Numerical method to compute the law of exit times for piecewise deterministic Markov processes

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Outline

Motivation

Piecewise deterministic Markov processes

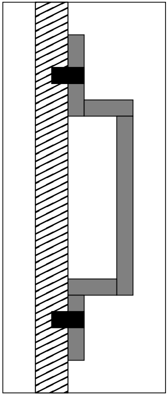
Exit time problem

Numerical results

Conclusion

Industrial problem from Astrium Space Transportation

Evaluate the service time of a material subject to corrosion



- support structure for other equipments
- small size: one point of measure
- long service life → monitor the thickness loss due to corrosion

Usage profile

Material subject to corrosion

Usage profile

Storage in 3 successive differents environments for random times

1. workshop
2. submarine in operation
3. submarine in dry dock

Strong safety requirements

⇓

Compute the service time

Degradation process

- Deterministic succession of environments : $1 \to 2 \to 3 \to 1 \cdots$

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$$d_t = \rho_i\Big(t - \eta_i + \eta_i exp\big(-\frac{t}{2\eta_i}\big)\Big)$$

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  - η_i deterministic time in environnement i.

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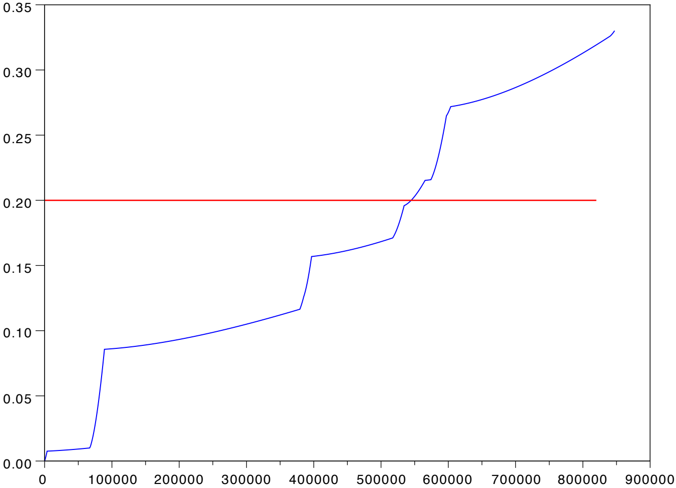
Inefficient material if $d_t \geq 0.2mm$

Simulated trajectories



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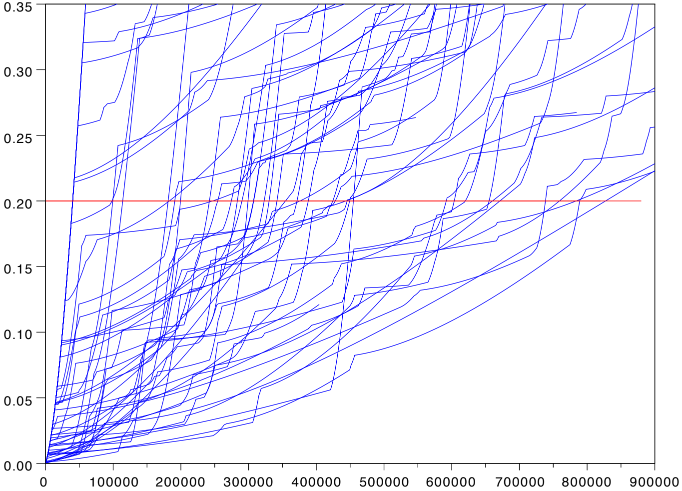


Aim of the collaboration with Astrium

Aims for Astrium

- compute the law of the service time
- assert that the structure will be safe for a given time with a given probability

Strategy

- model the corrosion problem with a PDMP
- give a general method to compute approximate exit time laws for PDMPs based on a special discretization of the process

Outline

Piecewise deterministic Markov processes

Davis (80's)

General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

Applications

Engineering systems, operations research, management science, economics, internet traffic, neurosciences, biology, dependability and safety, ...

