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EFFECTIVE MODELS OF GROUP SCHEMES

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Abstract

Let R be a discrete valuation ring with fraction field K and let X be a flat R-scheme. Given a faithful action of a K-group scheme G_K over the generic fibre X_K, we study models G of G_K acting on X. In various situations, we prove that if such a model G exists, then there exists another model G' that acts faithfully on X. This model is the schematic closure of G inside the fppf sheaf $\mathrm{Aut}_R(X)$; the major difficulty is to prove that it is representable by a scheme. For example, this holds if X is locally of finite type, separated, flat and pure and G is finite flat. Pure schemes (a notion recalled in the text) have many nice properties: in particular, we prove that they are the amalgamated sum of their generic fibre and the family of their finite flat closed subschemes. We also provide versions of our results in the setting of formal schemes.

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1. Introduction

The present paper is interested in the reduction of algebraic varieties with group action. Let us fix a discrete valuation ring R with fraction field K and residue field k. Algebraic and arithmetic geometers study all kinds of varieties, or varieties with additional structures, defined over K. In various situations, these objects have a unique model over R or over a finite extension; this is the case each time that one has a proper moduli space for the objects, but not only. Let us mention a few of these well-known models: stable models of curves (Deligne and Mumford [DM]), Néron models of abelian varieties (Néron [N]), semiabelic pairs as models of principally polarized varieties (Alexeev [Al]), stable maps as models of morphisms from a curve to a fixed variety (Abramovich and Vistoli [AV]). If a group G acts faithfully on the K-variety and the model satisfies some unicity property, the action extends to it.

Our concern is, in fact, exclusively in the reduction of the group action. The point is that even though most of the time the action of G extends as just indicated, in general the action on the special fibre is not faithful, and one wishes to consider other models of G whose action is better-behaved in reduction. For typical examples, assume that R has unequal characteristics $(0, p)$ and G is a finite p-group. If A is an abelian scheme over R, or the Néron model of an abelian variety A_K of dimension g, such that the p-torsion $A_K[p]$ is rational, then $G = \mathbb{Z}/p^{2g}\mathbb{Z}$ acts by translations. This action extends to A and, for lack of p-torsion points in characteristic p, the action has a nontrivial kernel on the special fibre. For another example, consider a smooth pointed curve (C_K, x_K) endowed with a faithful action of G leaving x_K fixed, and assume that (C_K, x_K) has a stable pointed model (C, x) over R. We wish to understand the reduction of the action, especially around the reduction x_k. We are led to focus on the orbit $Z \subset C_k$ of the irreducible component of x_k. After throwing away all components of C_k not in Z, we get an open R-curve, and we are asking for the best model for the induced action of G.

In the example above of an abelian scheme A, the R-group scheme of p-torsion $G' = A[p]$ is the obvious choice of a good model. We can recover it as follows: to the action of G is associated a morphism of R-group schemes $G_R \to \mathrm{Aut}_R(A)$, where G_R is the constant R-group scheme defined by G. Then G' is the schematic image of this morphism; the special properties of schematic images and closures over a discrete valuation ring ensure that G' is flat over R. In the examples of a Néron model or an open curve, we would like to do the same thing. But there comes a problem: these schemes are not proper, and the automorphism functor is not representable by a scheme or an algebraic space. Still, it is a sheaf for the fppf topology, and Raynaud has given a definition of schematic closures in this setting; but representability of these closures is by no means obvious, and indeed, it does not happen in general. The main theorems of this article prove that these schematic images are often representable by flat group schemes when we consider actions on *pure* schemes, the notion of purity being a (very) weak version of properness. For example, faithfully flat R-schemes with geometrically irreducible fibres without embedded components are pure. Here are some of our most striking results:

Theorem A. (i) *Let X be an R-scheme locally of finite type, separated, flat and pure. Let G be a proper flat R-group scheme acting on X, faithfully on the generic fibre. Let N denote the kernel of the action. Then the schematic image of G in $\mathrm{Aut}_R(X)$ is representable by a flat group scheme of finite type G' if and only if N_k is finite. Moreover, in this case G' is proper.*

(ii) *Let X be an affine R-scheme, equal to the spectrum of a ring A such that the map $A \to \Pi\, A/I_\lambda$ to the product of the finite flat quotients of A is universally injective. Let G be an R-group scheme locally of finite type, flat and pure, acting on X, faithfully on the generic fibre. Then the schematic image of G in $\mathrm{Aut}_R(X)$ is representable by a flat R-group scheme G'. If G is quasi-compact, or affine, or finite, then G' has the same property.*

When it is representable, we call the schematic image the *effective model* of G for its action on X. We also have versions of these results in the setting of formal schemes.

The affine version in case (ii) is interesting because it applies not only to rings of finite type, flat and pure (by Theorem B below), but also, for example, to rings arising from the completion of smooth R-schemes along a section, and also because the assumptions made on the group G are very light. Let us now focus on case (i). As it turns out, this result does not have much to do with groups. The crucial facts that govern the proof are the good properties of R-schemes locally of finite type and pure. Such a scheme X is the amalgamated sum of its generic fibre X_K and the family of all its closed subschemes finite flat over R, the latter family being schematically dense in a very strong sense. In fact, we prove the following theorem.

Theorem B. *Assume that R is Henselian. Let X be an R-scheme locally of finite type, flat and pure. Then, the family of all closed subschemes $Z_\lambda \subset X$ finite flat over R is R-universally schematically dense, and for all separated*

R-schemes Y and all diagrams in solid arrows

$$\begin{tikzcd}
\amalg Z_{\lambda,K} \arrow[r] \arrow[d] & X_K \arrow[d] \arrow[ddr] & \\
\amalg Z_\lambda \arrow[r] \arrow[drr] & X \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}$$

there exists a unique morphism $X \to Y$ making the full diagram commutative.

Here also, there is an analogue for formal schemes. Using Theorem B, we prove representability results for schematic images of schemes or formal schemes inside functors of the type $\mathrm{Hom}_R(X,Y)$. Theorem A above is essentially an application of the particular case $X = Y$.

The effective models defined in the present article have been studied in full detail, for the cyclic group of order p^2 in unequal characteristics, in the recent Ph.D. thesis of D. Tossici (see [To1] and [To2]). His results provide more examples of effective models, and show some of their general features. Also, related to this work is the note [Ab] of Abramovich. There, some group schemes over stable curves are considered. They are not unrelated with our effective models, and we plan to compare the two approaches more precisely in the near future. This will hopefully lead to some new insights on the reduction of the moduli space of admissible Galois covers of stable curves (see [BR]). The latter question is open at the moment, and it was the most important motivation for the present work.

1.1. Overview of the article. Here is a short description of the contents of the article, together with precise references to the statements of the main results. In section 2, we recall some results on purity and provide some complements. We prove openness results for some properties of the fibres of morphisms of finite presentation, flat and pure, that have some independent interest (Theorem 2.2.1). In section 3 we study schematically dominant families of morphisms from flat schemes to a fixed scheme X. We prove the density of finite flat closed subschemes (Theorem 3.2.4) as well as the amalgam property (Propositions 3.1.6, 3.2.5, and 3.2.6) which together give the statement of Theorem B. In the beginning of section 4 we introduce schematic images and we prove some useful general results on kernels for scheme or group scheme actions. Then the stage is set to prove representability of schematic images in various situations: we start with images of schemes inside Hom functors and then we prove representability of images of groups in the scheme case (Theorems 4.3.4 and 4.3.5) and in the formal scheme case (Theorems 4.4.4

and 4.4.5). Theorem A is the combination of these results. We also give some properties enjoyed by the effective model of a finite flat group scheme (Proposition 4.3.9). Finally, in section 5 we give some examples. Notably, we compute explicitly the schematic image in two different cases of degeneration of torsors under the cyclic group of order p^2 in equal characteristic $p > 0$ (see 5.2). We observe in particular that for a normal subgroup $H \subset G$, the effective model of G/H acting on X/H may be different from G'/H'.

1.2. Notation and conventions. Everywhere in the paper, we abbreviate the notation of a discrete valuation ring R with fraction field K, residue field k, and chosen uniformizer π, by the tuple (R, K, k, π). In general, the residue characteristic is denoted $p \geq 0$. For schemes or morphisms defined over R, we use subscripts $(\cdot)_K$ and $(\cdot)_k$ to denote the restrictions to the generic and the special fibre.

When R is complete, we also consider formal R-schemes. A formal scheme X may be identified with a direct system of ordinary schemes X_n over the ring $R_n = R/(\pi^n)$. We refer to [BL1] for basic facts on formal and rigid geometry, and, in particular, for the notion of *admissible formal blowing-up*. *Admissible formal schemes* in [BL1] are flat formal schemes locally of finite type. Raynaud's theorem (see [BL1], Theorem 4.1) asserts that there is an equivalence between the category of quasi-compact admissible formal R-schemes, localised by admissible formal blowing-ups, and the category of quasi-compact and quasi-separated rigid K-spaces. The K-space associated to a formal scheme X is called its *generic fibre* and denoted X_{rig}.

2. Complements on purity

2.1. Purity, projectivity and adic topologies. We first recall some definitions from Raynaud-Gruson [RG].

2.1.1. Definition. Let $X \to S$ be a morphism of schemes and $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module.

(i) The *relative assassin* of $\mathcal{M}$ over S, denoted $\mathrm{Ass}(\mathcal{M}/S)$ is the union over all $s \in S$ of the associated points $x \in X \otimes k(s)$ of $\mathcal{M} \otimes k(s)$. If $\mathcal{M} = \mathcal{O}_X$, we set $\mathrm{Ass}(X/S) = \mathrm{Ass}(\mathcal{M}/S)$.

(ii) Assume that $X \to S$ is locally of finite type and $\mathcal{M}$ is of finite type. For each $s \in S$, let $(\widetilde{S}, \widetilde{s})$ be a Henselization of (S, s). We say that $\mathcal{M}$ is *pure along* $X \otimes k(s)$ if the closure of any point $\widetilde{x} \in \mathrm{Ass}(\mathcal{M} \times_S \widetilde{S}/\widetilde{S})$ meets $X \otimes k(\widetilde{s})$. We say that $\mathcal{M}$ is *S-pure* if it is pure along $X \otimes k(s)$ for all $s \in S$. Finally, we say simply that X is *S-pure* if $\mathcal{O}_X$ is S-pure.

2.1.2. Examples. (1) If $X \to S$ is proper, then it is pure.

(2) If $X \to S$ is faithfully flat with geometrically irreducible fibres without embedded components, then it is pure.

(3) Let R be a Henselian discrete valuation ring and $X_1 = \mathrm{Spec}(R[\varepsilon, x]/(\varepsilon^2, \varepsilon x))$. Let X be the complement in X_1 of the closed point defined by the ideal (π, ε, x). Then X is not pure over R.

Here is one of the main results of [RG] (théorème 3.3.5 in part I of *loc. cit.*):

2.1.3. Theorem (Raynaud and Gruson). *Let A be a ring, B an A-algebra of finite presentation, M a B-module of finite presentation, flat over A. Then M is a projective A-module if and only if it is pure over A.*

In what follows, we shall provide some complements on the notion of purity. In particular, given an S-scheme X, we will explain the relation between purity of X and the property that X may have an open covering by affine schemes with function rings separated for some adic topologies coming from S (in particular, when S is a local scheme, the maximal-adic topology). We also give some applications.

2.1.4. Lemma. *Let S be a scheme and X, Y be S-schemes locally of finite type. Let $f\colon X \to Y$ be an fppf morphism over S. Then Y is S-pure if X is S-pure. If furthermore f is pure, then the converse holds.*

Proof. We may assume that S is a local Henselian scheme and since the locus of the base where a map is pure is open ([RG], I.3.3.8), it is enough to test purity at the closed point $s \in S$. Now let $y \in \mathrm{Ass}(Y/S)$. Choose some associated point $x \in X_y$ so $x \in \mathrm{Ass}(X/S)$. Then there exists $a \in X_s$ meeting the closure of x, so $f(a)$ meets the closure of y. So Y is S-pure.

Conversely, assume that f is pure and let $x \in \mathrm{Ass}(X/S)$ and $y = f(x)$. Thus $x \in \mathrm{Ass}(X/Y)$ and $y \in \mathrm{Ass}(Y/S)$ (see [RG], I.3.2.4). Since Y is S-pure, the closure of y meets Y_s at some point b. Let $(\widetilde{Y}, \widetilde{b})$ be a Henselization of (Y, b), let $\widetilde{X} = X \times_Y \widetilde{Y}$, and $\widetilde{x} = (x, \widetilde{b}) \in \widetilde{X}$ so that $\widetilde{x} \in \mathrm{Ass}(\widetilde{X}/\widetilde{Y})$ by [RG], I.3.2.3. Thus the closure of $\widetilde{x}$ inside $\widetilde{X}$ meets $\widetilde{X}_{\widetilde{b}}$ at a point $\widetilde{a}$. The image of $\widetilde{a}$ in X lies in the closure of x and above $\widetilde{b}$, thus in X_s. Therefore, X is S-pure. $\square$

2.1.5. Definition. Let $n \geq 1$ be an integer. We say that a morphism of schemes $X \to S$ is *of type* $(\mathrm{FA})_n$ if every set of n points of X whose images lie in an affine open set of S lie in an affine open set of X. We say that $X \to S$ is *of type* (FA) if it is of type $(\mathrm{FA})_n$ for all $n \geq 1$.

