2A).

Remark 2.6.1. Let $(X,\mathcal{O}_X)$ be a ringed space, and let $A = \Gamma(X,\mathcal{O}_X)$. Then for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, $\Gamma(X,\mathcal{F})$ has a natural structure of A-module. In particular, since we can calculate cohomology using resolutions in the category $\mathfrak{Mod}(X)$, all the cohomology groups of $\mathcal{F}$ have a natural structure of A-module; the associated exact sequences are sequences of A-modules, and so forth. Thus for example, if X is a scheme over Spec B for some ring B, the cohomology groups of any $\mathcal{O}_X$-module $\mathcal{F}$ have a natural structure of B-module.

A Vanishing Theorem of Grothendieck

Theorem 2.7 (Grothendieck [1]). *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X,\mathcal{F}) = 0$.*

Before proving the theorem, we need some preliminary results, mainly concerning direct limits. If $(\mathcal{F}_\alpha)$ is a direct system of sheaves on X, indexed by a directed set A, then we have defined the direct limit $\varinjlim \mathcal{F}_\alpha$ (II, Ex. 1.10).

Lemma 2.8. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

PROOF. Let $(\mathscr{F}_\alpha)$ be a directed system of flasque sheaves. Then for any inclusion of open sets $V \subseteq U$, and for each α, we have $\mathscr{F}_\alpha(U) \to \mathscr{F}_\alpha(V)$ is surjective. Since $\varinjlim$ is an exact functor, we get

$$\varinjlim \mathscr{F}_\alpha(U) \to \varinjlim \mathscr{F}_\alpha(V)$$

is also surjective. But on a noetherian topological space, $\varinjlim \mathscr{F}_\alpha(U) = (\varinjlim \mathscr{F}_\alpha)(U)$ for any open set (II, Ex. 1.11). So we have

$$(\varinjlim \mathscr{F}_\alpha)(U) \to (\varinjlim \mathscr{F}_\alpha)(V)$$

is surjective, and so $\varinjlim \mathscr{F}_\alpha$ is flasque.

Proposition 2.9. *Let X be a noetherian topological space, and let $(\mathscr{F}_\alpha)$ be a direct system of abelian sheaves. Then there are natural isomorphisms, for each $i \geqslant 0$*

$$\varinjlim H^i(X,\mathscr{F}_\alpha) \to H^i(X,\varinjlim \mathscr{F}_\alpha).$$

PROOF. For each α we have a natural map $\mathscr{F}_\alpha \to \varinjlim \mathscr{F}_\alpha$. This induces a map on cohomology, and then we take the direct limit of these maps. For $i = 0$, the result is already known (II, Ex. 1.11). For the general case, we consider the category $\mathfrak{ind}_A(\mathfrak{Ab}(X))$ consisting of all directed systems of objects of $\mathfrak{Ab}(X)$, indexed by A. This is an abelian category. Furthermore, since $\varinjlim$ is an exact functor, we have a natural transformation of δ-functors

$$\varinjlim H^i(X,\cdot) \to H^i(X,\varinjlim \cdot)$$

from $\mathfrak{ind}_A(\mathfrak{Ab}(X))$ to $\mathfrak{Ab}$. They agree for $i = 0$, so to prove they are the same, it will be sufficient to show they are both effaceable for $i > 0$. For in that case, they are both universal by (1.3A), and so must be isomorphic.

So let $(\mathscr{F}_\alpha) \in \mathfrak{ind}_A(\mathfrak{Ab}(X))$. For each α, let $\mathscr{G}_\alpha$ be the sheaf of discontinuous sections of $\mathscr{F}_\alpha$ (II, Ex. 1.16e). Then $\mathscr{G}_\alpha$ is flasque, and there is a natural inclusion $\mathscr{F}_\alpha \to \mathscr{G}_\alpha$. Furthermore, the construction of $\mathscr{G}_\alpha$ is functorial, so the $\mathscr{G}_\alpha$ also form a direct system, and we obtain a monomorphism $u\colon(\mathscr{F}_\alpha) \to (\mathscr{G}_\alpha)$ in the category $\mathfrak{ind}_A(\mathfrak{Ab}(X))$. Now the $\mathscr{G}_\alpha$ are all flasque, so $H^i(X,\mathscr{G}_\alpha) = 0$ for $i > 0$ (2.5). Thus $\varinjlim H^i(X,\mathscr{G}_\alpha) = 0$, and the functor on the left-hand side is effaceable for $i > 0$. On the other hand, $\varinjlim \mathscr{G}_\alpha$ is also flasque by (2.8). So $H^i(X,\varinjlim \mathscr{G}_\alpha) = 0$ for $i > 0$, and we see that the functor on the right-hand side is also effaceable. This completes the proof.

Remark 2.9.1. As a special case we see that cohomology commutes with infinite direct sums.

Lemma 2.10. *Let Y be a closed subset of X, let $\mathscr{F}$ be a sheaf of abelian groups on Y, and let $j\colon Y \to X$ be the inclusion. Then $H^i(Y,\mathscr{F}) = H^i(X,j_*\mathscr{F})$, where $j_*\mathscr{F}$ is the extension of $\mathscr{F}$ by zero outside Y* (II, Ex. 1.19).

PROOF. If $\mathscr{I}^{\cdot}$ is a flasque resolution of $\mathscr{F}$ on Y, then $j_*\mathscr{I}^{\cdot}$ is a flasque resolution of $j_*\mathscr{F}$ on X, and for each i, $\Gamma(Y,\mathscr{I}^i) = \Gamma(X,j_*\mathscr{I}^i)$. So we get the same cohomology groups.

Remark 2.10.1. Continuing our earlier abuse of notation (II, Ex. 1.19), we often write $\mathscr{F}$ instead of $j_*\mathscr{F}$. This lemma shows there will be no ambiguity about the cohomology groups.

PROOF OF (2.7). First we fix some notation. If Y is a closed subset of X, then for any sheaf $\mathscr{F}$ on X we let $\mathscr{F}_Y = j_*(\mathscr{F}|_Y)$, where $j\colon Y \to X$ is the inclusion. If U is an open subset of X, we let $\mathscr{F}_U = i_!(\mathscr{F}|_U)$, where $i\colon U \to X$ is the inclusion. In particular, if $U = X - Y$, we have an exact sequence (II, Ex. 1.19)

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

We will prove the theorem by induction on $n = \dim X$, in several steps.

Step 1. Reduction to the case X irreducible. If X is reducible, let Y be one of its irreducible components, and let $U = X - Y$. Then for any $\mathscr{F}$ we have an exact sequence

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

From the long exact sequence of cohomology, it will be sufficient to prove that $H^i(X,\mathscr{F}_Y) = 0$ and $H^i(X,\mathscr{F}_U) = 0$ for $i > n$. But Y is closed and irreducible, and $\mathscr{F}_U$ can be regarded as a sheaf on the closed subset $\bar{U}$, which has one fewer irreducible components than X. Thus using (2.10) and induction on the number of irreducible components, we reduce to the case X irreducible.

Step . Suppose X is irreducible of dimension 0. Then the only open subsets of X are X and the empty set. For otherwise, X would have a proper irreducible closed subset, and $\dim X$ would be $\geqslant 1$. Thus $\Gamma(X,\cdot)$ induces an equivalence of categories $\mathfrak{Ab}(X) \to \mathfrak{Ab}$. In particular, $\Gamma(X,\cdot)$ is an exact functor, so $H^i(X,\mathscr{F}) = 0$ for $i > 0$, and for all $\mathscr{F}$.

Step 3. Now let X be irreducible of dimension n, and let $\mathscr{F} \in \mathfrak{Ab}(X)$. Let $B = \bigcup_{U\subseteq X} \mathscr{F}(U)$, and let A be the set of all finite subsets of B. For each $\alpha \in A$, let $\mathscr{F}_\alpha$ be the subsheaf of $\mathscr{F}$ generated by the sections in α (over various open sets). Then A is a directed set, and $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$. So by (2.9), it will be sufficient to prove vanishing of cohomology for each $\mathscr{F}_\alpha$. If α' is a subset of α, then we have an exact sequence

$$0 \to \mathscr{F}_{\alpha'} \to \mathscr{F}_\alpha \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is a sheaf generated by $\#(\alpha - \alpha')$ sections over suitable open sets. Thus, using the long exact sequence of cohomology, and induction on $\#(\alpha)$, we reduce to the case that $\mathscr{F}$ is generated by a single section over some open set U. In that case $\mathscr{F}$ is a quotient of the sheaf $\mathbf{Z}_U$ (where $\mathbf{Z}$ denotes the constant sheaf $\mathbf{Z}$ on X). Letting $\mathscr{R}$ be the kernel, we have an exact sequence

$$0 \to \mathscr{R} \to \mathbf{Z}_U \to \mathscr{F} \to 0.$$

Again using the long exact sequence of cohomology, it will be sufficient to prove vanishing for $\mathscr{R}$ and for $\mathbf{Z}_U$.