Dynamics

Hybrid process $X_t = (m_t, y_t)$

- discrete mode $m_t \in \{1, 2, \dots, p\}$
- Euclidean state variable $y_t \in \mathbb{R}^n$

Local characteristics for each mode m

- E_m open subset of $\mathbb{R}^d$, ∂E_m its boundary and $\overline{E}_m$ its closure
- Flow ϕ_m: $\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ deterministic motion between jumps, one-parameter group of homeomorphisms
- Intensity λ_m: $\overline{E}_m \to \mathbb{R}_+$ intensity of random jumps
- Markov kernel Q_m on $(\overline{E}_m, \mathcal{B}(\overline{E}_m))$ selects post-jump location

Two types of jumps

- $t^*(m, y)$ deterministic exit time starting from (m, y)

$$t^*(m, y) = \inf\{t > 0 : \phi_m(y, t) \in \partial E_m\}$$

- law of the first jump time T_1 starting from (m, y)

$$\mathbb{P}_{(m,y)}(T_1 > t) = \begin{cases} \mathrm{e}^{-\int_0^t \lambda_m\left(\phi_m(y,s)\right)ds} & \text{if} \quad t < t^*(m, y) \\ 0 & \text{if} \quad t \geq t^*(m, y) \end{cases}$$

Remark

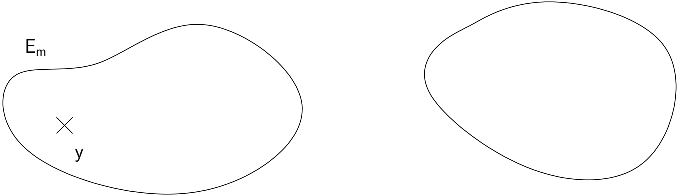
T_1 has a density on $[0, t^*(m, y)[$ but has an atom at $t^*(m, y)$

$$\mathbb{P}_{(m,y)}(T_1 = t^*(m, y)) > 0$$

Iterative construction

Starting point

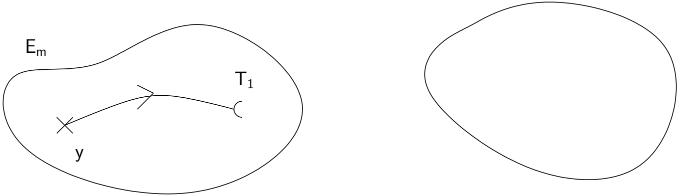
$$X_0 = Z_0 = (m, y)$$



Iterative construction

X_t follows the deterministic flow until the first jump time $T_1 = S_1$

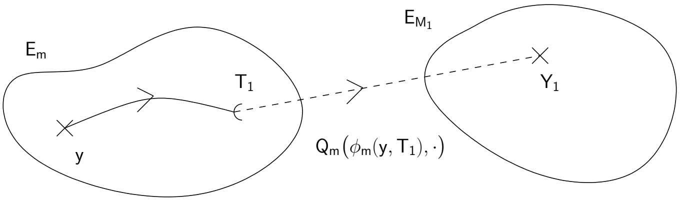
$$X_t = \big(m, \phi_m(y,t)\big), \quad t < T_1$$



Iterative construction

Post-jump location $Z_1 = (M_1, Y_1)$ selected by the Markov kernel

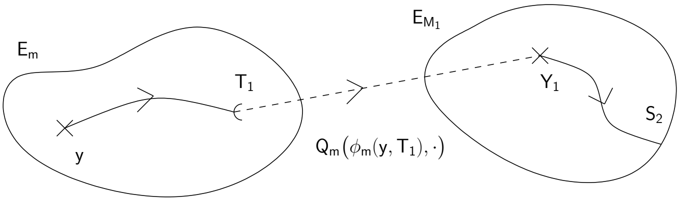
$$Q_m(\phi_m(y, T_1), \cdot)$$



Iterative construction

X_t follows the flow until the next jump time $T_2 = T_1 + S_2$

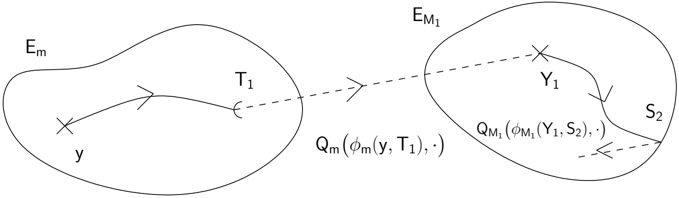
$$X_{T_1+t} = \big(M_1, \phi_{M_1}(Y_1, t)\big), \quad t < S_2$$