2.1.6. Lemma. *Assume that S is affine. Let $X \to S$ be of finite presentation and of type $(\mathrm{FA})_n$. Then there exists a scheme S_0 of finite type over $\mathbb{Z}$ and a morphism $S \to S_0$, an S_0-scheme X_0 of finite presentation of type $(\mathrm{FA})_n$, such that $X \simeq X_0 \times_{S_0} S$.*

Proof. Since $S = \mathrm{Spec}(A)$ is affine and $X \to S$ is quasicompact, to say that $X \to S$ is of type $(\mathrm{FA})_n$ means that there exists a finite cover by open affine schemes U_i $(1 \leq i \leq m)$ such that $\amalg(U_i)^n \to X^n$ is surjective. Thus, writing A as the inductive limit of its subrings finitely generated over $\mathbb{Z}$ and using [EGA], IV.8.10.5(iii)–(vi), we see that there exists a scheme S_0 of finite type over $\mathbb{Z}$, an S_0-scheme X_0 of finite presentation, and an open cover $U_{0,i}$ of X_0 such that $U_i \simeq U_{0,i} \times_{S_0} S$ for all i, $X \simeq X_0 \times_{S_0} S$, and $\amalg(U_{i,0})^n \to (X_0)^n$ is surjective. $\square$

In the next lemma, we relate the notion of purity for a scheme over a Noetherian Henselian local ring (R, m) with the property of separation of the function rings with respect to the m-adic topology. We will say that an R-algebra A is *strongly separated for the m-adic topology* if and only if for all prime ideals $q \subset m$, the ring A/qA is separated for the m/q-adic topology.

2.1.7. Lemma. *Let R be a Noetherian Henselian local ring with maximal ideal m. Let X be a scheme locally of finite type and flat over R. Consider the following conditions:*

(i) *X is R-pure.*
(ii) *X has an open covering by affine schemes whose function algebras are free R-modules.*
(iii) *X has an open covering by affine schemes whose function algebras are strongly m-adically separated.*

Then, we have (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). *Assume, moreover, that R is a discrete valuation ring and X is of type* $(\mathrm{FA})_{n+1}$, *where n is the number of associated points of the generic fibre. Then all three conditions* (i), (ii), *and* (iii) *are equivalent. Furthermore, we may choose an open covering $\{U_i\}$ as in* (ii)–(iii) *so that all intersections $U_i \cap U_j$ are R-pure again. Finally, if moreover, X is quasicompact, then the R-module $H^0(X, \mathcal{O}_X)$ is free.*

Proof. The fact that (ii) implies (iii) is clear since any free R-module is strongly separated for the m-adic topology. Let us check that (iii) implies (i). Let $x \in \mathrm{Ass}(X/R)$ and $U = \mathrm{Spec}(A)$ be an open affine containing x, with A strongly m-adically separated. Let $p \subset A$ (resp. $q \subset R$) be the prime ideal corresponding to x (resp. the image of x in S) and let $k(q) = R_q/qR_q$ be the residue field of q. If the closure of x in U does not meet the special fibre, there exist $u \in p$ and $v \in mA$ such that $1 = u + v$. But by assumption, there is $a \in A$ such that the image of p in $A \otimes k(q)$ is the annihilator $\mathrm{Ann}_{A\otimes k(q)}(a)$. Hence there exists $s \in R \setminus q$ such that $sua \in qA$. In the ring A/qA, we get $sa = sav = sav^n$ for all $n \geq 1$; hence, sa lies in $\bigcap_{n\geq 0} (m/q)^n(A/qA)$. The latter intersection is zero by assumption; hence, $sa = 0$ in A/qA and $a = 0$ in $A \otimes k(q)$. This is impossible. By contrapositive, X is pure.

We now prove that under the additional assumptions, we have (i) $\Rightarrow$ (ii). Call $x_1, \dots, x_n$ the associated points of the generic fibre of X. By purity, for each i the closure of x_i meets the closed fibre in at least one point x'_i. Since it is assumed that $X \to \mathrm{Spec}(R)$ is of type $(\mathrm{FA})_{n+1}$, for each $x \in X$ we may find an open affine $U_x = \mathrm{Spec}(A)$ containing $x, x'_1, \dots, x'_n$. Obviously U_x is R-pure, so it follows from 2.1.3 that A is a projective R-module, i.e. a free R-module since R is a principal ideal domain. Since X is quasicompact, we can extract from $\{U_x\}$ a finite open cover, and since each of them contains $x'_1, \dots, x'_n$, the intersections $U_i \cap U_j$ are R-pure.

Finally, we prove that $H^0(X, \mathcal{O}_X)$ is free. Let $U_i = \mathrm{Spec}(A_i)$ be an open covering by affine schemes whose function algebras are free R-modules. Since X is quasi-compact, finitely many of the U_i are sufficient. Since a submodule of a free R-module is free, the injection $H^0(X, \mathcal{O}_X) \hookrightarrow \Pi\, H^0(U_i, \mathcal{O}_{U_i})$ gives the desired result. $\square$

2.1.8. Remark. The special case where X is affine of finite type and flat over a discrete valuation ring will be useful later in the paper. In this case, the proof above shows that X is pure if and only if $\Gamma(X, \mathcal{O}_X)$ is a free R-module, if and only if $\Gamma(X, \mathcal{O}_X)$ is separated for the π-adic topology.

We now point out some features of pure schemes over a discrete valuation ring, and in particular a relation between purity and the topology of the neighbourhoods of the special fibre. Note that the notions of schematic density and schematic dominance will receive a more complete treatment in section 3; we refer to it for more details.

2.1.9. Lemma. *Let (R, K, k, π) be a discrete valuation ring. The following properties hold.*

(1) *Let $f : Z \to X$ be a morphism of R-schemes with X flat over R. Then f is schematically dominant if and only if f_K is schematically dominant.*
(2) *Let X be an R-scheme locally of finite type and pure. Then any open neighbourhood of the closed fibre X_k is schematically dense in X. If, moreover, X is flat over R, then such a neighbourhood is R-universally schematically dense.*
(3) *Let X, Y be R-schemes of finite type with X pure and Y separated. Let $\widehat{R}$, $\widehat{X}$, $\widehat{Y}$ be the π-adic formal completions of R, X, Y. Then, the completion map*

$$\mathrm{Hom}_R(X, Y) \to \mathrm{Hom}_{\widehat{R}}(\widehat{X}, \widehat{Y}), \quad f \mapsto \widehat{f}$$

is injective.

Proof. (1) This is clear, since X_K is schematically dense in X.

(2) Let U be an open neighbourhood of the closed fibre X_k. To prove that U is schematically dense, we may replace R by its Henselization and hence assume that R is Henselian. Then it is enough to prove that $\mathrm{Ass}(X) \subset U$. If $x \in \mathrm{Ass}(X)$, then by [EGA], IV.3.3.1, it is an associated point in its fibre X_s, where s is the image of x in $\mathrm{Spec}(R)$. Since X is pure, the closure of x meets X_k, hence it meets U, so $x \in U$ and we are done. If, moreover, X is flat over R, then using point (1) we see that U_K is schematically dense in X_K. Since $U_k = X_k$ is schematically dense in X_k and U is flat over R, it is R-universally schematically dense by [EGA] IV, 11.10.9.

(3) Let $f, g : X \to Y$ be such that $\widehat{f} = \widehat{g}$. By [EGA], I.10.9.4, there is an open neighbourhood $U \subset X$ of X_k where f and g are equal. It follows from (2) that U is schematically dense in X. Since $f = g$ on a schematically dense open subscheme of X, we get $f = g$ on X. □

2.1.10. Remark. Point (2) of this lemma allows us to compare pure schemes with other schemes by looking at "how close" a scheme is to its special fibre. If we arrange R-schemes by increasing distance to their special fibre, we have k-schemes, then formal R-schemes, then pure R-schemes, then general R-schemes.

2.1.11. Lemma. *Let $X \to S$ be a morphism of schemes. Assume that X is locally oetherian and S is affine. Let $s \in S$ and let $p \subset \Gamma(S, \mathcal{O}_S)$ be the corresponding ideal. Then there is an open neighbourhood of the fibre X_s that is covered by affine schemes whose function ring is separated for the p-adic topology.*

Proof. Let $x \in X_s$ and let $U_1 = \mathrm{Spec}(A_1)$ be an affine neighbourhood with A_1 Noetherian. Let $m \subset A_1$ be the prime ideal corresponding to x, so $pA_1 \subset m$. Let $I_1 = \bigcap_{n \geq 0} p^n A_1$. Since $\mathcal{O}_{X,x}$ is local Noetherian, it is separated for the p-adic topology, hence I_1 lies in the kernel of the localization morphism $A_1 \to \mathcal{O}_{X,x}$. Since I_1 is finitely generated, there is $s_1 \in A_1 \setminus m$ such that $s_1 I_1 = 0$. In other words, if we set $A_2 = A_1[1/s_1]$, then I_1 maps to 0 under $A_1 \to A_2$. By induction, after we have defined A_r, we let $I_r = \bigcap_{n \geq 0} p^n A_r$, we argue that there is $s_r \in A_r \setminus m$ such that $s_r I_r = 0$, and we define $A_{r+1} = A_r[1/s_r]$. Because A_1 is Noetherian, the increasing sequence of ideals $K_r = \ker(A_1 \to A_r)$ must stabilise at some ρ. One checks that $I_\rho = 0$, that is, A_ρ is separated for the p-adic topology. □

2.2. Application to the fibres of morphisms. We now mention an application of these results to the study of the fibres of morphisms of schemes. Namely, one can weaken the assumptions in some theorems of [EGA] IV, §12.2, by requiring purity instead of properness.

2.2.1. Theorem. *Let $f\colon X \to S$ be of finite presentation, flat and pure, and let $n \geq 1$ be an integer. Then the following sets are open in S:*

(i) *The locus of points $s \in S$ such that the fibre X_s is geometrically reduced.*
(ii) *The locus of points $s \in S$ such that the geometric fibre $X_{\overline{s}}$ is reduced with less than n connected components.*
(iii) *The locus of points $s \in S$ such that the geometric fibre $X_{\overline{s}}$ is reduced and has less than n irreducible components.*

Proof. The assertions to be proven are local on S so we may assume $S = \mathrm{Spec}(R)$ affine. By limit arguments using [RG], corollaire 3.3.10, and other usual results of [EGA] IV, §§8–11, we reduce to the case where R is Noetherian.

Let P be one of the properties *reduced*, *reduced with less than n connected components*, or *reduced with less than n irreducible components*. The loci we are interested in are constructible so it is enough to prove that they are stable under generization. By [EGA], II.7.1.7, one reduces to $R = (R, K, k, \pi)$ equal to a discrete valuation ring, which we may assume is Henselian. Then we assume that the closed fibre has the geometric property P, and we have to prove that the generic fibre has it also. For this it is enough to prove that for all finite field extensions L/K, the scheme $X \otimes_K L$ has property P. Replacing R by its integral closure in L we reduce to $K = L$. We now consider the three cases separately.

(i) By Lemma 2.1.11, there is an open neighbourhood U of the special fibre of X that is covered by open affine subschemes with function ring separated for the π-adic topology, i.e. pure. By Lemma 2.1.9, this U is universally schematically dense so if U has reduced generic fibre, then X is also. Therefore, we may replace X by U and hence assume that X is covered by pure open affine subschemes. Let $V = \mathrm{Spec}(A)$ be such an open affine, it is enough to prove that A is reduced. Since A is separated for the π-adic topology and has no π-torsion, if x is a nonzero nilpotent we may assume that $x \notin \pi A$. But then we have a contradiction with the fact that A_k is reduced. So A is reduced.

(ii) From (i) we know that X_K is reduced. Then we may as in (i) reduce to the case where X is covered by pure open affine subschemes. We shall prove that the number of connected components of X_K is less than that of X_k. Let $B = H^0(X, \mathcal{O}_X)$. From the injection $B_k \hookrightarrow H^0(X_k, \mathcal{O}_{X_k})$ we learn that B_k is reduced. This, together with an easy calculation, proves that the idempotents of B and those of B_K are the same. So X_K and X have the same number of connected components; call it u. Then B splits as a product of rings $B_1 \times \cdots \times B_u$, with $B_i \neq 0$ for $i = 1, \dots, u$. Since B is a free R-module

(Lemma 2.1.7), each of the B_i is free, and hence $B_{i,k} \neq 0$. Hence, B_k has at least 2^u idempotents, so X_k has at least u connected components.

(iii) From (i) we know that X_K is reduced. It is enough to prove that for any irreducible component W of X_k, there is a unique irreducible component Z of X_K whose closure in X contains W. For this, we may remove from X all irreducible components $W' \neq W$ of X_k and all irreducible components Y of X that do not contain W (they are closures in X of irreducible components of X_K). Hence, we may assume that X_k is integral, and we have to prove that X_K is integral also. We may as in (i) reduce to the case where X is covered by pure open affine subschemes. It is then enough to prove that all such open affines $V = \mathrm{Spec}(A)$ are integral. But if $xy = 0$ in A, and x, y are nonzero, we may as in (i) assume that they do not belong to πA. Then this contradicts the fact that A_k is integral. $\square$

2.2.2. Counter-examples. Obviously, the corollary does not extend to all properties listed in [EGA], IV, §12.2. We give counter-examples for some of them. Let (R, K, k, π) be a discrete valuation ring.