Step 4. Let U be an open subset of X and let $\mathscr{R}$ be a subsheaf of $\mathbf{Z}_U$. For each $x \in U$, the stalk $\mathscr{R}_x$ is a subgroup of $\mathbf{Z}$. If $\mathscr{R} = 0$, skip to Step 5. If not, let d be the least positive integer which occurs in any of the groups $\mathscr{R}_x$. Then there is a nonempty open subset $V \subseteq U$ such that $\mathscr{R}|_V \cong d \cdot \mathbf{Z}|_V$ as a subsheaf of $\mathbf{Z}|_V$. Thus $\mathscr{R}_V \cong \mathbf{Z}_V$ and we have an exact sequence

$$0 \to \mathbf{Z}_V \to \mathscr{R} \to \mathscr{R}/\mathbf{Z}_V \to 0.$$

Now the sheaf $\mathscr{R}/\mathbf{Z}_V$ is supported on the closed subset $(U - V)^-$ of X, which has dimension $<n$, since X is irreducible. So using (2.10) and the induction hypothesis, we know $H^i(X,\mathscr{R}/\mathbf{Z}_V) = 0$ for $i \geqslant n$. So by the long exact sequence of cohomology, we need only show vanishing for $\mathbf{Z}_V$.

Step 5. To complete the proof, we need only show that for any open subset $U \subseteq X$, we have $H^i(X,\mathbf{Z}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathbf{Z}_U \to \mathbf{Z} \to \mathbf{Z}_Y \to 0.$$

Now $\dim Y < \dim X$ since X is irreducible, so using (2.10) and the induction hypothesis, we have $H^i(X,\mathbf{Z}_Y) = 0$ for $i \geqslant n$. On the other hand, $\mathbf{Z}$ is flasque, since it is a constant sheaf on an irreducible space (II, Ex. 1.16a). Hence $H^i(X,\mathbf{Z}) = 0$ for $i > 0$ by (2.5). So from the long exact sequence of cohomology we have $H^i(X,\mathbf{Z}_U) = 0$ for $i > n$. q.e.d.

Historical Note: The derived functor cohomology which we defined in this section was introduced by Grothendieck [1]. It is the theory which is used in [EGA]. The use of sheaf cohomology in algebraic geometry started with Serre [3]. In that paper, and in the later paper [4], Serre used Čech cohomology for coherent sheaves on an algebraic variety with its Zariski topology. The equivalence of this theory with the derived functor theory follows from the "theorem of Leray" (Ex. 4.11). The same argument, using Cartan's "Theorem B" shows that the Čech cohomology of a coherent analytic sheaf on a complex analytic space is equal to the derived functor cohomology. Gunning and Rossi [1] use a cohomology theory computed by fine resolutions of a sheaf on a paracompact Hausdorff space. The equivalence of this theory with ours is shown by Godement [1, Thm. 4.7.1, p. 181 and Ex. 7.2.1, p. 263], who shows at the same time that both theories coincide with his theory which is defined by a canonical flasque resolution. Godement also shows [1, Thm. 5.10.1, p. 228] that on a paracompact Hausdorff space, his theory coincides with Čech cohomology. This provides a bridge to the standard topological theories with constant coefficients, as developed in the book of Spanier [1]. He shows that on a paracompact Hausdorff space, Čech cohomology and Alexander cohomology and singular cohomology all agree (see Spanier [1, pp. 314, 327, 334]).

The vanishing theorem (2.7) was proved by Serre [3] for coherent sheaves on algebraic curves and projective algebraic varieties, and later [5] for abstract algebraic varieties. It is analogous to the theorem that singular cohomology on a (real) manifold of dimension n vanishes in degrees $i > n$.

Exercises

2.1. (a) Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k. Let P,Q be distinct closed points of X, and let $U = X - \{P,Q\}$. Show that $H^1(X,\mathbf{Z}_U) \neq 0$.

***(b) More generally, let $Y \subseteq X = \mathbf{A}_k^n$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X,\mathbf{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.**

2.2. Let $X = \mathbf{P}_k^1$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathscr{O} \to \mathscr{K} \to \mathscr{K}/\mathscr{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathscr{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X,\mathscr{O}) = 0$ for all $i > 0$.

2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X,\mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X,\cdot)$ by $H_Y^i(X,\cdot)$. They are the *cohomology groups of X with supports* in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, with $\mathscr{F}'$ flasque, show that

$$0 \to \Gamma_Y(X,\mathscr{F}') \to \Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(X,\mathscr{F}'') \to 0$$

is exact.

(c) Show that if $\mathscr{F}$ is flasque, then $H_Y^i(X,\mathscr{F}) = 0$ for all $i > 0$.

(d) If $\mathscr{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}) \to \Gamma(X - Y,\mathscr{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathscr{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H_Y^0(X,\mathscr{F}) \to H^0(X,\mathscr{F}) \to H^0(U,\mathscr{F}|_U) \to \\ &\to H_Y^1(X,\mathscr{F}) \to H^1(X,\mathscr{F}) \to H^1(U,\mathscr{F}|_U) \to \\ &\to H_Y^2(X,\mathscr{F}) \to \dots . \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathscr{F}$,

$$H_Y^i(X,\mathscr{F}) \cong H_Y^i(V,\mathscr{F}|_V).$$

2.4. *Mayer–Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\begin{aligned} \dots &\to H_{Y_1 \cap Y_2}^i(X,\mathscr{F}) \to H_{Y_1}^i(X,\mathscr{F}) \oplus H_{Y_2}^i(X,\mathscr{F}) \to H_{Y_1 \cup Y_2}^i(X,\mathscr{F}) \to \\ &\to H_{Y_1 \cap Y_2}^{i+1}(X,\mathscr{F}) \to \dots . \end{aligned}$$

2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j: X_P \to X$ be the inclusion, and for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_P = j^*\mathscr{F}$. Show that for all i, $\mathscr{F}$, we have

$$H^i_P(X,\mathscr{F}) = H^i_P(X_P,\mathscr{F}_P).$$

2.6. Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha\in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective. [*Hints*: First show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathscr{R} \subseteq \mathbf{Z}_U$, and for every map $f: \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map of $\mathbf{Z}_U \to \mathscr{I}$. Secondly, show that any such sheaf $\mathscr{R}$ is finitely generated, so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.]

2.7. Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.
(a) Show that $H^1(S^1,\mathbf{Z}) \cong \mathbf{Z}$, using our definition of cohomology.
(b) Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1,\mathscr{R}) = 0$.

3 Cohomology of a Noetherian Affine Scheme

In this section we will prove that if $X = \operatorname{Spec} A$ is a noetherian affine scheme, then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves $\mathscr{F}$ of $\mathscr{O}_X$-modules. The key point is to show that if I is an injective A-module, then the sheaf $\tilde{I}$ on $\operatorname{Spec} A$ is flasque. We begin with some algebraic preliminaries.

Proposition 3.1A (Krull's Theorem). *Let A be a noetherian ring, let $M \subseteq N$ be finitely generated A-modules, and let $\mathfrak{a}$ be an ideal of A. Then the $\mathfrak{a}$-adic topology on M is induced by the $\mathfrak{a}$-adic topology on N. In particular, for any $n > 0$, there exists an $n' \geqslant n$ such that $\mathfrak{a}^nM \supseteq M \cap \mathfrak{a}^{n'}N$.*

PROOF. Atiyah–Macdonald [1, 10.11] or Zariski–Samuel [1, vol. II, Ch. VIII, Th. 4].

Recall (II, Ex. 5.6) that for any ring A, and any ideal $\mathfrak{a} \subseteq A$, and any A-module M, we have defined the submodule $\Gamma_\mathfrak{a}(M)$ to be $\{m \in M | \mathfrak{a}^n m = 0$ for some $n > 0\}$.