Iterative construction

Post-jump location $Z_2 = (M_2, Y_2)$ selected by Markov kernel

$$Q_{M_1}(\phi_{M_1}(Y_1, S_2), \cdot) \ldots$$



Embedded Markov chain

$\{X_t\}$ strong Markov process [Davis 93]

Natural embedded Markov chain

- Z_0 starting point, $T_0 = 0$
- Z_n new mode and location after n-th jump
  T_n date of n-th jump

Proposition

(Z_n, T_n) is a discrete-time Markov chain
Only source of randomness of the PDMP

Outline

Exit time problem

U open subset of E. Exit time from U

$$\tau = \inf\{s \geq 0 \ : \ X_s \notin U\}$$

$X_0 \in U$, $\mathbb{P}(\tau < \infty) = 1$

Aim

Propose a general numerical method to approximate the law of τ

$$s \longmapsto \mathbb{P}(\tau > s)$$

Recursive formulation

$$\{\tau \leq T_{k+1}\} = \{\tau \leq T_k\} \cup \{T_k < \tau \leq T_{k+1}\}$$

Consequence : recursive computation of $\mathbb{P}(\tau > s|\tau \leq T_n)$

$$\mathbb{P}(\tau > s|\tau \leq T_{k+1}) = \frac{\mathbb{P}(\tau > s|\tau \leq T_k)\mathbb{P}(\tau \leq T_k) + \mathbb{P}(\{\tau > s\} \cap \{T_k < \tau \leq T_{k+1}\})}{\mathbb{P}(\tau \leq T_{k+1})}$$

$$\mathbb{P}(\tau > s|\tau \leq T_0) = 0$$

New formulation

Compute two quantities

- $\mathbb{P}(\tau \leq T_k)$
- $\mathbb{P}(\{\tau > s\} \cap \{T_k < \tau \leq T_{k+1}\})$

Strategy

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Strategy

- make appear the Markov chain (Z_n, T_n)

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- $\mathbb{P}(\{\tau > s\} \cap \{T_k < \tau \leq T_{k+1}\})$

Strategy

- make appear the Markov chain (Z_n, T_n)
- Discretize la chaîne de Markov (Z_n, T_n)

Formulation with the embedded Markov chain

- $\mathbb{P}(\tau \leq T_k)$

- $\mathbb{P}(\{\tau > s\} \cap \{T_k < \tau \leq T_{k+1}\})$

Formulation with the embedded Markov chain

- $\mathbb{P}(\tau \leq T_k) = \mathbb{E}[\mathbb{1}_{U^c}(Z_k)]$
  provided the process does not return to U after the first exit
  $\longrightarrow$ kill the process when it exits U
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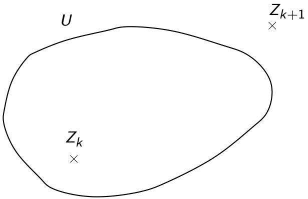

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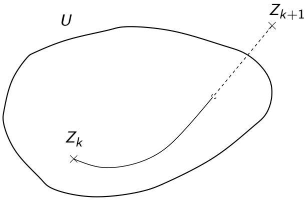

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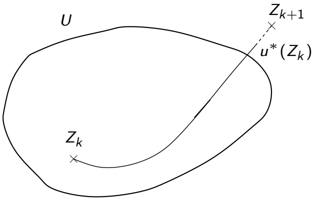


Quantization

[Pagès 98], [Pagès, Pham, Printems 04]...