(1) *Geometrically connected.* Let $A = R[t]/(t^2 - \pi t)$ and $X = \mathrm{Spec}(A)$. Then X_k is geometrically connected, but X_K has two connected components.
(2) *Geometrically pointwise integral.* Let $A = R[e, x, y]/I$ where I is the ideal generated by the four elements xy, $e^2 - e + \pi$, $(1 - e)x - \pi x$, $ey - \pi y$. Let $X = \mathrm{Spec}(A)$. Then X_k is geometrically pointwise integral (with two connected components), but X_K is not, for it is geometrically connected and A_K has zero divisors x, y.
(3) *Smooth, geometrically normal, etc.* Let X be a flat finite type R-scheme with geometrically integral fibres without embedded components. Let U be the complement in X of the singular locus of X_k. Then U is again pure over R, with smooth special fibre, but the generic fibre can be chosen to have arbitrary singularities.

3. Reconstructing a scheme from flat closed subschemes

In this section, we consider two types of situations:

(I) *Ordinary*: a discrete valuation ring (R, K, k, π) and an R-scheme X with a family of morphisms of R-schemes $Z_\lambda \to X$ indexed by a set L.

(II) *Formal*: a complete discrete valuation ring (R, K, k, π) and a formal R-scheme X with a family of morphisms of formal R-schemes $Z_\lambda \to X$ indexed by a set L.

Most of the time, we write this family as a single morphism $f : \amalg Z_\lambda \to X$. After some generalities in subsection 3.1, we specialise in subsection 3.2 to the case where f is the family of all (formal) closed subschemes of X finite flat over R. The general theme is to find some conditions under which X is the amalgamated sum of its generic fibre X_K and the subschemes Z_λ along the subschemes $Z_{\lambda,K}$ (in the formal case, the generic fibres are the rigid analytic spaces X_{rig} and $Z_{\lambda,\mathrm{rig}}$). As a matter of notation, when no confusion seems possible, we will allow ourselves a slight abuse by maintaining the letter f to denote the restriction $Z_{\lambda_0} \hookrightarrow \amalg Z_\lambda \to X$, for a given $\lambda_0 \in L$. For example, we will write $f_*\mathcal{O}_{Z_\lambda}$ instead of $(f_{|Z_\lambda})_*\mathcal{O}_{Z_\lambda}$.

3.1. Schematically dominant morphisms. We will need various notions of dominant morphisms; see also [EGA], IV.11.10.

3.1.1. Definitions. Let $f : \amalg Z_\lambda \to X$, $\lambda \in L$, be a family of morphisms of R-schemes.

(1) If X is affine, f is called *affinely dominant* if the intersection of the kernels of the maps $\Gamma(X, \mathcal{O}_X) \to \Gamma(Z_\lambda, \mathcal{O}_{Z_\lambda})$ is 0. If X is arbitrary, f is called *weakly schematically dominant* if there exists a covering of X by open affine subschemes U_i such that $f^{-1}(U_i) \to U_i$ is affinely dominant for all i.
(2) The map f is called *schematically dominant* if the intersection of the kernels of the maps of sheaves $\mathcal{O}_X \to (f_\lambda)_*\mathcal{O}_{Z_\lambda}$ is 0, or equivalently, if for *all* open affine subschemes $U \subset X$, the map $f^{-1}(U) \to U$ is affinely dominant.

If one of these properties is true after any base change $R \to R'$, we say that it is true universally.

The family of maps $\amalg_{n\geq 0} \operatorname{Spec}(R/\pi^n) \to \operatorname{Spec}(R)$ is affinely dominant, hence weakly schematically dominant, but not schematically dominant.

If X is affine, it is equivalent to say that f is affinely dominant or that for any two morphisms $u, v\colon X \to X'$ to an affine R-scheme X', $u \circ f = v \circ f$ implies $u = v$. If X is arbitrary, it is equivalent to say that f is schematically dominant or that for any open set $U \subset X$, and any two morphisms $u, v\colon U \to X'$ to a separated R-scheme X', if the compositions of u and v with the restriction $f^{-1}(U) \to U$ are equal, then $u = v$. In the case where each $f_{|Z_\lambda}$ is an immersion, this gives the notion of a *schematically dense* family of subschemes.

If we consider a family of morphisms of formal R-schemes $f : \amalg Z_\lambda \to X$, $\lambda \in L$, the same definitions and remarks apply word for word.

In the sequel, we will meet one particular case where weakly schematically dominant are schematically dominant. In order to explain this, we recall the following standard notation: if I, J are ideals in a ring A, we write $(I : J)_A$ or simply $(I : J)$ for the ideal of elements $a \in A$ such that $aJ \subset I$, and we write $(I : J^\infty)$ for the increasing union of the ideals $(I : J^n)$. The following definition applies in the case of schemes or formal schemes.

3.1.2. Definition. We say that *the torsion in $f_*\mathcal{O}_{Z_\lambda}$ is bounded uniformly in λ* if and only if for all $U \subset X$ open, for all $t \in \mathcal{O}_X(U)$, there exists an integer $c \geq 1$ such that for all $\lambda \in L$, we have $(0 : t^\infty) = (0 : t^c)$ as ideals of $(f_*\mathcal{O}_{Z_\lambda})(U)$.

3.1.3. Lemma. *Let $f : \amalg Z_\lambda \to X$, $\lambda \in L$, be a family of morphisms of R-schemes or formal R-schemes. Assume that either L is finite, or the torsion in $f_*\mathcal{O}_{Z_\lambda}$ is bounded uniformly in λ. Then f is schematically dominant if and only if it is weakly schematically dominant.*

Proof. Only the *if* part needs a proof. Let $U = \mathrm{Spec}(A)$ in the scheme case, resp. $U = \mathrm{Spf}(A)$ in the formal scheme case, be an open affine such that $f^{-1}(U) \to U$ is affinely dominant. Let $B_\lambda = (f_*\mathcal{O}_{Z_\lambda})(U)$, $\varphi_\lambda : A \to B_\lambda$ the map corresponding to f_λ, and $I_\lambda = \ker(\varphi_\lambda)$. The intersection of the ideals I_λ is zero and we have to prove that for all $t \in A$, the intersection of the kernels of the maps $\varphi_\lambda[1/t] : A[1/t] \to B_\lambda[1/t]$ is zero. Let a be in this intersection. Clearly it is enough to take $a \in A$. For all λ there is an integer $c_\lambda \geq 0$ such that $t^{c_\lambda}\varphi_\lambda(a) = 0$. If the torsion in $f_*\mathcal{O}_{Z_\lambda}$ is bounded uniformly in λ, there is an integer c such that for all λ we have $t^c\varphi_\lambda(a) = 0$. If L is finite, this is also true with $c = \sup\{c_\lambda\, ,\, \lambda \in L\}$. It follows that $t^c a$ is in the intersection of the I_λ, hence zero by assumption. Thus $a = 0$ in $A[1/t]$. □

3.1.4. Remark. We will also use this lemma in the case where the base ring R is a field (*cf.* proof of Theorem 3.2.4), and it is clear that it holds true also in this context.

We now use more specifically the properties of flat modules over the discrete valuation ring R. The first lemma below is stated as a useful observation to keep in mind. Then we continue with some properties of schemes dominated by flat families.

3.1.5. Lemma. *For a morphism of R-modules $u : M \to N$ with N flat, the following conditions are equivalent:*

(1) *u is universally injective.*
(2) *u is injective and u_k is injective.*
(3) *u is injective and* $\mathrm{coker}(u)$ *is flat.*

If N is a direct product of flat modules N_λ, $\lambda \in L$, and we denote by I_λ the kernel of $M \to N_\lambda$, these conditions are also equivalent to:

(4) $\bigcap_{\lambda\in L} I_\lambda = 0$ *and* $\bigcap_{\lambda\in L} I_{\lambda,k} = 0$.

Proof. See for instance [EGA], $\mathrm{O_{III}}$.10.2, notably [EGA], $\mathrm{O_{III}}$.10.2.4. □

The main point of the following result is to say that X satisfies the property of the amalgamated sum of X_K and the Z_λ along their respective generic fibres, for morphisms to affine R-schemes Y.

3.1.6. Proposition. *Let $f : \amalg Z_\lambda \to X$ be a family of morphisms of R-schemes with Z_λ flat over R, for all $\lambda \in L$. Assume, moreover, that we are in one of the following cases.*

(i) *X has a covering by open affine schemes U_i whose function algebras are π-adically separated and the restriction of f_k to $f^{-1}(U_i)_k$ is affinely dominant.*
(ii) *X is locally Noetherian and f_k is schematically dominant.*

Then the following properties hold:

(1) *X is flat over R.*
(2) *f, equivalently f_K, is weakly schematically dominant (in case* (ii) *one needs to assume also that X is locally of finite type and pure).*
(3) *For all affine R-schemes Y and all diagrams in solid arrows,*

$$\begin{tikzcd}
\amalg Z_{\lambda,K} \arrow[r] \arrow[d] & X_K \arrow[d] \arrow[ddr] & \\
\amalg Z_\lambda \arrow[r] \arrow[drr] & X \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}$$

there exists a unique morphism $X \to Y$ making the full diagram commutative.

Note that the equivalence in point (2) between the fact that one of the two morphisms f or f_K is weakly schematically dominant is granted by Lemma 2.1.9 (although that lemma is not stated for weakly schematically dominant morphisms, it is clear that it holds for these morphisms, with the same proof).

Proof. Observe that after we have proven that X is flat, in order to prove the amalgamated sum property to affine schemes, since X is flat and Y is separated, the map $g\colon X \to Y$ is unique if it exists. Thus we may define it locally on X and glue. It follows that all assertions to be established are local.

In case (i) we are immediately reduced to the situation where $X = \mathrm{Spec}(A)$ with A separated for the π-adic topology. We keep the notation of the proof of Lemma 3.1.3 and we also set $B = \Pi\, B_\lambda$, $\varphi = \Pi\, \varphi_\lambda$ and $I = \ker(\varphi)$. From the injection $A/I \hookrightarrow B$ it follows that A/I has no π-torsion hence is flat over R. If $a \in I$, then since φ_k is injective, there exists $a_1 \in A$ such that $a = \pi a_1$. Since A/I has no π-torsion, a_1 himself lies in I, and by induction we obtain $a \in \bigcap \pi^n A$. So $a = 0$ by the assumption on A. This proves that A is torsion-free, hence flat over R, and also that f is weakly schematically dominant. Now we have a diagram with all morphisms injective:

$$\begin{array}{ccc} B & \longrightarrow & B_K \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_K \end{array}$$

Obviously, in order to prove the amalgamated sum property for maps to affine schemes, it is enough to show that A is isomorphic to the fibred product $A_K \times_{B_K} B$. Since A is separated for the π-adic topology, a nonzero element in $B \cap A_K$ may be written a/π^d with $a \in A$ and $d \in \mathbb{Z}$ minimal, such that there exists $b \in B$ with $a = \pi^d b$ in B. If $d \geqslant 1$, reducing modulo π we find that the image of a vanishes in B_k. Since $A_k \to B_k$ is injective, it follows that $a \in \pi A$, and this contradicts the minimality of d. Hence $d \leq 0$, so $a/\pi^d \in A$ and we are done.

In case (ii), in order to prove flatness it is enough to look at points of the special fibre X_k. By Lemma 2.1.11, such a point has an affine neighbourhood $\mathrm{Spec}(A)$ with A separated for the π-adic topology. From case (i) it follows that X is flat. Also, in this way we have found a neighbourhood U of the special fibre which is covered by open affine schemes whose function algebras are π-adically separated. From case (i) it follows that the restriction of f to $f^{-1}(U)$ is weakly schematically dominant. So, if X is locally of finite type and pure, U is schematically dense in X by Lemma 2.1.9; hence, f itself is weakly schematically dominant. Finally, to prove the amalgamated sum property, it is enough to define g in a neighbourhood of all closed points $x \in X_k$. By Lemma 2.1.11, we may choose a neighbourhood $\mathrm{Spec}(A)$ where A is π-adically separated. Then we are reduced to case (i). □

It is possible to formulate an analogue of the amalgamated sum property for formal schemes finite type, using the definition of the generic fibre as a rigid analytic K-space as in [BL1]. Since we have to impose the assumption of finite type, the direct formal analogue of the affine version 3.1.6 is not relevant. Hence we will content ourselves with a statement of the properties needed in order to prove 3.2.6.

3.1.7. Proposition. *Assume that R is complete. Let $f : \amalg Z_\lambda \to X$ be a family of morphisms of formal R-schemes locally of finite type, with Z_λ flat over R for all $\lambda \in L$, such that f_k is schematically dominant. Then,*

(1) *X is flat over R.*
(2) *f (equivalently f_K) is weakly schematically dominant.*

Proof. (1) We may restrict to an open affine formal subscheme $\mathrm{Spf}(A)$. Then A is π-adically separated and the arguments of the proof of point (1) in Proposition 3.1.6 carry on.