Lemma 3.2. *Let A be a noetherian ring, let $\mathfrak{a}$ be an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_\mathfrak{a}(I)$ is also an injective A-module.*

PROOF. To show that J is injective, it will be sufficient to show that for any ideal $\mathfrak{b} \subseteq A$, and for any homomorphism $\varphi: \mathfrak{b} \to J$, there exists a homomorphism $\psi: A \to J$ extending φ. (This is a well-known criterion for an injective module—Godement [1, I, 1.4.1]). Since A is noetherian, $\mathfrak{b}$ is finitely generated. On the other hand, every element of J is annihilated by some

power of $\mathfrak{a}$, so there exists an $n > 0$ such that $\mathfrak{a}^n\varphi(\mathfrak{b}) = 0$, or equivalently, $\varphi(\mathfrak{a}^n\mathfrak{b}) = 0$. Now applying (3.1A) to the inclusion $\mathfrak{b} \subseteq A$, we find that there is an $n' \geqslant n$ such that $\mathfrak{a}^n\mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^{n'}$. Hence $\varphi(\mathfrak{b} \cap \mathfrak{a}^{n'}) = 0$, and so the map $\varphi: \mathfrak{b} \to J$ factors through $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$. Now we consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] & A/\mathfrak{a}^{n'} \arrow[dr, dashed] \arrow[drr, dashed, "\psi'"] & & \\
\mathfrak{b} \arrow[u, hook] \arrow[r] \arrow[rr, bend right, "\varphi"'] & \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'}) \arrow[u, hook] \arrow[r] & J \arrow[r] & I
\end{tikzcd}$$

Since I is injective, the composed map of $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$ to I extends to a map $\psi': A/\mathfrak{a}^{n'} \to I$. But the image of ψ' is annihilated by $\mathfrak{a}^{n'}$, so it is contained in J. Composing with the natural map $A \to A/\mathfrak{a}^{n'}$, we obtain the required map $\psi: A \to J$ extending φ.

Lemma 3.3. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$, the natural map of I to its localization I_f is surjective.*

PROOF. For each $i > 0$, let $\mathfrak{b}_i$ be the annihilator of f^i in A. Then $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \ldots$, and since A is noetherian, there is an r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \ldots$. Now let $\theta: I \to I_f$ be the natural map, and let $x \in I_f$ be any element. Then by definition of localization, there is a $y \in I$ and an $n \geqslant 0$ such that $x = \theta(y)/f^n$. We define a map φ from the ideal (f^{n+r}) of A to I by sending f^{n+r} to f^ry. This is possible, because the annihilator of f^{n+r} is $\mathfrak{b}_{n+r} = \mathfrak{b}_r$, and $\mathfrak{b}_r$ annihilates f^ry. Since I is injective, φ extends to a map $\psi: A \to I$. Let $\psi(1) = z$. Then $f^{n+r}z = f^ry$. But this implies that $\theta(z) = \theta(y)/f^n = x$. Hence θ is surjective.

Proposition 3.4. *Let I be an injective module over a noetherian ring A. Then the sheaf $\tilde{I}$ on $X = \operatorname{Spec} A$ is flasque.*

PROOF. We will use noetherian induction on $Y = (\operatorname{Supp} \tilde{I})^-$. See (II, Ex. 1.14) for the notion of support. If Y consists of a single closed point of X, then $\tilde{I}$ is a skyscraper sheaf (II, Ex. 1.17) which is obviously flasque.

In the general case, to show that $\tilde{I}$ is flasque, it will be sufficient to show, for any open set $U \subseteq X$, that $\Gamma(X,\tilde{I}) \to \Gamma(U,\tilde{I})$ is surjective. If $Y \cap U = \varnothing$, there is nothing to prove. If $Y \cap U \neq \varnothing$, we can find an $f \in A$ such that the open set $X_f = D(f)$ (II, §2) is contained in U and $X_f \cap Y \neq \varnothing$. Let $Z = X - X_f$, and consider the following diagram:

$$\begin{tikzcd}
\Gamma(X,\tilde{I}) \arrow[r] & \Gamma(U,\tilde{I}) \arrow[r] & \Gamma(X_f,\tilde{I}) \\
\Gamma_Z(X,\tilde{I}) \arrow[u, hook] \arrow[r] & \Gamma_Z(U,\tilde{I}), \arrow[u, hook] &
\end{tikzcd}$$

where Γ_Z denotes sections with support in Z (II, Ex. 1.20). Now given a section $s \in \Gamma(U,\tilde{I})$, we consider its image s' in $\Gamma(X_f,\tilde{I})$. But $\Gamma(X_f,\tilde{I}) = I_f$ (II, 5.1), so by (3.3), there is a $t \in I = \Gamma(X,\tilde{I})$ restricting to s'. Let t' be the restriction of t to $\Gamma(U,\tilde{I})$. Then $s - t'$ goes to 0 in $\Gamma(X_f,\tilde{I})$, so it has support in Z. Thus to complete the proof, it will be sufficient to show that $\Gamma_Z(X,\tilde{I}) \to \Gamma_Z(U,\tilde{I})$ is surjective.

Let $J = \Gamma_Z(X,\tilde{I})$. If $\mathfrak{a}$ is the ideal generated by f, then $J = \Gamma_{\mathfrak{a}}(I)$ (II, Ex. 5.6), so by (3.2), J is also an injective A-module. Furthermore, the support of $\tilde{J}$ is contained in $Y \cap Z$, which is strictly smaller than Y. Hence by our induction hypothesis, $\tilde{J}$ is flasque. Since $\Gamma(U,\tilde{J}) = \Gamma_Z(U,\tilde{I})$ (II, Ex. 5.6), we conclude that $\Gamma_Z(X,\tilde{I}) \to \Gamma_Z(U,\tilde{I})$ is surjective, as required.

Theorem 3.5. *Let X = Spec A be the spectrum of a noetherian ring A. Then for all quasi-coherent sheaves $\mathscr{F}$ on X, and for all $i > 0$, we have $H^i(X,\mathscr{F}) = 0$.*

PROOF. Given $\mathscr{F}$, let $M = \Gamma(X,\mathscr{F})$, and take an injective resolution $0 \to M \to I^{\cdot}$ of M in the category of A-modules. Then we obtain an exact sequence of sheaves $0 \to \tilde{M} \to \tilde{I}^{\cdot}$ on X. Now $\mathscr{F} = \tilde{M}$ (II, 5.5) and each $\tilde{I}^i$ is flasque by (3.4), so we can use this resolution of $\mathscr{F}$ to calculate cohomology (2.5.1). Applying the functor Γ, we recover the exact sequence of A-modules $0 \to M \to I^{\cdot}$. Hence $H^0(X,\mathscr{F}) = M$, and $H^i(X,\mathscr{F}) = 0$ for $i > 0$.

Remark 3.5.1. This result is also true without the noetherian hypothesis, but the proof is more difficult [EGA III, 1.3.1].

Corollary 3.6. *Let X be a noetherian scheme, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $\mathscr{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathscr{G}$.*

PROOF. Cover X with a finite number of open affines U_i = Spec A_i, and let $\mathscr{F}|_{U_i} = \tilde{M}_i$ for each i. Embed M_i in an injective A_i-module I_i. For each i, let $f: U_i \to X$ be the inclusion, and let $\mathscr{G} = \bigoplus f_*(\tilde{I}_i)$. For each i we have an injective map of sheaves $\mathscr{F}|_{U_i} \to \tilde{I}_i$. Hence we obtain a map $\mathscr{F} \to f_*(\tilde{I}_i)$. Taking the direct sum over i gives a map $\mathscr{F} \to \mathscr{G}$ which is clearly injective. On the other hand, for each i, $\tilde{I}_i$ is flasque (3.4) and quasi-coherent on U_i. Hence $f_*(\tilde{I}_i)$ is also flasque (II, Ex. 1.16d) and quasi-coherent (II, 5.8). Taking the direct sum of these, we see that $\mathscr{G}$ is flasque and quasi-coherent.