Quantization of a random variable $X \in L^p(\mathbb{R}^d)$

Approximate X by $\widehat{X}$ taking finitely many values such that $\|X - \widehat{X}\|_p$ is minimum

- Find a finite weighted grid Γ with $|\Gamma| = K$
- Set $\widehat{X} = p_\Gamma(X)$ closest neighbor projection

Asymptotic properties

If $E[|X|^{p+\eta}] < +\infty$ for some $\eta > 0$ then

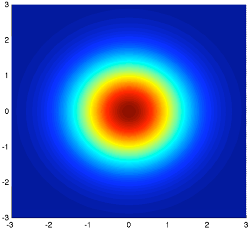
$$\lim_{K\to\infty} K^{1/d} \min_{|\Gamma|\leq K} \|X - \widehat{X}^\Gamma\|_p = C$$

Algorithms

There exist algorithms providing

- Γ
- law of $\widehat{X}$
- transition probabilities for quantization of Markov chains

Example: $\mathcal{N}(0, I_2)$:



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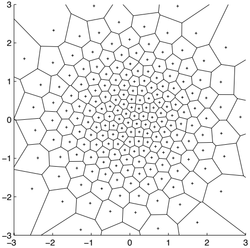

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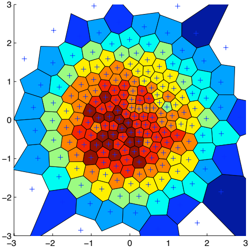
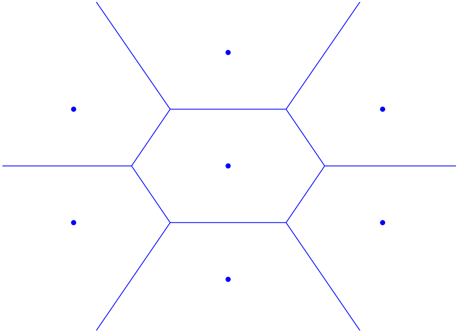


Grids construction

Model $\longrightarrow$ simulator of trajectories $\longrightarrow$ grids



Grids construction

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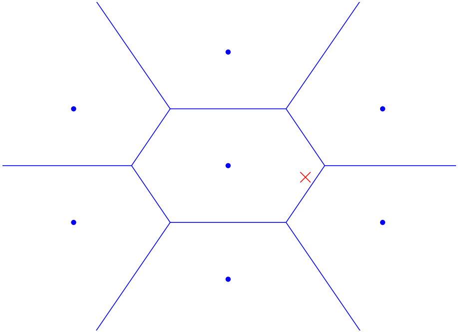

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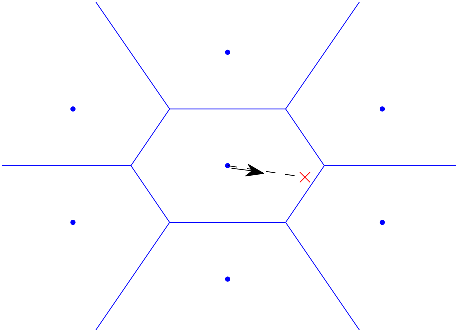

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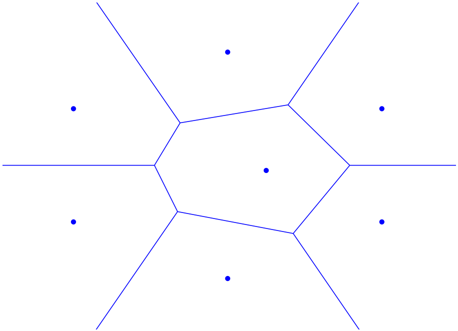


Assets and drawbacks of quantization

Assets

- a simulator of the target law is enough to build the grids
- automatic construction of grids
- convergence rate for $\mathbb{E}[|f(X) - f(\widehat{X}|]$ if f lipschitz

Drawbacks

- computation time
- curse of dimension
- open questions of convergence of the algorithms

Numerical scheme

Replace (Z_n, T_n) by its quantized approximation $(\widehat{Z}_n, \widehat{T}_n)$

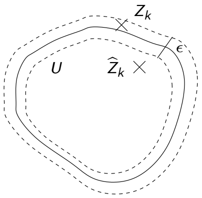
$$\begin{aligned}
\widehat{\mathbb{P}}(\tau > s | \tau \leq T_0) &= 0 \\
\widehat{\mathbb{P}}(\tau > s | \tau \leq T_{k+1}) &= \frac{1}{\mathbb{E}[\mathbb{1}_{U^c}(\widehat{Z}_{k+1})]}\Big(\widehat{\mathbb{P}}(\tau > s | \tau \leq T_k)\mathbb{E}[\mathbb{1}_{U^c}(\widehat{Z}_k)] \\
&\quad + \mathbb{E}[\mathbb{1}_{\{\widehat{T}_k + u^*(\widehat{Z}_k) \wedge \widehat{T}_{k+1} > s\}} \mathbb{1}_U(\widehat{Z}_k) \mathbb{1}_{U^c}(\widehat{Z}_{k+1})]\Big)
\end{aligned}$$

if $\mathbb{E}[\mathbb{1}_{U^c}(\widehat{Z}_{k+1})] \neq 0$, $\widehat{\mathbb{P}}(\tau > s | \tau \leq T_{k+1}) = 0$ otherwise