(2) The arguments are the same as in point (2) in Proposition 3.1.6. □

In the sequel of the paper, we will be mainly interested in the case where L is infinite. Concerning the case where L is finite (this is essentially the case where L has just one element, for, one may consider $Z = \amalg Z_\lambda$), the following property is still worth recording:

3.1.8. Proposition. *Let S be a scheme and let $f : Z \to X$ be a morphism of flat S-schemes of finite presentation. Assume that X is pure. Let $S_0 \subset S$ be the locus of points $s \in S$ such that f_s is schematically dominant, $X_0 = X \times_S S_0$, $Z_0 = Z \times_S S_0$. Then S_0 is open in S an $f_{|Z_0} : Z_0 \to X_0$ is S_0-universally schematically dominant.*

Proof. As in the proof of Theorem 2.2.1, one reduces to the case where S is the spectrum of a Henselian discrete valuation ring R with uniformizer π, and f_k is schematically dominant. By Lemma 2.1.11, there is an open neighbourhood U of the special fibre of X that is covered by open affine schemes whose function algebras are π-adically separated. By 3.1.6(2) and 3.1.3, the restriction of f to U is schematically dominant. Since U is schematically dense in X by Lemma 2.1.9, then f is schematically dominant. The fact that $f_{|Z_0} : Z_0 \to X_0$ is S_0-universally schematically dominant is a consequence of [EGA] IV, 11.10.9. □

3.2. Gluing along the finite flat subschemes. We continue with the ordinary (I) and formal (II) situations presented at the beginning of section 3. From now on, the family Z_λ will always be the family of all closed subschemes of X in case (I), resp. closed formal subschemes of X in case (II), that are finite and flat over R. We denote this family by $\mathcal{F}(X)$. Under some mild conditions, we will prove that this family is R-universally schematically dense in X and we will improve Proposition 3.1.6 by extending the amalgamated sum property to morphisms to arbitrary separated (formal) schemes Y.

We keep the notation $f : \amalg Z_\lambda \to X$ for the canonical morphism induced by the inclusions $Z_\lambda \subset X$. Note that $\mathcal{F}(X)$ is naturally an inductive system, if we consider it together with the closed immersions $Z_\lambda \hookrightarrow Z_\mu$. Moreover, we can define the union of two finite flat closed subschemes by the intersection of the defining ideals; this is again a finite flat closed subscheme. In this way, we see that $\mathcal{F}(X)$ is filtering.

Let us start our program. We start with a well-known property.

3.2.1. Lemma. *Consider one of the two situations:*

(I) *X is an R-scheme locally of finite type. Assume that X is flat over R, or more generally that X_{red} is flat over R.*

(II) *R is complete and X is a flat formal R-scheme locally of finite type.*

Then $\mathcal{F}(X)$ attains all the closed points of X_k. In case (I) *the converse is true: if $\mathcal{F}(X)$ attains all the closed points of X_k, then X_{red} is flat over R.*

Proof. In case (I), first note that X_{red} is flat if and only if no irreducible component of X is included in the special fibre. Hence if X_{red} is flat, for each closed point $x \in X_k$, there is an irreducible component $W \subset X$ at x that is not contained in X_k. Then the claim follows from Proposition 10.1.36 of [Liu] applied to W. Conversely, if X_{red} is not flat, then there is an irreducible component included in the special fibre, and it is clear that this component contains at least one point not lying on any $Z \in \mathcal{F}(X)$. In case (II) this is just [BL1], Proposition 3.5. □

For the sequel, a crucial ingredient is a theorem of Eisenbud and Hochster (see [EH]) which we recall for convenience:

3.2.2. Theorem (Eisenbud and Hochster). *Let A be a ring, and let P be a prime ideal of A. Let $\mathcal{N}$ be a set of maximal ideals m such that A_m/P_m is a regular local ring, and such that*

$$\bigcap_{m\in\mathcal{N}} m = P.$$

If M is a finitely generated P-coprimary module annihilated by P^e, then

$$\bigcap_{m\in\mathcal{N}} m^e M = 0.$$

As a preparation for the proof of Theorem 3.2.4 below, we first establish a lemma. We refer to Bruns-Herzog [BH] for more details on the following notions. Let (A, m) be a Noetherian local ring of dimension r, and write $\lg_A(M)$ or simply $\lg(M)$ for the length of an A-module M. For an arbitrary ideal of definition $q \subset A$, one defines the Hilbert-Samuel multiplicity $\mathfrak{e}(q)$ as the coefficient of $i^r/r!$ in the polynomial-like function $i \mapsto \lg_A(A/q^i)$. The Hilbert-Samuel multiplicity of A itself is defined to be $\mathfrak{e}(m)$. If A is Cohen-Macaulay and q is a parameter ideal (that is, an ideal generated by a system of parameters), we have $\mathfrak{e}(q) = \lg(A/q)$. If, moreover, the residue field is infinite, there exists a parameter ideal q such that $\mathfrak{e}(q) = \mathfrak{e}(m)$ (see exercise 4.5.14 in [BH]).

3.2.3. Lemma. *Let k be a separably closed field and $X = \mathrm{Spec}(A)$ an affine scheme of finite type over k. Then there exists an integer $c \geq 1$, a set of Cohen-Macaulay closed points $\mathcal{M} \subset X$, and for all points $x \in \mathcal{M}$ a parameter ideal $q_x \subset \mathcal{O}_{X,x}$ satisfying* $\dim_k(\mathcal{O}_{X,x}/q_x) \leq c$*, such that*

$$\bigcap_{x \in \mathcal{M}} q'_x = 0$$

where q'_x is the preimage of q_x in A.

Proof. Let $0 = I_1 \cap \cdots \cap I_r$ be a primary decomposition of the sub-A-module $0 \subset A$, where I_j is a P_j-primary ideal, $P_j = \sqrt{I_j}$. For each $1 \leq j \leq r$, let e_j be such that $(P_j)^{e_j} \subset I_j$. The closed subscheme Z_j defined by the ideal P_j is a variety, in particular it is reduced. On one hand, by classical properties of schemes of finite type over a field, there is a dense open set $U_j \subset Z_j$ of points that are regular in Z_j and Cohen-Macaulay in X. On the other hand, let k^{a} be an algebraic closure of k, and let S_j be the smooth locus of the reduced subscheme of $Z_j \otimes_k k^{\mathrm{a}}$. It is defined over a finite purely inseparable extension ℓ_j/k, whose degree we call γ_j. Hence there is a smooth ℓ_j-scheme V_j whose pullback to k^{a} is S_j. Since ℓ_j is separably closed, the set of ℓ_j-rational points of V_j is dense. Therefore, the set $\mathcal{M}_j = U_j \cap V_j(\ell_j)$ is dense in Z_j. By Theorem 3.2.2 applied with $\mathcal{N} = \mathcal{M}_j$ and $M = A/I_j$, we have

$$\bigcap_{x \in \mathcal{M}_j} m^{e_j} A \subset I_j$$

where m denotes the maximal ideal of A corresponding to the point x. We call $e = \max(e_j)$, $\mathcal{M} = \bigcup \mathcal{M}_j$, $\gamma = \max(\gamma_j)$. Then, for all $x \in \mathcal{M}$, we have $[k(x) : k] \leq \gamma$, and

$$\bigcap_{x \in \mathcal{M}} m^e A \ \subset \ I_1 \cap \cdots \cap I_r = 0.$$

We now choose suitable parameter ideals q_x. For $x \in X$ we let $\mathfrak{e}(x)$ denote the Hilbert-Samuel multiplicity of the local ring at x. This is an upper-semicontinuous function, hence it is bounded on X by some constant α. By the remarks preceding the lemma, for each Cohen-Macaulay closed point $x \in X$, we can find a parameter ideal $q = (r_1, \ldots, r_s)$ with $\mathfrak{e}(q) = \mathfrak{e}(x)$, where $s = \dim(\mathcal{O}_{X,x}) \leq n = \dim(X)$. Now $q_x := ((r_1)^e, \ldots, (r_s)^e)$ is again a parameter ideal, with $q_x \subset m^e$. It follows from the above that if q'_x denotes the preimage of q_x in A, then

$$\bigcap_{x \in \mathcal{M}} q'_x = 0.$$

Furthermore, one sees readily that if $\beta = s(e-1)+1$, then $q^\beta \subset q_x$. Thus,

$$\lg(\mathcal{O}_{X,x}/q_x) = \mathfrak{e}(q_x) \leq \mathfrak{e}(q^\beta) = \beta^s \mathfrak{e}(q) \leq \beta^s \alpha \ .$$

Finally, since the degree of the residue fields of points $x \in \mathcal{M}$ is bounded by γ, we have

$$\dim_k(\mathcal{O}_{X,x}/q_x) = [k(x):k]\ \lg(\mathcal{O}_{X,x}/q_x) \le \gamma\beta^s\alpha \le \gamma(n(e-1)+1)^n\alpha.$$

If we set $c := \gamma(n(e-1)+1)^n\alpha$, we have proven all the assertions of the lemma. □

3.2.4. Theorem. *Consider one of the two situations:*

(I) *R is Henselian and X is an R-scheme locally of finite type, flat and pure.*

(II) *R is complete and X is a flat formal R-scheme locally of finite type.*

Then the family $\mathcal{F}(X)$ of all closed (formal) subschemes $Z_\lambda \subset X$ finite flat over R is R-universally schematically dense.

Proof. We start with case (I). We first assume that R is strictly Henselian. By Lemma 2.1.11, there is an open neighbourhood U of the special fibre of X that is covered by open affine subschemes with function ring separated for the π-adic topology. Lemma 2.1.9 implies that U is R-universally schematically dense in X. Therefore we may replace X by U and hence assume that X is covered by open affine subschemes with function ring separated for the π-adic topology. Since the result is local on X we may finally assume that X is affine, with function ring A of finite type over R, separated for the π-adic topology (and in fact free, by Remark 2.1.8).

By Lemma 3.2.3, there exists a constant $c \ge 1$, a set of Cohen-Macaulay closed points $\mathcal{M} \subset X_k$, and parameter ideals $q_x \subset \mathcal{O}_{X_k,x}$ satisfying $\dim_k(\mathcal{O}_{X_k,x}/q_x) \le c$ and such that the ideals $q'_x = q_x \cap A_k$ have zero intersection. We let $\{Z^c_\lambda\}$, $\lambda \in L^c$, denote the family of all closed subschemes of X, finite flat over R, of degree less than c, and we write $f^c : \amalg Z^c_\lambda \to X$ for the canonical morphism.

The ideal q_x is generated by a regular sequence $r = (r_1, \dots, r_s)$, where $s = \dim(\mathcal{O}_{X_k,x})$. Let $\widetilde{r}$ be a sequence obtained by lifting the r_i in $\mathcal{O}_{X,x}$ and let $Y = \mathrm{Spec}(\mathcal{O}_{X,x}/(\widetilde{r}))$. As r is a regular sequence, it follows that Y is flat over R. Furthermore, Y_k is Artinian, hence Y is quasi-finite over R. Since R is Henselian, Y is in fact finite over R. Thus $Y \to X$ is a proper monomorphism, hence a closed immersion. So Y is one of the schemes Z^c_λ.

Since the k-algebras of functions of $Z^c_{\lambda,k}$ are free of rank less than c, the Cayley-Hamilton theorem implies that in the terminology of Definition 3.1.2, the torsion in $(f^c_k)_*\mathcal{O}_{Z^c_{\lambda,k}}$ is bounded uniformly in λ (in a strong form, since the bound c is independent of the local sections t). As the intersection of the ideals $q'_x = q_x \cap A_k$ is zero, Lemma 3.1.3 applies and proves that f^c_k is schematically dominant. Moreover, the R-algebras of functions of Z^c_λ are free of rank less than c, so the argument used above works again and by Proposition 3.1.6 we get that f^c and f^c_K are schematically dominant. Applying [EGA] IV, 11.10.9, it follows that f^c is R-universally schematically dominant. A fortiori, the family $\mathcal{F}(X)$ is R-universally schematically dense.

It remains to treat the case of a general Henselian discrete valuation ring R. Let R^{sh} be a strict Henselization, and $X^{\mathrm{sh}} = X \otimes_R R^{\mathrm{sh}}$. By the preceding discussion we know that $\mathcal{F}(X^{\mathrm{sh}})$ is universally schematically dense in X^{sh}. Since R^{sh} is an integral extension of R, the canonical morphism $j\colon X^{\mathrm{sh}} \to X$ is integral. Thus the schematic image of any finite R^{sh}-flat closed subscheme $Z^{\mathrm{sh}} \subset X^{\mathrm{sh}}$ is an R-flat closed subscheme Z of X, integral over R, hence a finite flat R-scheme. This proves that the family $\{j^{-1}(Z)\}$, with $Z \in \mathcal{F}(X)$, is a cofinal subfamily of $\mathcal{F}(X^{\mathrm{sh}})$, thus it is universally schematically dense in X^{sh}. By faithfully flat descent (see [EGA], IV.11.10.5), so is $\mathcal{F}(X)$ in X.

In case (II), we follow the same strategy of proof. We start with the case where R is strictly Henselian. We reduce to the formal affine case $X = \mathrm{Spf}(A)$, with A topologically of finite type over R. Such an A is automatically separated for the π-adic topology. Then we consider the family $\{Z^c_\lambda\}$ of all closed formal subschemes of X, finite flat over R, of degree less than c. We apply Lemma 3.2.3 again, and as before, for each Cohen-Macaulay closed point x in $\mathcal{M} \subset X_k$, we can realize the subscheme defined by the parameter ideal $q_x \subset \mathcal{O}_{X_k,x}$ as the special fibre of some Z^c_λ. Then we use Proposition 3.1.7 to get that f^c and f^c_K are schematically dense. It makes no difficulty to adapt [EGA], IV, 11.10.9, to formal schemes and conclude that f^c and a fortiori $\mathcal{F}(X)$ is R-universally schematically dense. Also, the argument from [EGA] to descend from the strict Henselization to R is easily adapted. $\square$

3.2.5. Proposition. *Let X be an R-scheme locally of finite type and flat. Let $\{Z_\lambda\}$ be a family of closed subschemes of X finite flat over R, and assume that the family $\{Z_{\lambda,k}\}$ is schematically dense in X_k and attains all closed points (e.g. R is Henselian, X is pure and $\{Z_\lambda\}$ is the family of all closed subschemes of X finite flat over R, by Lemma* 3.2.1 *and Theorem* 3.2.4*). Then for all separated R-schemes Y and all diagrams in solid arrows,*

$$\begin{tikzcd}
\amalg Z_{\lambda,K} \arrow[r] \arrow[d] & X_K \arrow[d] \arrow[ddr, "\beta"] & \\
\amalg Z_\lambda \arrow[r] \arrow[drr, "\alpha"'] & X \arrow[dr, dotted] & \\
& & Y
\end{tikzcd}$$

there exists a unique morphism $g : X \to Y$ making the full diagram commutative.