Theorem 3.7 (Serre [5]). *Let X be a noetherian scheme. Then the following conditions are equivalent:*

(i) *X is affine;*
(ii) *$H^i(X,\mathscr{F}) = 0$ for all $\mathscr{F}$ quasi-coherent and all $i > 0$;*
(iii) *$H^1(X,\mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$.*

PROOF. (i) ⇒ (ii) is (3.5). (ii) ⇒ (iii) is trivial, so we have only to prove (iii) ⇒ (i). We use the criterion of (II, Ex. 2.17). First we show that X can

be covered by open affine subsets of the form X_f, with $f \in A = \Gamma(X,\mathcal{O}_X)$. Let P be a closed point of X, let U be an open affine neighborhood of P, and let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathscr{I}_{Y \cup \{P\}} \to \mathscr{I}_Y \to k(P) \to 0,$$

where $\mathscr{I}_Y$ and $\mathscr{I}_{Y \cup \{P\}}$ are the ideal sheaves of the closed sets Y and $Y \cup \{P\}$, respectively. The quotient is the skyscraper sheaf $k(P) = \mathcal{O}_P/\mathfrak{m}_P$ at P. Now from the exact sequence of cohomology, and hypothesis (iii), we get an exact sequence

$$\Gamma(X,\mathscr{I}_Y) \to \Gamma(X,k(P)) \to H^1(X,\mathscr{I}_{Y \cup \{P\}}) = 0.$$

So there is an element $f \in \Gamma(X,\mathscr{I}_Y)$ which goes to 1 in $k(P)$, i.e., $f_P \equiv 1$ (mod $\mathfrak{m}_P$). Since $\mathscr{I}_Y \subseteq \mathcal{O}_X$, we can consider f as an element of A. Then by construction, we have $P \in X_f \subseteq U$. Furthermore, $X_f = U_{\bar{f}}$, where $\bar{f}$ is the image of f in $\Gamma(U,\mathcal{O}_U)$, so X_f is affine.

Thus every closed point of X has an open affine neighborhood of the form X_f. By quasi-compactness, we can cover X with a finite number of these, corresponding to $f_1, \dots, f_r \in A$.

Now by (II, Ex. 2.17), to show that X is affine, we need only verify that $f_1, \dots, f_r$ generate the unit ideal in A. We use $f_1, \dots, f_r$ to define a map $\alpha: \mathcal{O}_X^r \to \mathcal{O}_X$ by sending $\langle a_1, \dots, a_r \rangle$ to $\sum f_i a_i$. Since the X_{f_i} cover X, this is a surjective map of sheaves. Let $\mathscr{F}$ be the kernel:

$$0 \to \mathscr{F} \to \mathcal{O}_X^r \xrightarrow{\alpha} \mathcal{O}_X \to 0.$$

We filter $\mathscr{F}$ as follows:

$$\mathscr{F} = \mathscr{F} \cap \mathcal{O}_X^r \supseteq \mathscr{F} \cap \mathcal{O}_X^{r-1} \supseteq \dots \supseteq \mathscr{F} \cap \mathcal{O}_X$$

for a suitable ordering of the factors of $\mathcal{O}_X^r$. Each of the quotients of this filtration is a coherent sheaf of ideals in $\mathcal{O}_X$. Thus using our hypothesis (iii) and the long exact sequence of cohomology, we climb up the filtration and deduce that $H^1(X,\mathscr{F}) = 0$. But then $\Gamma(X,\mathcal{O}_X^r) \xrightarrow{\alpha} \Gamma(X,\mathcal{O}_X)$ is surjective, which tells us that $f_1, \dots, f_r$ generate the unit ideal in A. q.e.d.

Remark 3.7.1. This result is analogous to another theorem of Serre in complex analytic geometry, which characterizes Stein spaces by the vanishing of coherent analytic sheaf cohomology.

EXERCISES

3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine. [*Hint*: Use (3.7), and for any coherent sheaf $\mathscr{F}$ on X, consider the filtration $\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \dots$, where $\mathscr{N}$ is the sheaf of nilpotent elements on X.]

3.2. Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.

(a) Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H^i_{\mathfrak{a}}(\cdot)$.

(b) Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X,\tilde{M}),$$

where $H^i_Y(X,\cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

(c) For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

3.4. *Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A-module, then *depth*$_{\mathfrak{a}}$ M is the maximum length of an M-regular sequence $x_1, \ldots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II, §8).

(a) Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geqslant 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated. [*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.]

(b) Show inductively, for M finitely generated, that for any $n \geqslant 0$, the following conditions are equivalent:

(i) $\operatorname{depth}_{\mathfrak{a}} M \geqslant n$;

(ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

For more details, and related results, see Grothendieck [7].

3.5. Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:

(i) $\operatorname{depth} \mathcal{O}_P \geqslant 2$;

(ii) if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

3.6. Let X be a noetherian scheme.

(a) Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.

*(b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will *not* work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathcal{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathcal{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines . . .]

(c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X,\cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

3.7. Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

(a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U,\tilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n,M).$$

(b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

3.8. Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0,x_1,x_2,\ldots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \ldots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

4 Čech Cohomology

In this section we construct the Čech cohomology groups for a sheaf of abelian groups on a topological space X, with respect to a given open covering of X. We will prove that if X is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups defined in §2. The value of this result is that it gives a practical method for computing cohomology of quasi-coherent sheaves on a scheme.

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \ldots \cap U_{i_p}$ by $U_{i_0,\ldots,i_p}$.

Now let $\mathcal{F}$ be a sheaf of abelian groups on X. We define a complex $C^{\cdot}(\mathfrak{U},\mathcal{F})$ of abelian groups as follows. For each $p \geqslant 0$, let

$$C^p(\mathfrak{U},\mathcal{F}) = \prod_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

Thus an element $\alpha \in C^p(\mathfrak{U},\mathcal{F})$ is determined by giving an element

$$\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p}),$$

for each $(p + 1)$-tuple $i_0 < \ldots < i_p$ of elements of I. We define the coboundary map $d\colon C^p \to C^{p+1}$ by setting

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}\big|_{U_{i_0,\ldots,i_{p+1}}}.$$

Here the notation $\hat{i}_k$ means omit i_k. Then since $\alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}$ is an element of $\mathcal{F}(U_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}})$, we restrict to $U_{i_0,\ldots,i_{p+1}}$ to get an element of $\mathcal{F}(U_{i_0,\ldots,i_{p+1}})$. One checks easily that $d^2 = 0$, so we have indeed defined a complex of abelian groups.

Remark 4.0.1. If $\alpha \in C^p(\mathfrak{U},\mathcal{F})$, it is sometimes convenient to have the symbol $\alpha_{i_0,\ldots,i_p}$ defined for *all* $(p + 1)$-tuples of elements of I. If there is a repeated index in the set $\{i_0,\ldots,i_p\}$, we define $\alpha_{i_0,\ldots,i_p} = 0$. If the indices are all distinct, we define $\alpha_{i_0,\ldots,i_p} = (-1)^{\sigma}\alpha_{\sigma i_0,\ldots,\sigma i_p}$, where σ is the permutation for which $\sigma i_0 < \ldots < \sigma i_p$. With these conventions, one can check that the formula given above for $d\alpha$ remains correct for any $(p + 2)$-tuple $i_0, \ldots, i_{p+1}$ of elements of I.

Definition. Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of abelian groups $\mathscr{F}$ on X, we define the pth *Čech cohomology group of* $\mathscr{F}$, *with respect to the covering* $\mathfrak{U}$, to be

$$\check{H}^p(\mathfrak{U},\mathscr{F}) = h^p(C^{\cdot}(\mathfrak{U},\mathscr{F})).$$

Caution 4.0.2. Keeping X and $\mathfrak{U}$ fixed, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves of abelian groups on X, we do *not* in general get a long exact sequence of Čech cohomology groups. In other words, the functors $\check{H}^p(\mathfrak{U},\cdot)$ do not form a δ-functor (§1). For example, if $\mathfrak{U}$ consists of the single open set X, then this results from the fact that the global section functor $\Gamma(X,\cdot)$ is not exact.

Example 4.0.3. To illustrate how well suited Čech cohomology is for computations, we will compute some examples. Let $X = \mathbf{P}^1_k$, let $\mathscr{F}$ be the sheaf of differentials Ω (II, §8), and let $\mathfrak{U}$ be the open covering by the two open sets $U = \mathbf{A}^1$ with affine coordinate x, and $V = \mathbf{A}^1$ with affine coordinate $y = 1/x$. Then the Čech complex has only two terms:

$$C^0 = \Gamma(U,\Omega) \times \Gamma(V,\Omega)$$
$$C^1 = \Gamma(U \cap V,\Omega).$$

Now

$$\Gamma(U,\Omega) = k[x]\, dx$$
$$\Gamma(V,\Omega) = k[y]\, dy$$
$$\Gamma(U \cap V,\Omega) = k\left[x,\frac{1}{x}\right] dx,$$

and the map $d: C^0 \to C^1$ is given by

$$x \mapsto x$$
$$y \mapsto \frac{1}{x}$$
$$dy \mapsto -\frac{1}{x^2}\, dx.$$

So ker d is the set of pairs $\langle f(x)\, dx, g(y)\, dy\rangle$ such that

$$f(x) = -\frac{1}{x^2}\, g\left(\frac{1}{x}\right).$$

This can happen only if $f = g = 0$, since one side is a polynomial in x and the other side is a polynomial in $1/x$ with no constant term. So $\check{H}^0(\mathfrak{U},\Omega) = 0$.