Error derivation: dealing with indicator functions

$$\mathbb{E}[|\mathbb{1}_U(Z_k) - \mathbb{1}_U(\widehat{Z}_k)|] \text{ small ?}$$

- $= 0$ if Z_k and $\widehat{Z}_k$ are on the same side of ∂U
- $\leq \mathbb{P}(|Z_k - \widehat{Z}_k| > \epsilon) + \mathbb{P}(d(Z_k, \partial U) \leq \epsilon)$ otherwise



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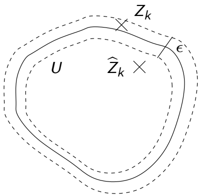

$\longrightarrow$ Control the probability to have a post-jump location close to the boundary of U

$$\mathbb{P}(d(Z_k, \partial U) \leq \epsilon) \leq C\epsilon^{\beta}$$

Error derivation: dealing with the inverse function

Lipschitz continuity assumption on the parameters of the PDMP

Main other difficulties

- Recurrence relation not Lipschitz because of denominator
  $\longrightarrow$ prove that $\mathbb{E}[\mathbb{1}_{U^c}(\widehat{Z}_{k+1})]$ is either 0 or $\geq q > 0$
- $Z_k \in U$ does not generally imply $\widehat{Z}_k \in U$
  $\longrightarrow$ U convex

Convergence result

Theorem

For all $s \geq 0$ and k smaller than the horizon N

$$|\widehat{\mathbb{P}}(\tau > s \mid \tau \leq T_k) - \mathbb{P}(\tau > s \mid \tau \leq T_k)| \longrightarrow 0$$

when the quantization error goes to 0

Outline

Practical problem for the corrosion model quantization

Parameters with different scales

- corrosion rate $\rho \sim 10^{-6}$ mm/h
- mean time spent in the submarine 2 : $\lambda_2^{-1} = 131\ 400$ h

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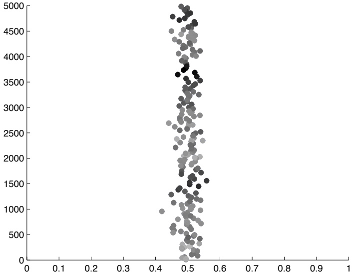
Uniform law on $[0,1] \times [0,5000]$

standard algorightm



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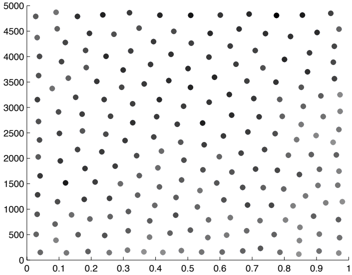
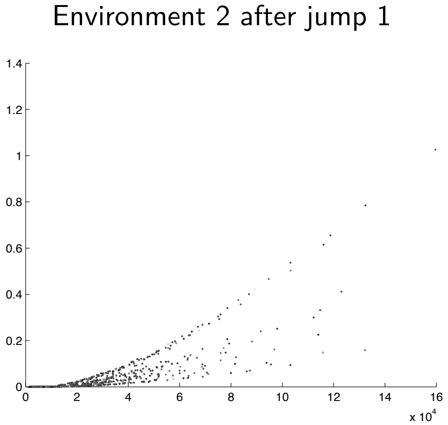
Uniform law on $[0,1] \times [0,5000]$

