Optimal stopping for partially observed piecewise deterministic Markov processes

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   ▶ Piecewise deterministic Markov processes
   ▶ Optimal stopping
   ▶ State of the art
2. Observation process
3. Filtering
4. Dynamic programming
5. Numerical method
   ▶ Quantization
   ▶ Convergence
   ▶ Example
Definition of piecewise deterministic Markov processes

Davis (80’s)

General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

Applications

Engineering systems, operations research, management science, economics, dependability and safety, maintenance,…
Dynamics

Hybrid process $X_t = (m_t, y_t)$

- **discrete mode** $m_t \in \{1, 2, \ldots, p\}$
- **Euclidean state variable** $y_t \in \mathbb{R}^n$

**Local characteristics for each mode $m$**

- $E_m$ open subset of $\mathbb{R}^d$, $\partial E_m$ its boundary and $\overline{E}_m$ its closure
- **Flow** $\phi_m : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ deterministic motion between jumps, one-parameter group of homeomorphisms
- **Intensity** $\lambda_m : \overline{E}_m \to \mathbb{R}_+$ intensity of random jumps
- **Markov kernel** $Q_m$ on $(\overline{E}_m, \mathcal{B}(\overline{E}_m))$ selects the post-jump location
Two types of jumps

- \( t^*(m, y) \) deterministic exit time

\[
t^*(m, y) = \inf\{ t > 0 : \phi_m(y, t) \in \partial E_m \}
\]

- law of the first jump time \( T_1 \)

\[
\mathbb{P}_{(m, y)}(T_1 > t) = \begin{cases} 
  e^{-\int_0^t \lambda_m(\phi_m(y, s)) \, ds} & \text{if} \quad t < t^*(m, y) \\
  0 & \text{if} \quad t \geq t^*(m, y)
\end{cases}
\]

- \( T_1 \) has a density on \([0, t^*(m, y)]\) but has an atom at \( t^*(m, y) \):

\[
\mathbb{P}_{(m, y)}(T_1 = t^*(m, y)) > 0
\]
Iterative construction

Starting point

\[ X_0 = Z_0 = (m, y) \]
Iterative construction

\( X_t \) follows the deterministic flow until the first jump time \( T_1 = S_1 \)

\[
X_t = (m, \phi_m(y, t)), \quad t < T_1
\]
Iterative construction

Post-jump location $Z_1 = (M_1, Y_1)$ selected by

$$Q_m(\phi_m(y, T_1), \cdot)$$
Iterative construction

$X_t$ follows the flow until the next jump time $T_2 = T_1 + S_2$

$$X_{T_1 + t} = (M_1, \phi_{M_1}(Y_1, t)), \quad t < S_2$$
Iterative construction

Post-jump location $Z_2 = (M_2, Y_2)$ selected by

$$Q_{M_1}(\phi_{M_1}(Y_1, S_2), \cdot) \ldots$$
Embedded Markov chain

\{X_t\} strong Markov process (M.H.A. Davis)

Natural embedded Markov chain

- $Z_0$ starting point, $S_0 = 0$, $S_1 = T_1$
- $Z_n$ new mode and location after $n$-th jump, $S_n = T_n - T_{n-1}$, time between two jumps

Important property

$(Z_n, S_n)$ is a discrete-time Markov chain
Only source of randomness of the PDMP
Optimal stopping problem

Stop the process in order to maximize a reward $g$

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_x[g(X_{\tau})]$$

- compute the value function $V$ best possible performance
- compute an optimal stopping time $\tau$

Classical problem

$\mathcal{M}$ set of stopping times for the natural filtration of $(X_t)$

- dynamic programming equation [Gugerli 86]
- numerical approximation based on a discretization of $(Z_n, S_n)$ [de Saporta, Dufour, Gonzalez 10]
Partial observations

Only a noisy observation of \((X_t)\) is available, \(\mathcal{M}\) set of stopping times for the natural filtration \((\mathcal{F}_t^O)\) of the observation process

Methodology

- introduce the filter process \(\Pi_t = \mathbb{E}[X_t | \mathcal{F}_t^O]\)
- transform the initial problem into a completely observed one for the filter process

Main drawback

- infinite dimension of the filter
State of the art

[Pham, Runggaldier, Sellami 05]
Optimal stopping under partial observation for discrete time Markov chains with finite state space

- absolute continuity assumption for the observation process
- reformulation as a standard optimal stopping problem for a continuous state space Markov chain
- numerical approximation of the value function based on discretization of the filter process
Specificities of PDMP’s

- **Continuous time** process
  - work with \((Z_n, S_n)\),
  - \(Z_n \in \{x_1, \ldots, x_q\}\)
  - stopping times remain **continuous**

- **Distribution of PDMP** has **singular** components
  - study of the filter not straightforward

- **Non standard** reformulated problem
  - derive new **dynamic programming** equations
  - operators **not** Lipschitz
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   ▶ Quantization
   ▶ Convergence
   ▶ Example
Observation process