To compute H^1, note that the image of d is the set of all expressions

$$\left(f(x) + \frac{1}{x^2}\, g\left(\frac{1}{x}\right)\right) dx,$$

where f and g are polynomials. This gives the subvector space of $k[x,1/x]\, dx$ generated by all $x^n\, dx$, $n \in \mathbf{Z}$, $n \neq -1$. Therefore $\check{H}^1(\mathfrak{U},\Omega) \cong k$, generated by the image of $x^{-1}\, dx$.

Example 4.0.4. Let S^1 be the circle (in its usual topology), let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$, and let $\mathfrak{U}$ be the open covering by two connected open semi-circles U, V, which overlap at each end, so that $U \cap V$ consists of two small intervals. Then

$$C^0 = \Gamma(U,\mathbf{Z}) \times \Gamma(V,\mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$$
$$C^1 = \Gamma(U \cap V,\mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$$

and the map $d: C^0 \to C^1$ takes $\langle a,b\rangle$ to $\langle b-a, b-a\rangle$. Thus $\check{H}^0(\mathfrak{U},\mathbf{Z}) = \mathbf{Z}$ and $\check{H}^1(\mathfrak{U},\mathbf{Z}) = \mathbf{Z}$. Since we know this is the right answer (Ex. 2.7), this illustrates the general principle that Čech cohomology agrees with the usual cohomology provided the open covering is taken fine enough so that there is no cohomology on any of the open sets (Ex. 4.11).

Now we will study some properties of the Čech cohomology groups.

Lemma 4.1. *For any* $X,\mathfrak{U},\mathscr{F}$ *as above, we have* $\check{H}^0(\mathfrak{U},\mathscr{F}) \cong \Gamma(X,\mathscr{F})$.

PROOF. $\check{H}^0(\mathfrak{U},\mathscr{F}) = \ker(d: C^0(\mathfrak{U},\mathscr{F}) \to C^1(\mathfrak{U},\mathscr{F}))$. If $\alpha \in C^0$ is given by $\{\alpha_i \in \mathscr{F}(U_i)\}$, then for each $i < j$, $(d\alpha)_{ij} = \alpha_j - \alpha_i$. So $d\alpha = 0$ says the sections α_i and α_j agree on $U_i \cap U_j$. Thus it follows from the sheaf axioms that ker $d = \Gamma(X,\mathscr{F})$.

Next we define a "sheafified" version of the Čech complex. For any open set $V \subseteq X$, let $f: V \to X$ denote the inclusion map. Now given $X,\mathfrak{U},\mathscr{F}$ as above, we construct a complex $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ of sheaves on X as follows. For each $p \geqslant 0$, let

$$\mathscr{C}^p(\mathfrak{U},\mathscr{F}) = \prod_{i_0 < \ldots < i_p} f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}}),$$

and define

$$d: \mathscr{C}^p \to \mathscr{C}^{p+1}$$

by the same formula as above. Note by construction that for each p we have $\Gamma(X,\mathscr{C}^p(\mathfrak{U},\mathscr{F})) = C^p(\mathfrak{U},\mathscr{F})$.

Lemma 4.2. *For any sheaf of abelian groups* $\mathscr{F}$ *on* X, *the complex* $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ *is a resolution of* $\mathscr{F}$, i.e., *there is a natural map* $\varepsilon: \mathscr{F} \to \mathscr{C}^0$ *such that the sequence of sheaves*

$$0 \to \mathscr{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \to \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \to \ldots$$

is exact.

PROOF. We define $\varepsilon:\mathscr{F} \to \mathscr{C}^0$ by taking the product of the natural maps $\mathscr{F} \to f_*(\mathscr{F}|_{U_i})$ for $i \in I$. Then the exactness at the first step follows from the sheaf axioms for $\mathscr{F}$.

To show the exactness of the complex $\mathscr{C}^{\cdot}$ for $p \geqslant 1$, it is enough to check exactness on the stalks. So let $x \in X$, and suppose $x \in U_j$. For each $p \geqslant 1$, we define a map

$$k:\mathscr{C}^p(\mathfrak{U},\mathscr{F})_x \to \mathscr{C}^{p-1}(\mathfrak{U},\mathscr{F})_x$$

as follows. Given $\alpha_x \in \mathscr{C}^p(\mathfrak{U},\mathscr{F})_x$, it is represented by a section $\alpha \in \Gamma(V,\mathscr{C}^p(\mathfrak{U},\mathscr{F}))$ over a neighborhood V of x, which we may choose so small that $V \subseteq U_j$. Now for any p-tuple $i_0 < \ldots < i_{p-1}$, we set

$$(k\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}},$$

using the notational convention of (4.0.1). This makes sense because $V \cap U_{i_0,\ldots,i_{p-1}} = V \cap U_{j,i_0,\ldots,i_{p-1}}$. Then take the stalk of $k\alpha$ at x to get the required map k. Now one checks that for any $p \geqslant 1$, $\alpha \in \mathscr{C}^p_x$,

$$(dk + kd)(\alpha) = \alpha.$$

Thus k is a homotopy operator for the complex $\mathscr{C}^{\cdot}_x$, showing that the identity map is homotopic to the zero map. It follows (§1) that the cohomology groups $h^p(\mathscr{C}^{\cdot}_x)$ of this complex are 0 for $p \geqslant 1$.

Proposition 4.3. *Let X be a topological space, let $\mathfrak{U}$ be an open covering, and let $\mathscr{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$.*

PROOF. Consider the resolution $0 \to \mathscr{F} \to \mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ given by (4.2). Since $\mathscr{F}$ is flasque, the sheaves $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ are flasque for each $p \geqslant 0$. Indeed, for any $i_0,\ldots,i_p$, $\mathscr{F}|_{U_{i_0,\ldots,i_p}}$ is a flasque sheaf on $U_{i_0,\ldots,i_p}$; f_* preserves flasque sheaves (II, Ex. 1.16d), and a product of flasque sheaves is flasque. So by (2.5.1) we can use this resolution to compute the usual cohomology groups of $\mathscr{F}$. But $\mathscr{F}$ is flasque, so $H^p(X,\mathscr{F}) = 0$ for $p > 0$ by (2.5). On the other hand, the answer given by this resolution is

$$h^p(\Gamma(X,\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F}))) = \check{H}^p(\mathfrak{U},\mathscr{F}).$$

So we conclude that $\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$ for $p > 0$.

Lemma 4.4. *Let X be a topological space, and $\mathfrak{U}$ an open covering. Then for each $p \geqslant 0$ there is a natural map, functorial in $\mathscr{F}$,*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \to H^p(X,\mathscr{F}).$$

PROOF. Let $0 \to \mathscr{F} \to \mathscr{I}^{\cdot}$ be an injective resolution of $\mathscr{F}$ in $\mathfrak{Ab}(X)$. Comparing with the resolution $0 \to \mathscr{F} \to \mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ of (4.2), it follows from a general result on complexes (Hilton and Stammbach [1, IV, 4.4]) that there is a morphism of complexes $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F}) \to \mathscr{I}^{\cdot}$, inducing the identity map on $\mathscr{F}$, and unique up to homotopy. Applying the functors $\Gamma(X,\cdot)$ and h^p, we get the required map.