Proof. In fact, the flatness of X follows from the other assumptions, by Proposition 3.1.6. Let $f : \amalg Z_\lambda \to X$, $\alpha : \amalg Z_\lambda \to Y$ and $\beta : X_K \to Y$ be the maps in the diagram. By the same argument, as in the proof of Proposition 3.1.6, the map $g : X \to Y$ is unique if it exists. Thus we may define it locally on X and glue. It is enough to define g in a neighbourhood of all closed points $x \in X_k$. By assumption f is surjective on closed points of the special fibre, so the given point x is equal to $f(z)$ for some λ and $z \in Z_\lambda$. Let $y = \alpha(x)$, let $V = \mathrm{Spec}(C)$ be an open affine neighbourhood of y in Y, and let U be an open subscheme of X containing x. We will prove that x does not belong to the closure in X of $X_K \setminus \beta^{-1}(V)$. Indeed, otherwise there is a point $\eta \in X_K \setminus \beta^{-1}(V)$ such that $x \in W := \overline{\{\eta\}}$. Thanks to Lemma 3.2.1 applied to W, we may replace η by a closed point of W_K and hence we assume that η is closed in X_K. In this case W is one of the Z_λ, so it makes sense to speak about the images of x and η under α. Then,

$$x \in \overline{\{\eta\}} \quad \text{implies that} \quad y = \alpha(x) \in \overline{\{\alpha(\eta)\}} = \overline{\{\beta(\eta)\}}$$

and this is a contradiction with the fact that $\beta(\eta) \notin V$. Therefore, we may shrink U and assume that $U_K \subset \beta^{-1}(V)$. Then by Lemma 2.1.11 we may shrink U further to the spectrum of a ring A separated for the π-adic topology. Therefore, we reduce to $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(C)$, and Proposition 3.1.6 applies. □

3.2.6. Proposition. *Assume that R is complete. Let X be a flat formal R-scheme of finite type. Let $\{Z_\lambda\}$ be a family of closed formal subschemes of X finite flat over R such that the family $\{Z_{\lambda,k}\}$ is schematically dense in X_k and attains all closed points (e.g. the family of all closed subschemes of X finite flat over R). Let $f : \amalg Z_\lambda \to X$ be the canonical map. Then the analogue of the amalgamated sum property of Proposition* 3.2.5 *holds, if we understand a morphism from a rigid analytic K-space Z to a formal R-scheme Y to be a morphism $Z \to Y_{\mathrm{rig}}$. More precisely, given*

- *a separated formal R-scheme Y,*
- *a morphism of formal R-schemes $\alpha : \amalg Z_\lambda \to Y$,*
- *a morphism of rigid spaces $\beta : X_{\mathrm{rig}} \to Y_{\mathrm{rig}}$*

such that $\alpha_{\mathrm{rig}} = \beta \circ f_{\mathrm{rig}}$, there exists a unique morphism $g : X \to Y$ such that $g_{\mathrm{rig}} = \beta$ and $g \circ f = \alpha$.

Proof. The proof of Proposition 3.2.5 works again in this setting, with some adaptations which we now sketch. If $g, g' : X \to Y$ are two solutions to the problem, then in particular $g_{\mathrm{rig}} = g'_{\mathrm{rig}}$. By Raynaud's theorem ([BL1], th. 4.1) there exists an admissible formal blowing-up $s : X' \to X$ such that $g \circ s = g' \circ s$. Since s is schematically dominant and Y is separated, we get $g = g'$. Because of this unicity statement, as far as existence is concerned, we may define g locally on X and glue. Also, we need to know that $\mathcal{F}(X)_k$ is schematically dense, which is granted by Theorem 3.2.4. Then, by the same method as above, we reduce to the affine case $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(C)$. Now the arguments of the proof of point (3) in Proposition 3.1.6 carry on. $\square$

4. Schematic images inside Hom and Aut functors

Throughout this section, we fix a discrete valuation ring (R, K, k, π). We first recall the definition of schematic closures and images for fppf sheaves over a discrete valuation ring R. After a brief discussion of kernels, we prove the main theorems of the paper on representability of schematic images.

4.1. Definitions. Recall that if $f\colon W \to X$ is a morphism of schemes, there exists a smallest closed subscheme $X' \subset X$ such that f factors through X'. We call it the *schematic image* of f. If $U = \mathrm{Spec}(A)$ is an open affine subscheme of X and $V = f^{-1}(U)$, then $X' \cap U$ is defined by the ideal which is the kernel of the map $A \to \Gamma(V, \mathscr{O}_V)$ induced by f. It is equivalent to say that the schematic image of f is X, or that f is schematically dominant.

If W is a closed subscheme of the generic fibre of X and f is the canonical immersion, then the schematic image is called the *schematic closure* of W in X. It is the unique closed subscheme of X which is flat over R and whose generic fibre is W (see [EGA], IV.2.8.5).

These definitions may be adapted to morphisms of sheaves as follows (see [Ra]):

4.1.1. Definitions. Let F be an fppf sheaf over the category of R-schemes.

(1) Let G be a subsheaf of the generic fibre F_K. Then the *schematic closure of G in F* is the fppf sheaf G' associated to the presheaf $G^\flat$ defined as follows. Given an R-scheme T, $G^\flat(T)$ is the set of all morphisms $f\colon T \to F$ such that there exists a factorization,

$$\begin{tikzcd} T \arrow[r] \arrow[dr, "f"'] & T' \arrow[d, "g"] \\ & F \end{tikzcd}$$

with T' a flat R-scheme and $g(T'_K) \subset G$.

(2) We say that F is *flat over R* if it is equal to the schematic closure of its generic fibre.

(3) Let $h\colon H \to F$ be a morphism of fppf sheaves over R, with H flat. Let G be the image sheaf of $h_K\colon H_K \to F_K$. Then the *schematic image* of H in F is defined to be the schematic closure of G inside F.

The following properties are formal consequences of the definitions. The formation of the schematic closure commutes with flat extensions of discrete valuation rings. Let F_1, F_2 be sheaves over the category of R-schemes. Let $G_1 \subset F_{1,K}$, $G_2 \subset F_{2,K}$ be subsheaves, and let G_1', G_2' be the schematic closures. For a morphism of sheaves $\alpha\colon F_1 \to F_2$ such that $\alpha(G_1) \subset G_2$, we have $\alpha(G_1') \subset G_2'$. As a consequence, the schematic closure of G in F is the only subsheaf of F which is flat over R and has generic fibre equal to G. Finally, the formation of the schematic closure commutes with products; it follows that if F is a group (resp. monoid) sheaf, i.e. a group (resp. monoid) object in the category of fppf sheaves, and G is a subgroup (resp. submonoid) sheaf of F_K, then the schematic closure G' is a subgroup (resp. submonoid) sheaf of F.

In general, even if F is representable by a scheme, one needs rather strong conditions on the monomorphism $G \to F_K$ if one wants representability of the schematic closure G' by a scheme. As we recalled above, one pleasant case is when $G \to F_K$ is a closed immersion; then $G' \to F$ is also a closed immersion. As another example, the following lemma shows that in the case of an open immersion, the schematic closure is only representable by an inductive limit of schemes.

4.1.2. Lemma. *Let X be an R-scheme, $U_K \subset X_K$ the complement of a Cartier divisor. Then the schematic closure U' of U_K in X is representable by an inductive limit of affine X-schemes.*

Proof. We first construct U'. Fix an integer $n \geq 0$. For each open affine $V = \mathrm{Spec}(A)$ in X, we may choose an equation $f \in A$ for $X_K \setminus U_K$. Define $U_{V,n}$ to be the spectrum of the ring

$$\left(\frac{A[x_n]}{x_n f - \pi^n}\right)_0$$

where the subscript 0 means the quotient by the π-torsion ideal $(0 : \pi^\infty)$. There are maps $U_{V,n} \to U_{V,n+1}$ given by $x_{n+1} \mapsto \pi x_n$, and we define U_V' to be the limit of the schemes $U_{V,n}$. This construction glues over all V to give an inductive limit of affine X-schemes U'. It is not hard to see that this is independent of the choice of local equations f, up to isomorphism. Finally, we check that U' is the desired schematic closure. Let $g\colon T \to X$ be a morphism with T flat over R and $g(T_K) \subset U_K$. Let $V = \mathrm{Spec}(A)$ be an open affine in X and $W = \mathrm{Spec}(B)$ an open affine in T, with $g(W) \subset V$; let $f \in A$ be an equation for $X_K \setminus U_K$. Then we have a morphism of rings $\varphi\colon A \to B$ such that $\varphi(f)$ is invertible in B_K, i.e. there exists $n \geq 0$ and $t \in B$ such that $\varphi(f)t = \pi^n$. Furthermore, since B is R-flat, t is uniquely determined, as well as the morphism of A-algebras

$$\left(\frac{A[x_n]}{x_n f - \pi^n}\right)_0 \to B$$

given by $x_n \mapsto t$. These morphisms glue to a unique map $T \to U'$. $\square$

4.2. Kernels. Let S be a base scheme and let Γ, X, Y be schemes over S. We consider a morphism of S-schemes $\varphi\colon \Gamma \times_S X \to Y$, which we view as an action of Γ on X with values in Y. Equivalently, we have a morphism of functors $\varphi'\colon \Gamma \to \mathrm{Hom}_S(X, Y)$. We say that Γ *acts faithfully on* X, or that φ *is faithful*, if φ' is a monomorphism. We can relate this to the morphism:

$$\varphi' \times \varphi'\colon \Gamma \times_S \Gamma \to \mathrm{Hom}_S(X, Y) \times_S \mathrm{Hom}_S(X, Y).$$

4.2.1. Definition. The *kernel* of φ is the preimage of the diagonal of $\mathrm{Hom}_S(X, Y) \times_S \mathrm{Hom}_S(X, Y)$ under the morphism $\varphi' \times \varphi'$. It is denoted $\ker(\varphi)$.

Obviously, it is equivalent to say that Γ acts faithfully on X, or that the natural monomorphism $\Delta \to \ker(\varphi)$ is an isomorphism, $\Delta \subset \Gamma \times_S \Gamma$ being the diagonal of Γ. When this holds, we shall also say, by abuse of notation, that $\ker(\varphi)$ *is trivial*.

If $X = Y$ and G is a group scheme acting on X, the relation between the kernel we have just defined and the usual kernel $H := (\varphi')^{-1}(\mathrm{id}_X)$ is given by the isomorphism $G \times H \to \ker(\varphi)$ taking (g, h) to (g, gh). We use the notation $\ker(\varphi)$ in both situations, because the context will never allow confusions.

The lemma below collects some cases where one knows that the kernel is representable by a closed subscheme of $\Gamma \times_S \Gamma$. One of this case involves essentially free morphisms of schemes, a notion which can be slightly (and fruitfully) generalized to essentially semireflexive (see [SGA3], Exposé VIII, §6 and [To2], §1). Recall that a module M over a ring A is called *semireflexive* if the natural morphism $M \to M^{\vee\vee}$ to the linear bidual is injective. It is equivalent to say that M can be embedded into a product module A^I, for some set I. A morphism of schemes $X \to S$ is called *essentially free* (resp. *essentially semireflexive*) over S, if there exists a covering of S by open affine schemes S_i, for all i an affine scheme S'_i faithfully flat over S_i, and a covering of $X'_i = X \times_S S'_i$ by open affine schemes $X'_{i,j}$, such that for all i, j the function ring of $X'_{i,j}$ is a free (resp. semireflexive) module over the function ring of S'_i. It is clear that an essentially free morphism is essentially semireflexive.

4.2.2. Lemma. *Let* $X \to S$ *be flat and* $Y \to S$ *separated. Then* $\ker(\varphi) \to \Gamma \times_S \Gamma$ *is a closed immersion in any of the following cases:*

(i) $X \to S$ *is essentially semireflexive,*

(ii) *S is regular Noetherian of dimension* 1 *and $X \to S$ is locally of finite type, flat and pure, and*

(iii) *$X \to S$ is proper and X, Y are locally of finite presentation over S.*

We see that under one of these three conditions, faithfulness of φ implies separation of Γ. We remark also that it is not hard to see that if $X \to S$ is flat and $Y \to S$ is separated, then $\ker(\varphi) \to \Gamma \times_S \Gamma$ satisfies the valuative criterion of properness. What is more difficult is to check that it is of finite type.