- $S_n$ perfectly observed
- $Z_n$ observed through a noise

\[ Y_n = \phi(Z_n) + W_n \]

- continuous time observation process

\[ Y_t = \sum_{n=0}^{\infty} 1_{[T_n, T_{n+1}[}(t) Y_n \]

- filtration $\mathcal{F}_t^O = \sigma(Y_s, s \leq t)$
Simple example

PDMP: $E = [0, 1],$

$\Phi(x, t) = x + t, \quad \lambda(x) = 3x, \quad Q(x, \cdot) = x\delta_{1/4}(\cdot) + (1 - x)\delta_{1/2}(\cdot)$

Observation process: $Y_n = Z_n + W_n, \quad W_n \sim \mathcal{N}(0, 0.1^2)$
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Filter process

Assumption

Finite number of possible values for $Z_n$, $x_1, \ldots, x_q$ with $t_i^* = t^*(x_i)$, $t_1^* \leq t_2^* \cdots \leq t_q^*$

Definition

Filter process $\Pi_n = (\Pi_1^n, \ldots, \Pi_q^n)$

$$\Pi_i^n = \mathbb{P}[Z_n = x_i \mid \mathcal{F}_T^n]$$
Recursive construction

\[ \Pi_n = \Psi(\Pi_{n-1}, Y_n, S_n) \]

with

\[
\Psi^i(\pi, y, s) = \sum_{m=0}^{q-1} \mathbb{1}_{\{s \in [t^*_m, t^*_{m+1}]\}} \frac{\psi^i_m(\pi, y, s)}{\psi^*_m(\pi, y, s)} + \sum_{m=1}^{q} \mathbb{1}_{\{s = t^*_m\}} \frac{\psi^*_m(y)}{\psi^*_m(y)}
\]

\[
\psi^i_m(\pi, y, s) = \sum_{j=m+1}^{q} \pi^j \lambda(\Phi(x_j, s)) e^{-\Lambda(x_j, s)} Q(\Phi(x_j, s), x_i) f_W(y - \varphi(x_i)),
\]

\[
\psi^*_m(y) = Q(\Phi(x_m, t^*_m), x_i) f_W(y - \varphi(x_i)),
\]

\[
\bar{\psi}^*_m(\pi, y, s) = \sum_{i=1}^{q} \psi^i_m(\pi, y, s), \quad \bar{\psi}^*_m(y) = \sum_{i=1}^{q} \psi^*_m(y).
\]
Recursive construction

\[ \Pi_n = \Psi(\Pi_{n-1}, Y_n, S_n) \]

with

\[ \psi^i(\pi, y, s) = \sum_{m=0}^{q-1} \mathbb{1}_{s \in ]t^*_m; t^*_{m+1}[} \frac{\psi^i_m(\pi, y, s)}{\psi_m(\pi, y, s)} + \sum_{m=1}^{q} \mathbb{1}_{s = t^*_m} \frac{\psi^*_m(y)}{\psi^*_m(y)} \]

Proof Use

\[ F^{\cap}_{T^n} = \sigma(Y_0, S_0, \ldots, Y_n, S_n) \]

- independence between \((Z_n, S_n)\) and \(W_n\)
- expression of conditional law of \((Z_n, S_n)\) w.r.t. \(Z_{n-1}\)
Example

PDMP: $E = [0, 1]$, $x_1 = 0$, $x_2 = 1/4$, $x_3 = 1/2$,
$\Phi(x, t) = x + t$, $\lambda(x) = 3x$, $Q(x, \cdot) = x\delta_{1/4}(\cdot) + (1 - x)\delta_{1/2}(\cdot)$
Observation process: $Y_n = Z_n + W_n$, $W_n \sim \mathcal{N}(0, 0.1^2)$

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Aims

Optimal stopping stopping problem

\[ V(\pi) = \sup_{\tau \in \mathcal{M}} \mathbb{E}[g(X_{\tau \wedge T_N}) \mid \Pi_0 = \pi] \]

\( \mathcal{M} \) set of \((\mathcal{F}_t^O)\) stopping times

- reformulate the problem
- derive dynamic programming equations
- propose a numerical approximation of the value function
- propose a numerical construction for an \( \epsilon \)-optimal stopping time
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Properties of the observation and filter processes

Property of the filter process

\((\Pi_n),(\Pi_n, S_n)\) and \((\Pi_n, S_n, Y_n)\) are Markov chains

\(\mathcal{F}^O\) stopping times

\[\sigma \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1}\] on \(\{\sigma \geq T_n\}\)

with \(R_n\) \(\mathcal{F}_{T_n}^O\)-measurable
Reformulated problem

Reformulated optimal stopping problem

- similar to optimal stopping for PDMP
- different because \((\Pi_n, S_n)\) not underlying Markov chain of some PDMP

Use

- Markov property for \((\Pi_n, S_n)\)
- Fine structure of \(\mathcal{F}^{\mathcal{O}}\) stopping times
Dynamic programming equations