Theorem 4.5. *Let X be a noetherian separated scheme, let $\mathfrak{U}$ be an open affine cover of X, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then for all $p \geqslant 0$, the natural maps of (4.4) give isomorphisms*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \xrightarrow{\sim} H^p(X,\mathscr{F}).$$

PROOF. For $p = 0$ we have an isomorphism by (4.1). For the general case, embed $\mathscr{F}$ in a flasque, quasi-coherent sheaf $\mathscr{G}$ (3.6), and let $\mathscr{R}$ be the quotient:

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

For each $i_0 < \ldots < i_p$, the open set $U_{i_0,\ldots,i_p}$ is affine, since it is an intersection of affine open subsets of a separated scheme (II, Ex. 4.3). Since $\mathscr{F}$ is quasi-coherent, we therefore have an exact sequence

$$0 \to \mathscr{F}(U_{i_0,\ldots,i_p}) \to \mathscr{G}(U_{i_0,\ldots,i_p}) \to \mathscr{R}(U_{i_0,\ldots,i_p}) \to 0$$

of abelian groups, by (3.5) or (II, 5.6). Taking products, we find that the corresponding sequence of Čech complexes

$$0 \to C^{\cdot}(\mathfrak{U},\mathscr{F}) \to C^{\cdot}(\mathfrak{U},\mathscr{G}) \to C^{\cdot}(\mathfrak{U},\mathscr{R}) \to 0$$

is exact. Therefore we get a long exact sequence of Čech cohomology groups. Since $\mathscr{G}$ is flasque, its Čech cohomology vanishes for $p > 0$ by (4.3), so we have an exact sequence

$$0 \to \check{H}^0(\mathfrak{U},\mathscr{F}) \to \check{H}^0(\mathfrak{U},\mathscr{G}) \to \check{H}^0(\mathfrak{U},\mathscr{R}) \to \check{H}^1(\mathfrak{U},\mathscr{F}) \to 0$$

and isomorphisms

$$\check{H}^p(\mathfrak{U},\mathscr{R}) \xrightarrow{\sim} \check{H}^{p+1}(\mathfrak{U},\mathscr{F})$$

for each $p \geqslant 1$. Now comparing with the long exact sequence of usual cohomology for the above short exact sequence, using the case $p = 0$, and (2.5), we conclude that the natural map

$$\check{H}^1(\mathfrak{U},\mathscr{F}) \to H^1(X,\mathscr{F})$$

is an isomorphism. But $\mathscr{R}$ is also quasi-coherent (II, 5.7), so we obtain the result for all p by induction.

EXERCISES

4.1. Let $f:X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geqslant 0$,

$$H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F}).$$

[*Hint*: Use (II, 5.8).]

4.2. Prove Chevalley's theorem: Let $f:X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f:X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha:\mathscr{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta: f_*\mathscr{G} \rightarrow \mathscr{F}^r$ which is an isomorphism at the generic point of Y. [*Hint*: Apply $\mathscr{H}om(\cdot,\mathscr{F})$ to α and use (II, Ex. 5.17e).]

(c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider ker β and coker β, and use noetherian induction on Y.

4.3. Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$, and let $U = X - \{(0,0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U,\mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^iy^j | i,j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—cf. (I, Ex. 3.6).)

4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j\in J}$, together with a map $\lambda: J \rightarrow I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i: \check{H}^i(\mathfrak{U},\mathscr{F}) \rightarrow \check{H}^i(\mathfrak{V},\mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U},\mathscr{F}).$$

(b) For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U},\mathscr{F}) \rightarrow H^i(X,\mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U},\mathscr{F}) \rightarrow H^1(X,\mathscr{F})$$

is an isomorphism. [*Hint*: Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$, and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$, so that we have an exact sequence

$$0 \rightarrow \mathscr{F} \rightarrow \mathscr{G} \rightarrow \mathscr{R} \rightarrow 0.$$

Define a complex $D^{\cdot}(\mathfrak{U})$ by

$$0 \rightarrow C^{\cdot}(\mathfrak{U},\mathscr{F}) \rightarrow C^{\cdot}(\mathfrak{U},\mathscr{G}) \rightarrow D^{\cdot}(\mathfrak{U}) \rightarrow 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D^{\cdot}(\mathfrak{U}) \rightarrow C^{\cdot}(\mathfrak{U},\mathscr{R}),$$

and see what happens under refinement.]

4.5. For any ringed space $(X,\mathscr{O}_X)$, let Pic X be the group of isomorphism classes of invertible sheaves (II, §6). Show that Pic $X \cong H^1(X,\mathscr{O}_X^*)$, where $\mathscr{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U,\mathscr{O}_X)$, with multiplication as the group operation. [*Hint*: For any invertible sheaf $\mathscr{L}$ on X, cover X by open sets U_i on which $\mathscr{L}$ is free, and fix isomorphisms $\varphi_i: \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathscr{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U},\mathscr{O}_X^*)$. Now use (Ex. 4.4).]

4.6. Let $(X,\mathscr{O}_X)$ be a ringed space, let $\mathscr{I}$ be a sheaf of ideals with $\mathscr{I}^2 = 0$, and let X_0 be the ringed space $(X,\mathscr{O}_X/\mathscr{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \rightarrow \mathscr{I} \rightarrow \mathscr{O}_X^* \rightarrow \mathscr{O}_{X_0}^* \rightarrow 0,$$

where $\mathscr{O}_X^*$ (respectively, $\mathscr{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathscr{O}_X$ (respectively, $\mathscr{O}_{X_0}$); the map $\mathscr{I} \rightarrow \mathscr{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathscr{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \rightarrow H^1(X,\mathscr{I}) \rightarrow \operatorname{Pic} X \rightarrow \operatorname{Pic} X_0 \rightarrow H^2(X,\mathscr{I}) \rightarrow \ldots.$$

4.7. Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0,x_1,x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1,0,0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U,\mathscr{O}_X) \oplus \Gamma(V,\mathscr{O}_X) \rightarrow \Gamma(U \cap V,\mathscr{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X,\mathscr{O}_X) = 1,$$
$$\dim H^1(X,\mathscr{O}_X) = \frac{1}{2}(d-1)(d-2).$$

4.8. *Cohomological Dimension* (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\operatorname{cd}(X)$, to be the least integer n such that $H^i(X,\mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\operatorname{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\operatorname{cd}(X) \leqslant \dim X$.

(a) In the definition of $\operatorname{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

(c) Suppose X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\operatorname{cd}(X) \leqslant r$.

*(d) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r + 1$ open affine subsets. Conclude (independently of (2.7)) that $\operatorname{cd}(X) \leqslant \dim X$.

(e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\operatorname{cd}(X - Y) \leqslant r - 1$.

4.9. Let $X = \operatorname{Spec} k[x_1,x_2,x_3,x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure

$\bar{Y}$ in $\mathbf{P}_k^4$ is also not a set-theoretic complete intersection. [*Hints*: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P, \mathcal{O}_X) \neq 0$.]

***4.10.** Let X be a nonsingular variety over an algebraically closed field k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathscr{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathscr{F} \otimes \mathscr{T})$, where $\mathscr{T}$ is the tangent sheaf of X (II,§8). [*Hint*: Use (II, Ex. 8.6) and (4.5).]

4.11. This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathscr{F}|_V) = 0$. Then prove that for all $p \geqslant 0$, the natural maps

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

5 The Cohomology of Projective Space

In this section we make explicit calculations of the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space, by using Čech cohomology for a suitable open affine covering. These explicit calculations form the basis for various general results about cohomology of coherent sheaves on projective varieties.

Let A be a noetherian ring, let $S = A[x_0, \ldots, x_r]$, and let $X = \operatorname{Proj} S$ be the projective space $\mathbf{P}_A^r$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre (II, §5). For any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$, we denote by $\Gamma_*(\mathscr{F})$ the graded S-module $\bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathscr{F}(n))$ (see II, §5).

Theorem 5.1. *Let A be a noetherian ring, and let $X = \mathbf{P}_A^r$, with $r \geqslant 1$. Then:*

(a) *the natural map $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;*

(b) *$H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbf{Z}$;*

(c) *$H^r(X, \mathcal{O}_X(-r-1)) \cong A$;*

(d) *The natural map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbf{Z}$.

PROOF. Let $\mathscr{F}$ be the quasi-coherent sheaf $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)$. Since cohomology commutes with arbitrary direct sums on a noetherian topological space (2.9.1), the cohomology of $\mathscr{F}$ will be the direct sum of the cohomology of the sheaves $\mathcal{O}(n)$. So we will compute the cohomology of $\mathscr{F}$, and keep track

of the grading by n, so that we can sort out the pieces at the end. Note that all the cohomology groups in question have a natural structure of A-module (2.6.1).

For each $i = 0, \ldots, r$, let U_i be the open set $D_+(x_i)$. Then each U_i is an open affine subset of X, and the U_i cover X, so we can compute the cohomology of $\mathscr{F}$ by using Čech cohomology for the covering $\mathfrak{U} = (U_i)$, by (4.5). For any set of indices $i_0, \ldots, i_p$, the open set $U_{i_0, \ldots, i_p}$ is just $D_+(x_{i_0} \cdots x_{i_p})$, so by (II, 5.11) we have

$$\mathscr{F}(U_{i_0, \ldots, i_p}) \cong S_{x_{i_0} \cdots x_{i_p}},$$

the localization of S with respect to the element $x_{i_0} \cdots x_{i_p}$. Furthermore, the grading on $\mathscr{F}$ corresponds to the natural grading of $S_{x_{i_0} \cdots x_{i_p}}$ under this isomorphism. Thus the Čech complex of $\mathscr{F}$ is given by

$$C^{\cdot}(\mathfrak{U}, \mathscr{F}) : \prod S_{x_{i_0}} \to \prod S_{x_{i_0} x_{i_1}} \to \ldots \to S_{x_0 \cdots x_r},$$

and the modules all have a natural grading compatible with the grading on $\mathscr{F}$.