Proof. For case (i) we refer to [SGA3], Exposé VIII, §6 and [To2], Lemma 1.16. In case (iii), the functor $\mathrm{Hom}_S(X, Y)$ is a separated algebraic space, by Artin's theorems, so the result is clear. It remains to consider case (ii). We may assume that S is the spectrum of a Henselian discrete valuation ring R. By Lemma 2.1.11, there is an open neighbourhood U of the special fibre of X that is covered by open affine subschemes U_i with function ring A_i separated for the π-adic topology. Besides, U_i is pure over R and A_i is a free R-module, by Remark 2.1.8. It follows that U is essentially free over S, hence the kernel $N_U := \ker(\Gamma \times_S U \to Y)$ is a closed subscheme of $\Gamma \times_S \Gamma$ by case (i). Consider the map induced by the action:

$$\psi : N_U \times_S X \to Y \times_S Y$$

given on the points by

$$(\gamma_1, \gamma_2, x) \mapsto (\varphi(\gamma_1)(x), \varphi(\gamma_2)(x)).$$

By definition, the restriction of ψ to $N_U \times_S U$ factors through the diagonal of Y. Since U is R-universally schematically dense in X (Lemma 2.1.9), then $N_U \times_S U$ is schematically dense in $N_U \times_S X$. Thus ψ factors through the diagonal, that is, $N_U \to \Gamma \times_S \Gamma$ factors through the kernel $N := \ker(\Gamma \times_S X \to Y)$. This gives an inverse for the obvious morphism $N \to N_U$ and proves that $N \simeq N_U$. In particular, N is a closed subscheme of $\Gamma \times_S \Gamma$, as claimed. □

4.2.3. Lemma. *Let X, Y, Γ be R-schemes. Consider one of the two situations:*

(1) *R is Henselian, X is locally of finite type, flat and pure, Y is separated, Γ is Noetherian.*

(2) *X is affine and the family of its closed subschemes finite flat over R is universally affinely dominant (Definition* 3.1.1*), Y is affine, Γ is Noetherian.*

Consider an action $\varphi : \Gamma \times X \to Y$ faithful on the generic fibre. Then there exists a finite R-flat closed subscheme $Z \subset X$ such that the induced action $\Gamma \times Z \to Y$ has the same kernel as φ.

Proof. Note that in case (2), the scheme X is semireflexive, so that in both cases the kernels are closed subschemes of $\Gamma \times \Gamma$ by Lemma 4.2.2. Let $N = \ker(\varphi)$. Let $Z_1 \subset X$ be a finite flat closed subscheme and let N_1 be the kernel of the restricted action $\Gamma \times Z_1 \to Y$. If $N_1 \neq N$, there exists $Z_2 \subset X$ with $Z_1 \subset Z_2$ such that $N_1 \supsetneq N_2$. For, otherwise, N_1 would act trivially on all the finite flat closed subschemes $Z \supset Z_1$, which are universally schematically dense in X (by Theorem 3.2.4 in case (1)), hence N_1 would act trivially on X; a contradiction. For $s \geq 1$, as long as $N_s \neq N$, we iterate this process and obtain a sequence $N_1 \supsetneq N_2 \supsetneq N_3 \supsetneq \dots$ Since $\Gamma \times \Gamma$ is Noetherian, for some s, we obtain that $N_s = N$. We can choose $Z = Z_s$. $\square$

4.3. Representability of schematic images. We now come to the main results of this paper.

4.3.1. Lemma. *Let R be a discrete valuation ring. Let X, Y be R-schemes locally of finite type, with X flat and pure and Y separated. Consider a finite flat R-scheme Γ and an action $\varphi : \Gamma \times X \to Y$ faithful on the generic fibre. Then the schematic image of Γ in $\mathrm{Hom}_R(X, Y)$ is representable by a finite flat R-scheme Γ'.*

We stress again that $\mathrm{Hom}_R(X, Y)$ is far from being representable, in general.

Proof. We start with the case where R is Henselian. By Lemma 4.2.3, there is a finite R-flat closed subscheme $Z_0 \subset X$ such that Γ_K acts faithfully on $Z_{0,K}$. Let $\{Z_\lambda\}_{\lambda \in L}$ be the family of all finite R-flat closed subschemes of X containing Z_0. This family carries the filtering order by inclusion of subschemes: $\lambda \leq \mu$ if and only if $Z_\lambda \subset Z_\mu$. Since Z_λ is finite flat over R, the functor $\mathrm{Hom}_R(Z_\lambda, Y)$ is representable by a scheme. Moreover, since $Z_\lambda \supset Z_0$ and Γ is finite, the map $\Gamma_K \to \mathrm{Hom}_K(Z_{\lambda,K}, Y_K)$ is a closed immersion. For each λ we define Γ'_λ to be the schematic image of the map $\Gamma \to \mathrm{Hom}_R(Z_\lambda, Y)$. If $\lambda \leq \mu$ in L, there is a restriction morphism $\mathrm{Hom}_R(Z_\mu, Y) \to \mathrm{Hom}_R(Z_\lambda, Y)$ and taking schematic closures gives maps $\Gamma'_\mu \to \Gamma'_\lambda$. Let Γ' be the filtering projective limit of the system $\{\Gamma'_\lambda\}$. This is an affine, flat, integral R-scheme; it is dominated by Γ, hence finite over R. Applying Proposition 3.2.5 to the diagram,

$$\begin{tikzcd}
\coprod \Gamma'_K \times Z_{\lambda,K} \arrow[r, "\simeq"] \arrow[d] & \Gamma'_K \times \coprod Z_{\lambda,K} \arrow[r] & \Gamma'_K \times X_K \arrow[d] \arrow[ddr, bend left] & \\
\coprod \Gamma' \times Z_\lambda \arrow[r, "\simeq"] \arrow[drrr, bend right] & \Gamma' \times \coprod Z_\lambda \arrow[r] & \Gamma' \times X \arrow[dr, dotted] & \\
 & & & Y
\end{tikzcd}$$

we obtain an action of Γ' on X with values in Y. This action is clearly universally faithful (i.e. faithful after any base change), because the morphism $\amalg \Gamma' \times_R Z_\lambda \to \Gamma' \times_R X$ is universally schematically dominant (apply Theorem 3.2.4 to X and pull back to $\Gamma' \times_R X$). So Γ' has the characterizing properties of the schematic closure of Γ in $\mathrm{Hom}_R(X,Y)$, and this proves the theorem.

If R is an arbitrary discrete valuation ring, let R^{h} be a Henselization of R. By the preceding discussion, $\Gamma \otimes_R R^{\mathrm{h}}$ is representable by a finite flat R^{h}-scheme. So by descent using [BLR] 6.2/D.4, Γ is representable by a finite flat R-scheme. □

There is also a version in the affine case, where one can relax the assumptions of finite type. For example, it applies to rings arising from the completion of smooth R-schemes along a section.

4.3.2. Lemma. *Let X be an affine flat R-scheme such that the family of its closed subschemes finite flat over R is universally affinely dominant (Definition* 3.1.1*). Let Y be an affine R-scheme and Γ an R-scheme locally of finite type, flat and pure. Consider an action $\varphi : \Gamma \times X \to Y$ faithful on the generic fibre. Then the schematic image of Γ in $\mathrm{Hom}_R(X,Y)$ is representable by a flat R-scheme which is affine if Γ is, and finite if Γ is.*

Proof. Observe that the assumptions imply that X is semireflexive over R, therefore kernels of actions are representable by closed subschemes, by Lemma 4.2.2. Let $X = \mathrm{Spec}(A)$ and $Z_\lambda = \mathrm{Spec}(B_\lambda)$, $\lambda \in L$, be the family of the finite flat closed subschemes of X, and let $B = \Pi\, B_\lambda$. Note that since the family $\{Z_\lambda\}$ is universally affinely dominant, then the map $A \to B$ is injective and, in particular, A is separated for the π-adic topology. The proof goes in three steps.

First step: Γ is finite. In this case we follow the proof of Lemma 4.3.1. The reference to Theorem 3.2.4 is replaced by the assumption made on X. The reference to Proposition 3.2.5 is replaced by a reference to Proposition 3.1.6. The conclusion is that the schematic image is representable by a finite flat R-scheme Γ'.

Second step: Γ is affine. Let $\Gamma = \mathrm{Spec}(C)$ and call $\Delta_\mu = \mathrm{Spec}(D_\mu)$, $\mu \in M$ the family of all finite R-flat closed subschemes of Γ. By the first step, for all μ, the schematic image of Δ_μ in $\mathrm{Hom}_R(X,Y)$ is representable by a finite flat R-scheme $\Delta'_\mu = \mathrm{Spec}(D'_\mu)$. Let $D = \Pi\, D_\mu$, $D' = \Pi\, D'_\mu$. We have injective ring homomorphisms $C \hookrightarrow D$ and $D' \hookrightarrow D$. Let C' be the intersection of C and D' inside D, and $\Gamma' = \mathrm{Spec}(C')$. We claim that $\{\Delta'_\mu\}_{\mu \in M}$ is the family of all finite flat closed subschemes of Γ'. Indeed, it is easy to see that $C' \to D'_\mu$ is surjective, i.e. Δ'_μ is a finite flat closed subscheme of Γ'. Moreover, for each finite flat closed subscheme $T' \subset \Gamma'$, we can consider T'_K as a closed subscheme of Γ_K, we set Δ_μ equal to the schematic closure of T'_K in Γ, then obviously $T' = \Delta'_\mu$. Now we prove that Γ' acts on X. For this, note that $\operatorname{coker}(C' \to D')$ injects into $\operatorname{coker}(C \to D)$ and hence is R-flat. It follows from Lemma 3.1.5 that the family of finite flat closed subschemes of Γ' is universally affinely dominant. Then the affine scheme $\Gamma' \times X$ has a family of finite flat subschemes $\Delta'_\mu \times Z_\lambda$ which is universally affinely dominant. Using Proposition 3.1.6, one obtains an action $\Gamma' \times X \to Y$. It is clear that this action has trivial kernel, hence Γ' is the schematic image of Γ.

Third step: Γ is arbitrary. By Lemma 2.1.7 and Lemma 2.1.11, there is an open neighbourhood of the special fibre of Γ that is covered by pure open affine subschemes U_i. For each i, by the second step the schematic image of U_i is representable by an affine flat R-scheme U'_i. By unicity of the schematic image, the formation of U'_i is compatible with localisation, so that the various U'_i glue to give a flat R-scheme U'. Since $U'_K \simeq U_K$, we can glue U' and Γ_K along their intersection to get a flat R-scheme Γ'. It is clear that this is the schematic image of Γ. $\square$

In the sequel, we examine the most interesting case of images of *groups* acting on schemes by group homomorphisms. We introduce some terminology.

4.3.3. Definition. If an R-group scheme G acts on an R-scheme X in such a way that the action on the generic fibre is faithful, then the schematic image of G in $\operatorname{Aut}_R(X)$ is called the *effective model* of G for its action on X.

4.3.4. Theorem. *Let X be an affine flat R-scheme whose closed subschemes finite flat over R form a universally affinely dominant family. Let G be an R-group scheme locally of finite type, flat and pure, acting on X, faithfully on the generic fibre. Then the effective model G' of the action is representable by a flat R-group scheme. If G is quasi-compact, or affine, or finite, then G' has the same property.*

Proof. Let G'' be the schematic image of G inside $\operatorname{Hom}_R(X, X)$. By the previous lemma G'' is representable by a flat R-scheme. Since $\operatorname{Aut}_R(X)$ is an open subfunctor of $\operatorname{Hom}_R(X, X)$, the preimage of G'' in $\operatorname{Aut}_R(X)$ is flat over R and hence is the schematic image G'. It follows from the general remarks of subsection 4.1 that G' is a sub-R-group scheme of $\operatorname{Aut}_R(X)$.

If G is quasi-compact, let $(U'_i)_{i\in I}$ be an open cover of G'. Let U_i be the preimage of U'_i in G. By assumption, a finite number of open sets $U_1, \dots, U_n$ cover G. The scheme G' is covered by the schematic images of $U_1, \dots, U_n$ which are none other than $U'_1, \dots, U'_n$. It follows that G' is quasi-compact.

If G is affine, then G'' is affine by Lemma 4.3.2, hence G' is quasi-affine. Let H be the affine hull of G'. This is a flat group scheme containing G' as an open subgroup. Moreover, the special fibre G'_k is schematically dense in the special fibre H_k, and since these are k-group schemes, we have in fact $G'_k = H_k$. It follows that $G' = H$ is affine.

If G is finite, then $G \to G'$ is surjective and it follows easily that G' is finite. □

These representability results extend obviously to the case where X is covered by invariant open affine subschemes satisfying the relevant assumptions. When X is locally of finite type but not necessarily affine, it is more difficult to prove that schematic images are representable. In fact, it is easy to provide a group scheme G^c which is a candidate to be the image, but in order to prove that it acts on X using Proposition 3.2.5, one needs G^c to be of finite type. This is the major difficulty of our method. Moreover, it seems that in numerous situations one cannot expect the schematic image G' to be of finite type unless the kernel of the action of G is very small. The following two results give examples of this.

4.3.5. Theorem. *Let X be an R-scheme locally of finite type, separated, flat and pure. Let G be a flat proper R-group scheme acting on X, faithfully on the generic fibre. Let N denote the kernel of the action. Then the effective model G' is representable by a flat group scheme of finite type if and only if N_k is finite. Moreover, in this case G' is proper.*

Proof. First, assume that N_k is finite. We adapt the proof of Lemma 4.3.1. By Lemma 4.2.3, there is a finite R-flat closed subscheme $Z_0 \subset X$ such that G_K acts faithfully on $Z_{0,K}$. Let G_0 be the schematic image of G inside $\mathrm{Hom}_R(Z_0, X)$, which is representable since Z_0 is finite. We claim that the morphism $u \colon G \to G_0$ is finite. Indeed, on the special fibre u_k factors as the composition of the finite quotient $G_k \to G_k/N_k$ and the monomorphism $G_k/N_k \to G_{0,k}$ given by the embedding in $\mathrm{Hom}_k(Z_k, X_k)$. It follows that u is quasi-finite, hence finite since G is proper.