- **Initialization** \( v_N(\pi) = \sum_{i=1}^{q} g(x_i)\pi^i \)
- **Iteration** \( v_n(\pi) = L(v_{n+1}, g)(\pi), \) for \( n < N \)
- \( v_0(\pi) = V(\pi) \)
- recursive construction of \( \epsilon \)-optimal stopping time

\[
L(v, g)(\pi) = \max_{0 \leq m \leq q-1} \left\{ \sup_{t_m^* \leq u < t_{m+1}^*} \sum_{i=1}^{q} \mathbb{E} \left[ \prod_{i}^{i} h(\Phi(x_i, u)) 1\{s_{n+1} > u\} + v(\Pi_{n+1}) 1\{s_{n+1} \leq u\} | \Pi_n = \pi \right] \right\} \\
\quad \vee \mathbb{E}[v(\Pi_{n+1}) | \Pi_n = \pi]
\]
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Quantization

Quantization of a random variable $X \in L^p(\mathbb{R}^d)$

Approximate $X$ by $\hat{X}$ taking finitely many values such that $\|X - \hat{X}\|_p$ is minimum

- finite weighted grid $\Gamma$ with $|\Gamma| = K$
- $\hat{X} = p_\Gamma(X)$ closest neighbour projection

Asymptotic properties

If $E[|X|^{p+\eta}] < +\infty$ for some $\eta > 0$ then

$$\lim_{K \to \infty} \min_{|\Gamma| \leq K} \|X - \hat{X}_\Gamma\|_p \simeq K^d$$
Algorithms

There exist algorithms providing

- grids $\Gamma$
- law of $\hat{X}$
- transition probabilities for quantization of Markov chains

Example: $\mathcal{N}(0, I_2)$
Algorithms

There exist algorithms providing

- grids $\Gamma$
- law of $\hat{X}$
- transition probabilities for quantization of Markov chains

Example: $\mathcal{N}(0, I_2)$
Discretization of the dynamic programming operator

- recursion on functions $v_n$ turns into recursion on random variables $v_n(\Pi_n)$

\[
V_n(\Pi_n) = L(v_{n+1}, g)(\Pi_n)
\]

\[
= \max_{0 \leq m \leq q-1} \left\{ \sup_{t_m^* \leq u < t_{m+1}^*} \sum_{i=1}^{q} \mathbb{E}[\Pi^i_n h(\Phi(x_i, u)) \mathbb{1}_{\{s_{n+1} > u\}} + v_{n+1}(\Pi_{n+1}) \mathbb{1}_{\{s_{n+1} \leq u\}} | \Pi_n] \right\}
\]

\[
\vee \mathbb{E}[v(\Pi_{n+1}) | \Pi_n] \]
Discretization of the dynamic programming operator

- recursion on functions $v_n$ turns into recursion on random variables $v_n(\Pi_n)$
- discretize the intervals $[t^*_m; t^*_{m+1}]$ with regular grids $G_m$

\[ L^d(v, g)(\Pi_n) = \max_{0 \leq m \leq q-1} \left\{ \max_{u \in G_m} \sum_{i=1}^{q} \mathbb{E}\left[ \Pi^i_n h(\Phi(x_i, u)) \mathbb{I}_{\{s_{n+1} > u\}} + v(\Pi_{n+1}) \mathbb{I}_{\{s_{n+1} \leq u\}} | \Pi_n \right] \right\} \]
\[ \lor \mathbb{E}[v(\Pi_{n+1})|\Pi_n] \]
Discretization of the dynamic programming operator

- recursion on functions $\nu_n$ turns into recursion on random variables $\nu_n(\Pi_n)$
- discretize the intervals $[t^*_m; t^*_{m+1}]$ with regular grids $G_m$
- replace $(\Pi_n, S_n)$ by some quantized approximation

\[
\hat{L}^d(v, g)(\hat{\Pi}_n) = \max_{0 \leq m \leq q-1} \left\{ \max_{u \in G_m} \sum_{i=1}^{q} \mathbb{E} \left[ \hat{\Pi}_n^i h(\Phi(x_i, u)) \mathbb{I}\{\hat{s}_{n+1} > u\} + \nu(\Pi_{n+1}) \mathbb{I}\{\hat{s}_{n+1} \leq u\} | \hat{\Pi}_n \right] \right\} \\
\lor \mathbb{E}[\nu(\hat{\Pi}_{n+1}) | \hat{\Pi}_n]
\]
Convergence

**Theorem**
Lipschitz conditions

\[ \| \hat{v}_0(\hat{\Pi}_0) - V(\Pi_0) \|_p \leq cQE^{1/2} \]

Construction of a *computable* \( \epsilon \) stopping time
Numerical results

No theoretical value
Comparison with $\mathbb{E}[^\sup X_t] = 0.997$

<table>
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<th>Number of quantized points</th>
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<th>$\epsilon$-optimal stopping time</th>
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Stopped trajectories
Stopped trajectories
Merci