Now $H^0(X, \mathscr{F})$ is the kernel of the first map, which is just S, as we have seen earlier (II, 5.13). This proves (a).

Next we consider $H^r(X, \mathscr{F})$. It is the cokernel of the last map in the Čech complex, which is

$$d^{r-1} : \prod_k S_{x_0 \cdots \hat{x}_k \cdots x_r} \to S_{x_0 \cdots x_r}.$$

We think of $S_{x_0 \cdots x_r}$ as a free A-module with basis $x_0^{l_0} \cdots x_r^{l_r}$, with $l_i \in \mathbf{Z}$. The image of d^{r-1} is the free submodule generated by those basis elements for which at least one $l_i \geqslant 0$. Thus $H^r(X, \mathscr{F})$ is a free A-module with basis consisting of the "negative" monomials

$$\{x_0^{l_0} \cdots x_r^{l_r} \mid l_i < 0 \text{ for each } i\}.$$

Furthermore the grading is given by $\sum l_i$. There is only one such monomial of degree $-r-1$, namely $x_0^{-1} \cdots x_r^{-1}$, so we see that $H^r(X, \mathcal{O}_X(-r-1))$ is a free A-module of rank 1. This proves (c).

To prove (d), first note that if $n < 0$, then $H^0(X, \mathcal{O}_X(n)) = 0$ by (a), and $H^r(X, \mathcal{O}_X(-n-r-1)) = 0$ by what we have just seen, since in that case $-n-r-1 > -r-1$, and there are no negative monomials of that degree. So the statement is trivial for $n < 0$. For $n \geqslant 0$, $H^0(X, \mathcal{O}_X(n))$ has a basis consisting of the usual monomials of degree n, i.e., $\{x_0^{m_0} \cdots x_r^{m_r} \mid m_i \geqslant 0$ and $\sum m_i = n\}$. The natural pairing with $H^r(X, \mathcal{O}_X(-n-r-1))$ into $H^r(X, \mathcal{O}_X(-r-1))$ is determined by

$$(x_0^{m_0} \cdots x_r^{m_r}) \cdot (x_0^{l_0} \cdots x_r^{l_r}) = x_0^{m_0 + l_0} \cdots x_r^{m_r + l_r},$$

where $\sum l_i = -n-r-1$, and the object on the right becomes 0 if any $m_i + l_i \geqslant 0$. So it is clear that we have a perfect pairing, under which $x_0^{-m_0-1} \cdots x_r^{-m_r-1}$ is the dual basis element of $x_0^{m_0} \cdots x_r^{m_r}$.

It remains to prove statement (b), which we will do by induction on r. If $r = 1$ there is nothing to prove, so let $r > 1$. If we localize the complex $C^{\cdot}(\mathfrak{U},\mathscr{F})$ with respect to x_r, as graded S-modules, we get the Čech complex for the sheaf $\mathscr{F}|_{U_r}$ on the space U_r, with respect to the open affine covering $\{U_i \cap U_r | i = 0,\ldots,r\}$. By (4.5), this complex gives the cohomology of $\mathscr{F}|_{U_r}$ on U_r, which is 0 for $i > 0$ by (3.5). Since localization is an exact functor, we conclude that $H^i(X,\mathscr{F})_{x_r} = 0$ for $i > 0$. In other words, every element of $H^i(X,\mathscr{F})$, for $i > 0$, is annihilated by some power of x_r.

To complete the proof of (b), we will show that for $0 < i < r$, multiplication by x_r induces a bijective map of $H^i(X,\mathscr{F})$ into itself. Then it will follow that this module is 0.

Consider the exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{x_r} S \to S/(x_r) \to 0.$$

This gives the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

on X, where H is the hyperplane $x_r = 0$. Twisting by all $n \in \mathbf{Z}$ and taking the direct sum, we have

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_H \to 0,$$

where $\mathscr{F}_H = \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_H(n)$. Taking cohomology, we get a long exact sequence

$$\ldots \to H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F}) \to H^i(X,\mathscr{F}_H) \to \ldots .$$

Considered as graded S-modules, $H^i(X,\mathscr{F}(-1))$ is just $H^i(X,\mathscr{F})$ shifted one place, and the map $H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F})$ of the exact sequence is multiplication by x_r.

Now H is isomorphic to $\mathbf{P}_A^{r-1}$, and $H^i(X,\mathscr{F}_H) = H^i(H,\bigoplus\mathcal{O}_H(n))$ by (2.10). So we can apply our induction hypothesis to $\mathscr{F}_H$, and find that $H^i(X,\mathscr{F}_H) = 0$ for $0 < i < r - 1$. Furthermore, for $i = 0$ we have an exact sequence

$$0 \to H^0(X,\mathscr{F}(-1)) \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{F}_H) \to 0$$

by (a), since $H^0(X,\mathscr{F}_H)$ is just $S/(x_r)$. At the other end of the exact sequence we have

$$0 \to H^{r-1}(X,\mathscr{F}_H) \xrightarrow{\delta} H^r(X,\mathscr{F}(-1)) \xrightarrow{x_r} H^r(X,\mathscr{F}) \to 0.$$

Indeed, we have described $H^r(X,\mathscr{F})$ above as the free A-module with basis formed by the negative monomials in $x_0,\ldots,x_r$. So it is clear that x_r is surjective. On the other hand, the kernel of x_r is the free submodule generated by those negative monomials $x_0^{l_0}\cdots x_r^{l_r}$ with $l_r = -1$. Since $H^{r-1}(X,\mathscr{F}_H)$ is the free A-module with basis consisting of the negative monomials in $x_0,\ldots,x_{r-1}$, and δ is division by x_r, the sequence is exact. In particular, δ is injective.

Putting these results all together, the long exact sequence of cohomology shows that the map multiplication by $x_r : H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F})$ is bijective for $0 < i < r$, as required. q.e.d.

Theorem 5.2 (Serre [3]). *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over* Spec A. *Let $\mathscr{F}$ be a coherent sheaf on X. Then:*

(a) *for each $i \geqslant 0$, $H^i(X,\mathscr{F})$ is a finitely generated A-module;*

(b) *there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $H^i(X,\mathscr{F}(n)) = 0$.*

PROOF. Since $\mathcal{O}_X(1)$ is a very ample sheaf on X over Spec A, there is a closed immersion $i: X \to \mathbf{P}_A^r$ of schemes over A, for some r, such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbf{P}^r}(1)$—cf. (II, 5.16.1). If $\mathscr{F}$ is coherent on X, then $i_*\mathscr{F}$ is coherent on $\mathbf{P}_A^r$ (II, Ex. 5.5), and the cohomology is the same (2.10). Thus we reduce to the case $X = \mathbf{P}_A^r$.

For $X = \mathbf{P}_A^r$, we observe that (a) and (b) are true for any sheaf of the form $\mathcal{O}_X(q)$, $q \in \mathbf{Z}$. This follows immediately from the explicit calculations (5.1). Hence the same is true for any finite direct sum of such sheaves.

To prove the theorem for arbitrary coherent sheaves, we use descending induction on i. For $i > r$, we have $H^i(X,\mathscr{F}) = 0$, since X can be covered by $r + 1$ open affines (Ex. 4.8), so the result is trivial in this case.

In general, given a coherent sheaf $\mathscr{F}$ on X, we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E}$, which is a finite direct sum of sheaves $\mathcal{O}(q_i)$, for various integers q_i (II, 5.18). Let $\mathscr{R}$ be the kernel,

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0.$$

Then $\mathscr{R}$ is also coherent. We get an exact sequence of A-modules

$$\ldots \to H^i(X,\mathscr{E}) \to H^i(X,\mathscr{F}) \to H^{i+1}(X,\mathscr{R}) \to \ldots .$$

Now the module on the left is finitely generated because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$, as remarked above. The module on the right is finitely generated by the induction hypothesis. Since A is a noetherian ring, we conclude that the one in the middle is also finitely generated. This proves (a).

To prove (b), we twist and again write down a piece of the long exact sequence

$$\ldots \to H^i(X,\mathscr{E}(n)) \to H^i(X,\mathscr{F}(n)) \to H^{i+1}(X,\mathscr{R}(n)) \to \ldots .$$

Now for $n \gg 0$, the module on the left vanishes because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$. The module on the right also vanishes for $n \gg 0$ because of the induction hypothesis. Hence $H^i(X,\mathscr{F}(n)) = 0$ for $n \gg 0$. Note since there are only finitely many i involved in statement (b), namely $0 < i \leqslant r$, it is sufficient to determine n_0 separately for each i. This proves (b).