Now let $\{Z_\lambda\}_{\lambda\in L}$ be the family of all finite R-flat closed subschemes of X containing Z_0. For each λ, let G''_λ be the schematic image of the map $G \to \mathrm{Hom}_R(Z_\lambda, X)$. Since $G \to G''_\lambda \to G_0$ is finite and schematically dominant, then $G \to G''_\lambda$ and $G''_\lambda \to G_0$ are also finite schematically dominant. Let G'' be the filtering projective limit of the system $\{G''_\lambda\}$. This is a scheme which is finite over G_0. Also, $G \to G''$ is finite, thus G'' is of finite type over R by the Artin-Tate theorem (see [Ei], exercise 4.32). Applying Proposition 3.2.5 as in the proof of Lemma 4.3.1, we obtain an action of G'' on X with values in X. Let G' be the preimage of G'' under the inclusion $\mathrm{Aut}_R(X) \subset \mathrm{Hom}_R(X, X)$. This is the schematic image of G in $\mathrm{Aut}_R(X)$. Since $G \to G'$ is finite, then G' is proper.

Conversely, assume that G' is representable by a flat group scheme of finite type over R. A result of Anantharaman asserts that a separated morphism u

between flat R-group schemes of finite type such that u_K has affine kernel is affine ([An], chap. II, prop. 2.3.2). It follows that $G \to G'$ is affine. Since it is also proper, it is in fact finite. It follows easily that N_k is finite. □

4.3.6. Remark. It is a well-known fact that a proper flat group scheme over R is in fact projective. Here is one way to see it. Given a finite extension K^*/K, write G^* for the extension of G to R^*, the integral closure of R in K^*. By a result of Raynaud and Faltings ([PY], corollary A.4) there is a finite extension K^*/K such that the normalization morphism $(\widetilde{G^*})_{\mathrm{red}} \to (G^*)_{\mathrm{red}}$ is finite and $(\widetilde{G^*})_{\mathrm{red}}$ is smooth. Hence, it is the product of an abelian scheme by an étale finite group, hence projective. It follows that $(G^*)_{\mathrm{red}}$ is projective, hence also G^* and G itself. Another way to check that G is projective is to reduce to the connected case. Then G is commutative and one can apply [An], chap. II, prop. 2.2.1.

4.3.7. Remark. Under the assumptions of Theorem 4.3.5, it seems plausible that if N_k is finite, then G' is representable whether G is proper or not. The only point that needs a verification is that $u : G \to G_0$ is finite (with the notation of the proof of the proposition). Even though u_K and u_k are finite, we were not able to prove this.

4.3.8. Proposition. *Let X be an R-scheme locally of finite type, separated, flat and pure. Let G be a reductive R-group scheme acting on X, faithfully on the generic fibre. Assume furthermore that either k has characteristic $p \neq 2$, or that no normal subgroup of $G_{\overline{K}}$ is isomorphic to* SO_{2n+1} *for some $n \geq 1$. Let N denote the kernel of the action. Then the effective model G' is representable by a flat group scheme of finite type if and only if N is trivial.*

Proof. This is in fact a rigidity property of reductive groups. Assume that G' is representable by a flat group scheme of finite type. Since X is flat and separated, then $\mathrm{Aut}_R(X)$ is a separated sheaf. It follows that G' is separated. Then G' is affine by [An], chap. II, prop. 2.3.1. By corollary 1.3 of [PY], we obtain that $G \to G'$ is a closed immersion. It follows that G acts faithfully on X, in other words N is trivial. The converse is obvious. □

From this proposition it it follows that if G is a finite group scheme of order prime to $p = \mathrm{char}(k)$ acting on an R-scheme locally of finite type, separated, flat and pure X, then G acts faithfully as soon as G_K acts faithfully on X_K. Indeed, the effective model is a finite flat group scheme G' by Theorem 4.3.5. Since G is reductive by the assumption on its order, we get $N = 1$. We prove a refinement of this result in Proposition 4.3.9 below. There, we also give other properties of the effective model of a finite group scheme, especially in the case where the action is *admissible*, which means that X can be covered by G-stable open affine subschemes. In this case, there exist quotient schemes X/G and X/G', and we want to compare them.

4.3.9. Proposition. *Let X be an R-scheme satisfying the assumptions of Theorem* 4.3.4 *or of Theorem* 4.3.5. *Let G be a finite flat R-group scheme acting on X and let G' be its effective model. Then:*

(i) *Let W be a closed or an open subscheme of X. If W is G-stable, then it is G'-stable. In particular, if G acts admissibly, then G' also acts admissibly.*

(ii) *The effective model of a finite flat subgroup $H \subset G$, for the restricted action on X, is the schematic image of H in G'. If H is normal in G, then H' is normal in G'.*

(iii) *Assume that G is étale and let $p = \mathrm{char}(k)$. Let $N \lhd G$ be the (unique) subgroup of G such that N_k is the kernel of the action on X_k. Then, the effective model of N is a connected p-group.*

In the sequel, we assume that G acts admissibly on X.

(iv) *The identity of X induces an isomorphism $X/G \simeq X/G'$.*

(v) *Assume that there is an open subset $U \subset X$ which is universally schematically dense, such that G' acts freely on U. Then for any closed normal subgroup $H \lhd G$, the effective model of G/H acting on X/H is G'/H'.*

(vi) *Under assumptions* (iii) *and* (v), *the group G' has a connected-étale sequence*

$$1 \to N' \to G' \to G/N \to 1.$$

Proof. (i) If W is a closed subscheme of X, then it follows from the general remarks of subsection 4.1 that the morphism $G \times W \to W$ extends to a morphism $G' \times W \to W$. Now assume that W is open. It is enough to prove that the underlying set of W is stable under G'. Let $w \in W$ be a point and let Ω be its orbit, by which we mean the schematic image of $G \times \mathrm{Spec}(k(w))$ in X. This is a closed subscheme of X, hence G'-stable. Since $\Omega \subset W$, it follows that W is G'-stable.

(ii) This is clear.

(iii) Since the composition $N_k \to N'_k \hookrightarrow \mathrm{Aut}_k(X_k)$ is trivial as a morphism of sheaves, the morphism $N_k \to N'_k$ also is. Moreover, $N \to N'$ is dominant and closed, hence surjective. Hence N'_k is infinitesimal so N' is a p-group. Let us show that it is connected. We may and do assume that R is Henselian. Then N' has a connected-étale sequence whose étale quotient we denote by $N'_{\text{ét}}$. The composition $t : N \to N' \to N'_{\text{ét}}$ is trivial on the special fibre. Moreover, t is determined by its restriction to the special fibre because it is a morphism between étale schemes. So it is globally trivial. As t is dominant, we get $N'_{\text{ét}} = 1$, thus N' is connected.

(iv) The quotient $X \to X/G$ is described, locally on a G-stable open affine $U = \operatorname{Spec}(A)$, by the invariant ring $A^G = \{\, a \in A,\ \mu_G(a) = 1 \otimes a \,\}$ where $\mu_G \colon A \to RG \otimes A$ is the coaction. Now μ_G factors through the coaction $\mu_{G'}$ corresponding to the action of G':

$$A \to RG' \otimes A \hookrightarrow RG \otimes A.$$

Therefore, $A^{G'} = \{\, a \in A,\ \mu_{G'}(a) = 1 \otimes a \,\} = A^G$. The result follows.

(v) Clearly H acts admissibly, and $X/H \simeq X/H'$ by (ii). We just have to show that G'/H' acts faithfully on X/H'. This is true since G'/H' acts freely on the image of U in X/H', by the assumptions on U.

(vi) Apply (v) to $H = N$. □

In 5.2 and 5.3 below, we will give an example where the effective model G' does not act freely on some schematically dense open subscheme, and the claim in (v) does not hold.

4.4. Schematic images for formal schemes. The same methods as in subsection 4.3 yield analogous representability results in the category of formal schemes locally of finite type. Since the proofs are completely similar, we will simply indicate how the objects are defined and then state the results. In this subsection, the discrete valuation ring (R, K, k, π) is complete and we write $R_n := R/\pi^n$. With a slight abuse of notation, we use the notation i_n for both closed immersions $\operatorname{Spec}(R_n) \hookrightarrow \operatorname{Spec}(R_{n+1})$ and $\operatorname{Spec}(R_n) \hookrightarrow \operatorname{Spec}(R)$, since confusions are not likely to arise.

4.4.1. Formal sheaves. We first recall some notation and definitions. By a *presheaf* over R we mean a contravariant functor from the category of R-schemes to the category of sets. As usual, we have the notion of a *group presheaf* and most of what will be said hereafter is valid for group presheaves. Schemes over R are identified with their functor of points and hence can be viewed as presheaves. Presheaves over R form a category denoted PSh/R. Of course, what we just said works for any base ring.

Let $i_n^* : \operatorname{PSh}/R_{n+1} \to \operatorname{PSh}/R_n$ be the pullback defined by $i_n^* F = F \times_{\operatorname{Spec}(R_{n+1})} \operatorname{Spec}(R_n)$. An *fppf formal sheaf over R* is a functor from the category of formal R-schemes to the category of sets satisfying the sheaf condition for fppf coverings. It may be identified with a direct system of fppf sheaves over R_n, i.e. a sequence (F_n) such that $F_n = i_n^* F_{n+1}$ for all $n \geq 1$. Precisely, the identification goes as follows: to a formal sheaf F, we associate the direct system $F_n = i_n^* F$. To a direct system (F_n) of fppf sheaves over R_n, we associate the functor $F = \varinjlim F_n$ defined by $F(X) = \varprojlim F_n(X_n)$ where $X = (X_n)$. These mappings are inverse to each other. We say that F is locally of finite presentation (or locally of finite type, since R is Noetherian) if each F_n is locally of finite presentation, i.e. satisfies the usual condition of commutation with filtering direct limits of rings (see [EGA], IV.8.14.2).

4.4.2. Formal sheaves in groups. Given formal R-schemes of finite type X and Y, we have two important examples of formal sheaves locally of finite type: the *homomorphism sheaf* $\mathrm{Hom}_R(X,Y) = \varinjlim \mathrm{Hom}_{R_n}(X_n, Y_n)$ and the *automorphism sheaf* $\mathrm{Aut}_R(X) = \varinjlim \mathrm{Aut}_{R_n}(X_n)$.

Let G be a flat formal scheme in groups of finite type and X a flat separated formal scheme of finite type over R. An action of G on X is given by a morphism of formal schemes $G \times X \to X$ (satisfying the usual axioms) or equivalently by a morphism of formal sheaves in groups $G \to \mathrm{Aut}_R(X)$. The kernel N of the action is defined as usual. As in Lemma 4.2.2, one shows that N is representable by a closed formal subscheme of G. As in Lemma 4.2.3, one shows that there exists a finite R-flat formal closed subscheme $Z \subset X$ such that the induced action $G \times Z \to X$ has kernel equal to $G \times N$ (here the kernel is understood as a subobject of $G \times G$, see subsection 4.2). An action is faithful if and only if $N = 1$, and one can also define faithfulness by requiring that no nontrivial R-flat closed subscheme of G acts trivially on X.

4.4.3. Schematic images. Let $\mathrm{Rig}\,/K$ denote the category of quasi-compact, quasi-separated rigid analytic K-spaces. As we recalled, Raynaud's point of view gives an equivalence between $\mathrm{Rig}\,/K$ and the category of flat formal R-schemes of finite type localised by admissible formal blowing-ups. Using the existence of flat models for flat morphisms of rigid spaces (see [BL2]), one can set up a satisfactory theory of fppf descent in $\mathrm{Rig}\,/K$. It is not our intention to provide the details of such a theory. We quote these facts without further justification; they give a meaning to what an fppf sheaf on $\mathrm{Rig}\,/K$ is.

Recall that a *model* of a rigid K-space X_K is a pair (X, i) where X is a flat formal scheme of finite type and i is an isomorphism between X_{rig} and X_K. A map between models (X_1, i_1) and (X_2, i_2) is a morphism of formal schemes $X_1 \to X_2$ compatible with the given isomorphisms i_1, i_2. We define the *generic fibre* F_{rig} of an fppf formal sheaf locally of finite type F to be the fppf sheaf on $\mathrm{Rig}\,/K$ defined as follows. For any quasi-compact, quasi-separated rigid analytic space X_K, we set:

$$F_{\mathrm{rig}}(X_K) = \varinjlim_{X_{\mathrm{rig}} = X_K} F(X)$$

where the limit is taken with respect to all models X of X_K. If F is representable by a formal scheme locally of finite type, this definition coincides with the definition of the generic fibre of a formal scheme by [dJ], Proposition 7.1.7. Then the definitions of the schematic closure of a subsheaf G of the generic fibre F_{rig}, schematic image and related notions are the obvious extensions of the definitions in subsection 4.1. We can now state our results for formal schemes.