weighted algorithm

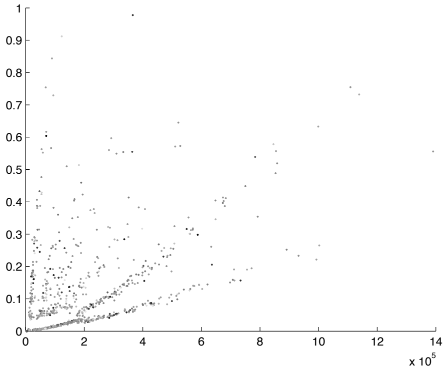


Quantization grids for the corrosion process



Quantization grids for the corrosion process

Environment 3 after jump 2



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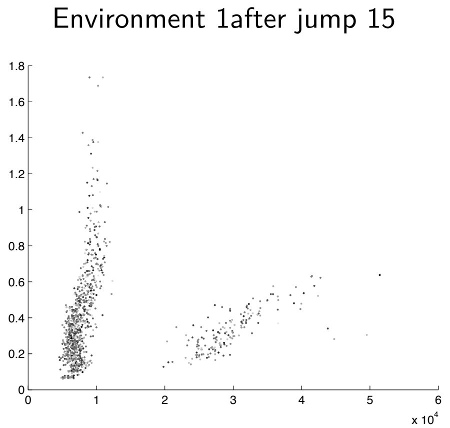
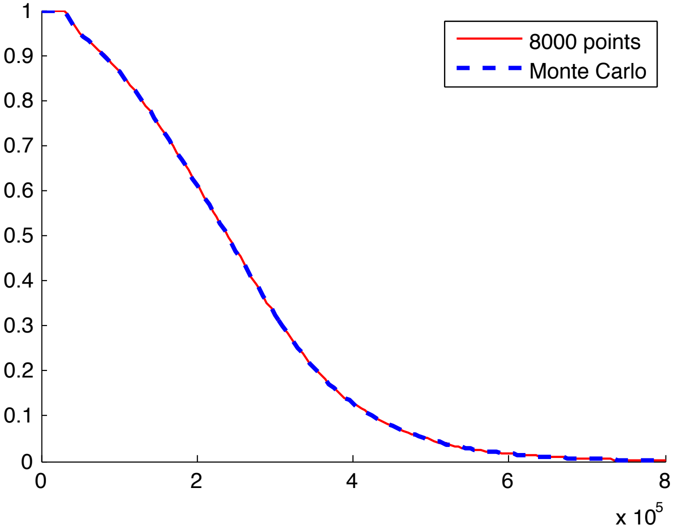


Numerical computation of the service time

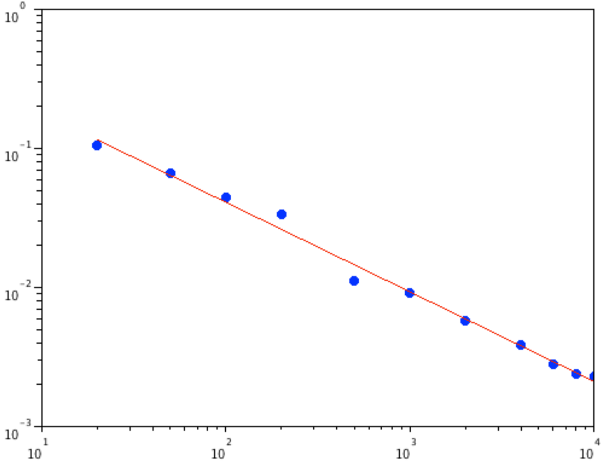
$U = [0, 0.2]mm$, τ=service time of the structure

Quantization grids	max of the error	max of relative error
20 points	0.1041	15%
50 points	0.0664	13%
100 points	0.0447	9.3%
200 points	0.0335	8%
500 points	0.0112	5.8%
1000 points	0.0091	4.6%
2000 points	0.0058	3.6%
4000 points	0.0039	2.8%
6000 points	0.0028	2.2%
8000 points	0.0024	2.0%
10000 points	0.0023	1.8%

Approximate service time



Empirical error



slope $-0.35 \longleftrightarrow$ quantization error for 3-dimensional process (thickness loss, corrosion rate, time)

Assets and drawbacks

Drawbacks

- not competitive with Monte Carlo simulation if only one computation is needed
- drawback of quantization: dimension, time to compute the grids

Assets

- same grids for different computations, do not change if U changes
- fast on-line computation once the grids are pre-computed

Thank you