Remark 5.2.1. As a special case of (a), we see that for any coherent sheaf $\mathscr{F}$ on X, $\Gamma(X,\mathscr{F})$ is a finitely generated A-module. This generalizes, and gives another proof of (II, 5.19).

As an application of these results, we give a cohomological criterion for an invertible sheaf to be ample (II, §7).

Proposition 5.3. *Let A be a noetherian ring, and let X be a proper scheme over* Spec A. *Let $\mathscr{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample;*
(ii) *For each coherent sheaf $\mathscr{F}$ on X, there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $H^i(X,\mathscr{F} \otimes \mathscr{L}^n) = 0$.*

PROOF. (i) $\Rightarrow$ (ii). If $\mathscr{L}$ is ample on X, then for some $m > 0$, $\mathscr{L}^m$ is very ample on X over Spec A, by (II, 7.6). Since X is proper over Spec A, it is necessarily projective (II, 5.16.1). Now applying (5.2) to each of the sheaves $\mathscr{F},\mathscr{F} \otimes \mathscr{L},\mathscr{F} \otimes \mathscr{L}^2,\ldots,\mathscr{F} \otimes \mathscr{L}^{m-1}$ gives (ii). Cf. (II, 7.5) for a similar technique of proof.

(ii) $\Rightarrow$ (i). To show that $\mathscr{L}$ is ample, we will show that for any coherent sheaf $\mathscr{F}$ on X, there is an integer n_0 such that $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for all $n \geqslant n_0$. This is the definition of ampleness (II, §7).

Given $\mathscr{F}$, let P be a closed point of X, and let $\mathscr{I}_P$ be the ideal sheaf of the closed subset $\{P\}$. Then there is an exact sequence

$$0 \to \mathscr{I}_P\mathscr{F} \to \mathscr{F} \to \mathscr{F} \otimes k(P) \to 0,$$

where $k(P)$ is the skyscraper sheaf $\mathscr{O}_X/\mathscr{I}_P$. Tensoring with $\mathscr{L}^n$, we get

$$0 \to \mathscr{I}_P\mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}^n \otimes k(P) \to 0.$$

Now by our hypothesis (ii), there is an n_0 such that $H^1(X,\mathscr{I}_P\mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n \geqslant n_0$. Therefore

$$\Gamma(X,\mathscr{F} \otimes \mathscr{L}^n) \to \Gamma(X,\mathscr{F} \otimes \mathscr{L}^n \otimes k(P))$$

is surjective for all $n \geqslant n_0$. It follows from Nakayama's lemma over the local ring $\mathscr{O}_P$, that the stalk of $\mathscr{F} \otimes \mathscr{L}^n$ at P is generated by global sections. Since it is a coherent sheaf, we conclude that for each $n \geqslant n_0$, there is an open neighborhood U of P, depending on n, such that the global sections of $\mathscr{F} \otimes \mathscr{L}^n$ generate the sheaf at every point of U.

In particular, taking $\mathscr{F} = \mathscr{O}_X$, we find there is an integer $n_1 > 0$ and an open neighborhood V of P such that $\mathscr{L}^{n_1}$ is generated by global sections over V. On the other hand, for each $r = 0,1,\ldots,n_1 - 1$, the above argument gives a neighborhood U_r of P such that $\mathscr{F} \otimes \mathscr{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_P = V \cap U_0 \cap \ldots \cap U_{n_1-1}.$$

Then over U_P, all of the sheaves $\mathscr{F} \otimes \mathscr{L}^n$, for $n \geqslant n_0$, are generated by global sections. Indeed, any such sheaf can be written as a tensor product

$$(\mathscr{F} \otimes \mathscr{L}^{n_0+r}) \otimes (\mathscr{L}^{n_1})^m$$

for suitable $0 \leqslant r < n_1$ and $m \geqslant 0$.

Now cover X by a finite number of the open sets U_P, for various closed points P, and let the new n_0 be the maximum of the n_0 corresponding to those points P. Then $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections over all of X, for all $n \geqslant n_0$. q.e.d.

EXERCISES

5.1. Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum(-1)^i \dim_k H^i(X,\mathscr{F}).$$

If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

5.2. (a) Let X be a projective scheme over a field k, let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbf{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbf{Z}$. We call P the *Hilbert polynomial* of $\mathscr{F}$ with respect to the sheaf $\mathscr{O}_X(1)$. [*Hints*: Use induction on dim Supp $\mathscr{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.]$$

(b) Now let $X = \mathbf{P}_k^r$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0,\ldots,x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

5.3. *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathscr{O}_X)-1).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X,\mathscr{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X,\mathscr{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X,\mathscr{O}_X).$$

[*Hint*: Use (I, 3.4).]

(b) If X is a closed subvariety of $\mathbf{P}_k^r$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geqslant 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

5.4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

(a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P:K(X) \to \mathbf{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X, $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

(b) Now let $X = \mathbf{P}_k^r$. For each $i = 0,1,\ldots,r$, let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0,\ldots,r\}$, and
(2) the map $P:K(X) \to \mathbf{Q}[z]$ is injective.

[*Hint*: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).]

5.5. Let k be a field, let $X = \mathbf{P}_k^r$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection (II, Ex. 8.4). Then:

(a) for all $n \in \mathbf{Z}$, the natural map

$$H^0(X,\mathcal{O}_X(n)) \to H^0(Y,\mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

(b) Y is connected;

(c) $H^i(Y,\mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbf{Z}$;

(d) $p_a(Y) = \dim_k H^q(Y,\mathcal{O}_Y)$.

[*Hint*: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

5.6. *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}_k^3$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that Pic $Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the *type* (a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a,b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$.

(a) Use the special cases $(q,0)$ and $(0,q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

(1) if $|a - b| \leqslant 1$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
(2) if $a,b < 0$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
(3) If $a \leqslant -2$, then $H^1(Q,\mathcal{O}_Q(a,0)) \neq 0$.

(b) Now use these results to show:

(1) if Y is a locally principal closed subscheme of type (a,b), with $a,b > 0$, then Y is connected;
(2) now assume k is algebraically closed. Then for any $a,b > 0$, there exists an irreducible nonsingular curve Y of type (a,b). Use (II, 7.6.2) and (II, 8.18).
(3) an irreducible nonsingular curve Y of type (a,b), $a,b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type (1,3), which is just the rational quartic curve (I, Ex. 3.18).

(c) If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y) = ab - a - b + 1$. [*Hint*: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q,0)$ which is a disjoint union of q copies of $\mathbf{P}^1$. See (V, 1.5.2) for another method.]

5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathcal{L}$ an invertible sheaf.

(a) If $\mathcal{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathcal{L}$ is ample on Y, where $i: Y \to X$ is the inclusion.

(b) $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{\text{red}} = \mathcal{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red}.

(c) Suppose X is reduced. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f:X \to Y$ be a finite surjective morphism, and let $\mathcal{L}$ be an invertible sheaf on Y. Then $\mathcal{L}$ is ample on Y if and only if $f^*\mathcal{L}$ is ample on X. [*Hints*: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

(b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f:\tilde{X} \to X$ be the projection. Let $\mathcal{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathcal{L}(D) \cong \mathcal{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathcal{L}_0$ on X with $f^*\mathcal{L}_0 \cong \mathcal{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

(c) If X is reduced, but not necessarily irreducible, let $X_1,\ldots,X_r$ be the irreducible components of X. Use (Ex. 4.5) to show Pic $X \to \bigoplus$ Pic X_i is surjective. Then use (Ex. 5.7c) to show X is projective.

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that Pic $X \to$ Pic X_{red} is surjective. Then use (Ex. 5.7b) to show X is projective.

5.9. *A Nonprojective Scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbf{P}_k^2$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X,\omega \otimes \mathcal{T})$ defined as follows (Ex. 4.10). Let x_0,x_1,x_2 be the homogeneous coordinates of X, let U_0,U_1,U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1, and since dim $X = 2$, we have $\omega \otimes \mathcal{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\ldots \to H^1(X,\omega) \to \text{Pic } X' \to \text{Pic } X \xrightarrow{\delta} H^2(X,\omega) \to \ldots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X,\omega) \cong k$. Since char $k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X,\omega) = 0$, we see that Pic $X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω, such that X' is not projective over k.