4.4.4. Theorem. *Let X be an affine flat formal R-scheme of finite type. Let G be a flat formal R-scheme in groups of finite type acting on X, faithfully on the generic fibre. Then the effective model G' of the action is representable by a flat formal R-scheme in groups which is not necessarily of finite type. If G is quasi-compact, or affine, or finite, then G' has the same property.*

4.4.5. Theorem. *Let X be a flat, separated formal R-scheme of finite type. Let G be a proper flat formal R-scheme in groups acting on X, faithfully on the generic fibre. Let N denote the kernel of the action and assume that N_k is finite. Then the effective model G' is representable by a proper flat formal R-group scheme.*

5. Examples

5.1. Schematic closure of a K-group scheme. When it is representable, it is clear that the schematic image G' depends only on the generic fibre of G. One may start from an action of a finite K-group scheme G_K and wonder if its schematic closure in $\mathrm{Aut}_R(X)$ is representable by a finite flat R-scheme. This is not true in general, simply because the action of G may fail to extend to the special fibre. For an example of this, consider the ring of power series $R = k[[\lambda]]$ over a field of characteristic 0. Consider the projective completion of the affine R-curve with equation $y^2 = x(x-1)(x-\lambda)$, and let E/R be the complement of the unique singular point of the special fibre. Thus E_K is the Legendre elliptic curve over K. The 2-torsion $E_K[2]$ is rational and contains, in particular, the point $A = (0,0)$ generating a group of translations $G_K \simeq (\mathbb{Z}/2\mathbb{Z})_K$. This point has singular reduction, and it is easy to see that the image of the nontrivial point of G_K under $G_K \to \mathrm{Aut}_R(E)$ is a closed point. Therefore, the schematic closure is the group obtained by gluing G_K and the unit section 1_R; it is not finite over R.

5.2. Two effective models of $\mathbb{Z}/p^2\mathbb{Z}$. The end of the paper is devoted to the computation of schematic images for the group $\mathbb{Z}/p^2\mathbb{Z}$. The degeneration of torsors under $\mathbb{Z}/p\mathbb{Z}$ is well understood; one observes the exceptional feature that the effective model tends to act freely on an R-universally dense open set. Recently, Saïdi studied degenerations of torsors under $\mathbb{Z}/p^2\mathbb{Z}$ in equal characteristics [Sa]. He computed equations for such degenerations; they inherit an action of $\mathbb{Z}/p^2\mathbb{Z}$. We will compute the effective model in two cases: one case where one gets a torsor structure, and one where this fails to happen. In the case of mixed characteristics, similar examples have been given by Tossici in his Ph.D. thesis using the Kummer-to-Artin-Schreier isogeny of Sekiguchi and Suwa in degree p^2 (see [To1] and [To2]).

We let (R, K, k, t) be a complete discrete valuation ring with equal characteristics $p > 0$, so $R \simeq k[[t]]$. Under this assumption, torsors under $\mathbb{Z}/p^2\mathbb{Z}$ are described by Witt theory.

5.2.1. Classical Witt theory First we briefly recall the notation of Witt theory in degree p^2 (see [DG], chap. V). The group scheme of Witt vectors of length 2 over R has underlying scheme $W_{2,R} = \mathrm{Spec}(R[u_1, u_2]) \simeq \mathbb{A}^2_R$ with multiplication law

$$(u_1, u_2) + (v_1, v_2) = \big(u_1 + v_1, u_2 + v_2 + \sum_{k=1}^{p-1} \{\begin{smallmatrix} p \\ k \end{smallmatrix}\}\, u_1^k v_1^{p-k}\big).$$

Here we put once and for all $\{\begin{smallmatrix} p \\ k \end{smallmatrix}\} := \frac{1}{p}\binom{p}{k}$, where $\binom{p}{k}$ is the binomial coefficient. The Frobenius morphism of W_2 is denoted by $F(u_1, u_2) = (u_1^p, u_2^p)$. Put $\phi := F - \mathrm{id}$. From the exact sequence

$$0 \to (\mathbb{Z}/p^2\mathbb{Z})_R \to W_{2,R} \xrightarrow{\phi} W_{2,R} \to 0$$

it follows that any étale torsor $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ under $(\mathbb{Z}/p^2\mathbb{Z})_R$ is given by an equation

$$F(X_1, X_2) - (X_1, X_2) = (a_1, a_2)$$

where $(a_1, a_2) \in W_2(A)$ is a Witt vector and the subtraction is that of Witt vectors. Furthermore, (a_1, a_2) is well defined up to the addition of elements of the form $F(c_1, c_2) - (c_1, c_2)$. Note that

$$F(X_1, X_2) - (X_1, X_2) = \big(X_1^p - X_1, X_2^p - X_2 + \sum_{k=1}^{p-1} \{\begin{smallmatrix} p \\ k \end{smallmatrix}\}\, (X_1)^{pk}(-X_1)^{p-k}\big).$$

We emphasize that the Hopf algebra of $(\mathbb{Z}/p^2\mathbb{Z})_R$ is

$$R[\mathbb{Z}/p^2\mathbb{Z}] = \frac{R[u_1, u_2]}{(u_1^p - u_1, u_2^p - u_2)}$$

with comultiplication that of W_2.

5.2.2. Twisted forms of W_2. Let λ, μ, ν be elements of R. We define a "twisted" group W_2^λ as the group with underlying scheme $\mathrm{Spec}(R[u_1, u_2])$ and multiplication law given by

$$(u_1, u_2) + (v_1, v_2) = \left(u_1 + v_1\,,\ u_2 + v_2 + \lambda \sum_{k=1}^{p-1} \{\begin{smallmatrix} p \\ k \end{smallmatrix}\}\, u_1^k v_1^{p-k}\right).$$

We have the following analogues of the scalar multiplication and the Frobenius of W_2:

$$\begin{aligned} I^{\nu}_{\lambda,\mu}:\ W_2^{\lambda} &\longrightarrow W_2^{\lambda\mu} \\ (u_1,u_2) &\longmapsto (\nu u_1, \mu\nu^p u_2) \end{aligned}$$

and

$$\begin{aligned} F_{\lambda}:\ W_2^{\lambda} &\longrightarrow W_2^{\lambda^p} \\ (u_1,u_2) &\longmapsto (u_1^p, u_2^p). \end{aligned}$$

In case $\mu = \lambda^{p-1}$ we define an isogeny

$$\phi_{\lambda,\nu} := F_\lambda - I^{\nu}_{\lambda,\lambda^{p-1}} : W_2^{\lambda} \to W_2^{\lambda^p}.$$

We have

$$\phi_{\lambda,\nu}(u_1,u_2) = \left(u_1^p - \nu u_1\,,\ u_2^p - \nu^p\lambda^{p-1}u_2 + \lambda^p \sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} u_1^{pk}(-\nu u_1)^{p-k} \right).$$

The kernel $\mathcal{K}_{\lambda,\nu} := \ker(\phi_{\lambda,\nu})$ is a finite flat group of rank p^2. If $p > 2$, its Hopf algebra is

$$R[\mathcal{K}_{\lambda,\nu}] = \frac{R[u_1,u_2]}{(u_1^p - \nu u_1, u_2^p - \nu^p\lambda^{p-1}u_2)}.$$

We now come to the examples. They arise from the following situation. Denote by $G = \mathbb{Z}/p^2\mathbb{Z}$ the constant group, and by $Y = \mathbb{A}^1_R = \operatorname{Spec}(R[w])$ the affine line over R. Let $m_1, m_2 \in \mathbb{Z}$ be integers. Let $f_K : X_K \to Y_K$ be the $(\mathbb{Z}/p^2\mathbb{Z})_K$-torsor over $Y_K = \mathbb{A}^1_K$ given by the equations:

$$\left\{ \begin{array}{lcl} T_1^p - T_1 &=& t^{m_1}w \\ T_2^p - T_2 &=& t^{m_2}w - \displaystyle\sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} (T_1)^{pk}(-T_1)^{p-k} \end{array} \right.$$

Depending on the values of the conductors m_1, m_2 this gives rise to different group degenerations.

5.2.3. First example. Assume $m_1 = 0$ and $m_2 = -p$. Then after the change of variables $Z_1 = T_1$, $Z_2 = tT_2$ the map f_K extends to a cover $X \to Y$ with equations:

$$\left\{ \begin{array}{lcl} Z_1^p - Z_1 &=& w, \\ Z_2^p - t^{(p-1)}Z_2 &=& w - t^p \displaystyle\sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} (Z_1)^{pk}(-Z_1)^{p-k}. \end{array} \right.$$

The scheme X is a smooth affine R-curve. It is quickly seen that the action of $\mathbb{Z}/p^2\mathbb{Z}$ extends to X. As is obvious from the expression of the isogeny $\phi_{\lambda,\nu}$ (see 5.2.2), the map $X \to Y$ is a torsor under $\mathcal{K}_{\lambda,\nu}$ for $\lambda = t$ and $\nu = 1$. Thus, the effective model is $G' = \mathcal{K}_{t,1}$.

5.2.4. Second example. Assume $m_1 = -p^2 n_1 < 0$ and $m_2 = 0$. Put $\widetilde{m}_1 = n_1(p(p-1)+1)$. Then after the change of variables $Z_1 = t^{pn_1}T_1$ and $Z_2 = t^{\widetilde{m}_1}T_2$ the map f_K extends to a cover $X \to Y$ with equations

$$\left\{\begin{array}{lcl} Z_1^p - t^{(p-1)pn_1}Z_1 & = & w, \\ Z_2^p - t^{(p-1)\widetilde{m}_1}Z_2 & = & t^{p\widetilde{m}_1}w - \displaystyle\sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} t^{pn_1(p-1)(p-1-k)}(Z_1)^{pk}(-Z_1)^{p-k}. \end{array}\right.$$

The scheme X is a flat R-curve with geometrically integral cuspidal special fibre. The action of $\mathbb{Z}/p^2\mathbb{Z}$ extends to this model as follows: for (u_1, u_2) a point of $G_R = (\mathbb{Z}/p^2\mathbb{Z})_R$,

$$(u_1, u_2).(Z_1, Z_2) = \Big(Z_1 + t^{pn_1}u_1, \\ Z_2 + t^{\widetilde{m}_1}u_2 + \sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} t^{n_1(p(p-1)+1-pk)}(Z_1)^k(u_1)^{p-k} \Big).$$

In order to find out the effective model G' we look at the subalgebra of RG generated by $v_1 = t^{n_1}u_1$ and $v_2 = t^{\widetilde{m}_1}u_2$:

$$RG' := R[v_1, v_2] \subset RG.$$

One computes that RG' inherits a comultiplication from RG:

$$(v_1, v_2) + (w_1, w_2) = \left(v_1 + w_1 \ , \ v_2 + w_2 + \sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} t^{n_1(p-1)^2} v_1^k w_1^{p-k} \right).$$

Thus if $p > 2$ we recognize $G' \simeq \mathcal{K}_{\lambda,\nu}$ for $\lambda = t^{n_1(p-1)^2}$ and $\nu = t^{n_1(p-1)}$. The action of G on X extends to an action of G' given by

$$(v_1, v_2).(Z_1, Z_2) = \Big(Z_1 + t^{(p-1)n_1}v_1, \\ Z_2 + v_2 + \sum_{k=1}^{p-1} \left\{ {p \atop k} \right\} t^{n_1(p-1)(p-1-k)} Z_1^k v_1^{p-k} \Big).$$

Here $X \to Y$ is not a torsor under G'. Indeed, on the special fibre we have $G'_k = (\alpha_p)^2$ and the action on X_k is

$$(v_1, v_2).(Z_1, Z_2) = \left(Z_1 \ , \ Z_2 + v_2 + v_1 Z_1^{p-1} \right).$$

This action is faithful as required, but any point $(z_1, z_2) \in X_k$ has a stabilizer of order p which is the subgroup of G'_k defined by the equation $v_2 + v_1 z_1^{p-1} = 0$.

5.3. Effective model of a quotient. We finish with a counter-example to point (v) in Proposition 4.3.9. For $\nu \in R$ we introduce the group scheme $\mathcal{M}_\nu$ which is the kernel of the isogeny $\psi_\nu : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ defined by $\psi_\nu(x) = x^p - \nu x$ (see [Ma], §3.2). This is a finite flat group scheme of order p.

We continue with the example in 5.2.4. Thus $G = (\mathbb{Z}/p^2\mathbb{Z})_R$ and $G' \simeq \mathcal{K}_{\lambda,\nu}$ where $\lambda = t^{n_1(p-1)^2}$ and $\nu = t^{n_1(p-1)}$. Let $H = (\mathbb{Z}/p\mathbb{Z})_R \subset G$ and let $H' \subset G'$ be its image. We have

$$H' = \operatorname{Spec}\left(\frac{R[v_2]}{(v_2^p - \nu^p \lambda^{p-1} v_2)}\right) \simeq \mathcal{M}_{\nu^p \lambda^{p-1}}$$

and

$$G'/H' = \operatorname{Spec}\left(\frac{R[v_1]}{(v_1^p - \nu v_1)}\right) \simeq \mathcal{M}_\nu .$$

The quotient scheme $X/H \simeq X/H'$ is the cover of Y given by the equation $Z_1^p - t^{(p-1)pn_1} Z_1 = w$, i.e. $Z_1^p - \nu^p Z_1 = w$. It has an action of G'/H' given by

$$v_1.Z_1 = Z_1 + \nu v_1 .$$

This action is not faithful on the special fibre. It is visible that the effective model of G'/H', or equivalently of G/H, acting on X/H' is the group whose Hopf algebra is equal to the subalgebra of $R[G'/H']$ generated by $s_1 = \nu v_1$. Therefore $(G/H)' \simeq \mathcal{M}_{\nu^p}$ and the map $G'/H' \to (G/H)' = (G'/H')'$ is not an isomorphism. We see that the effective model of the quotient is not the quotient of the effective models.

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