A Problem of Optimal Portfolio Allocation with Transaction Costs

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Outline

1. Introduction to Stochastic Control
2. Motivation
3. Framework
4. Study of the Value Function
   - Dynamic Programming Principle
   - Hamilton Jacobi Bellman Equations
5. Numerical Results
   - Value Function
   - An Efficient Strategy
   - Comparison of Strategies
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Stochastic Control Problem

- **State** of the system $W_t$ subject to stochastic dynamics
- **Control** process $\pi_t$
  - value chosen at each time depending on the available information: adapted
  - influence the dynamics of $W_t$
- Performance **Criterion** $J(W, \pi)$ to be maximised

### Value Function

$$V = \sup_{\pi} J(W, \pi)$$

### Aim

- Compute or identify the value function
- Find an optimal strategy (if it exists)
Example: Merton’s Problem
Statement I

Market:
- risk-free asset
- risky asset

\[
\begin{align*}
    dS^0_t &= S^0_t r dt \\
    dS_t &= \mu S_t dt + \sigma S_t dB_t
\end{align*}
\]

- **Control** process \( \pi_t \in [0; 1] \): proportion of the wealth invested in the risky asset

- **State** \( W_t \): wealth process

\[
\frac{dW^\pi_t}{W^\pi_t} = \pi_t \frac{dS_t}{S_t} + (1 - \pi_t) \frac{dS^0_t}{S^0_t} = \left( \pi_t \mu + (1 - \pi_t) r \right) dt + \pi_t \sigma dB_t
\]
Example: Merton’s Problem

Statement II

- **Criterion**: expected utility of the terminal wealth

\[ U(x) = x^\alpha, \quad 0 < \alpha < 1 \]

- **Value Function**

\[ V(t, x) = \sup_{\pi} \mathbb{E}[U(W_t^{x, \pi})] \]

- **Aim**

  - Compute or identify the value function
  - Find an optimal strategy (if it exists)
Example: Merton’s Problem
Hamilton-Jacobi-Bellman Equation

Hamilton-Jacobi-Bellman (HJB) Equation

\[
\frac{\partial \Phi}{\partial t}(t, x) + \sup_{p \in [0;1]} \mathcal{L}^p \Phi(t, x) = 0
\]

\[
\Phi(T, x) = U(x) = x^\alpha
\]

\[
\mathcal{L}^p \Phi(t, x) = x(p \mu + (1 - p)r) \frac{\partial \Phi}{\partial x}(t, x) + \frac{1}{2} \sigma^2 p^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}(t, x)
\]

As \( U(W_T^{x, \pi}) = U(x W_T^{x, 1, \pi}) = x^\alpha U(W_T^{x, 1, \pi}) \), we’re searching for a factorized solution \( \Phi(t, x) = x^\alpha \varphi(t) \)

\[
0 = \varphi'(t) + \varphi(t) \sup_{p \in [0;1]} \{ \alpha(p \mu + (1 - p)r) + \frac{\alpha(\alpha - 1)}{2} p^2 \sigma^2 \}
\]

\[
1 = \varphi(T)
\]
Example: Merton’s Problem
Solution of HJB Equation

\[ \Phi(t, x) = x^\alpha e^{\beta(T-t)} \]

with

\[ \beta = \sup_{p \in [0;1]} \{ \alpha(p\mu + (1-p)r) + \frac{\alpha(\alpha - 1)}{2} p^2 \sigma^2 \} \]

\[ = \alpha r + \frac{\alpha(\mu - r)^2}{2(1 - \alpha)\sigma^2} \]

reached at \( p^* = \frac{\mu - r}{(1 - \alpha)\sigma^2} \)
Example: Merton’s Problem

Interpretation of HJB Equation

Itô formula for \( \Phi \) between \( t \) and \( T \):

\[
\Phi(T, W_T^{t,x,\pi}) = \Phi(t, x) + \int_t^T \left( \frac{\partial \Phi}{\partial u} + \mathcal{L}^{\pi_u \Phi}(u, W_u^{t,x,\pi}) \right) du + \text{martingale}
\]

\[
U(W_T^{t,x,\pi}) \leq \Phi(t, x) + \int_t^T \left( \frac{\partial \Phi}{\partial u} + \sup_{p \in [0;1]} \mathcal{L}^p \Phi(u, W_u^{t,x,\pi}) \right) du + \text{martingale}
\]

\[
\leq \Phi(t, x) + \text{martingale}
\]

Hence

\[
\Phi(t, x) \geq \mathbb{E}[U(W_T^{t,x,\pi})]
\]

with equality when \( \pi_t = p^* \).
Example: Merton’s Problem

Conclusion

- \( V(t, x) = x^\alpha e^{\beta(T-t)} \)

- Constant optimal strategy \( \pi_t = \frac{\mu - r}{(1 - \alpha)\sigma^2} \)

- \( V \) is a solution of Hamilton Jacobi Bellman equation
Example: Merton’s Problem

Conclusion

- \( V(t, x) = x^\alpha e^{\beta(T-t)} \)
- Constant optimal strategy \( \pi_t = \frac{\mu - r}{(1-\alpha)\sigma^2} \)
- \( V \) is a solution of Hamilton Jacobi Bellman equation
Our Problem

Market:

- risk-free asset
- risky asset

\[
\begin{align*}
\text{risksfree asset:} & \quad dS_t^0 = S_t^0 r dt \\
\text{risky asset:} & \quad dS_t = \mu(t) S_t dt + \sigma S_t dB_t
\end{align*}
\]

- the drift alternately takes two different values
  \( \mu_1 < 0 \) and \( \mu_2 > 0 \)

- transaction costs
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Motivation

Main approaches to investment

- Fundamental approach
  - fundamental economic principles

- Technical Analysis approach
  - past prices behaviour

- Mathematical Approach
  - mathematical models

Aim

Compare the performance of technical analysis and miscalibrated mathematical models
Framework

Market: risk-free asset
      risky asset

\[ dS_t^0 = S_t^0 r dt \]
\[ dS_t = \mu(t) S_t dt + \sigma S_t dB_t \]

- \( B \) standard Brownian motion,
- \( \mu(t) \in \{\mu_1, \mu_2\} \) independent of \( B \),
- control process \( \pi_t \in \{0, 1\} \) proportion of the wealth invested in the risky asset
- state \( W_t^\pi \) wealth when strategy \( \pi \) is applied
- criterion expected utility of the terminal wealth

Aim
Maximise the expected utility of the terminal wealth
Technical Analyst strategy

Moving average

\[ M_t^\delta = \frac{1}{\delta} \int_{t-\delta}^{t} S_u du \]

- If \( S_t > M_t^\delta \) buy
- If \( S_t < M_t^\delta \) sell

\( \mu_1 = -0.2, \mu_2 = 0.2, \sigma = 0.15, \delta = 0.8. \)
BLANCHET, DIOP, GIBSON, KAMINSKI, TALAY, TAMRÉ (2005)

risk-free asset \[ dS^0_t = S^0_0 rdt, \]
risky asset \[ dS_t = \mu(t)S_t dt + \sigma S_t dB_t, \]

One change of drift
- \( \mu(t) = \mu_1 \) if \( t < \tau \)
- \( \mu(t) = \mu_2 \) if \( t \geq \tau \)

with \( \mathbb{P}(\tau > t) = e^{-\lambda t} \)

Strategy
detect \( \tau \)

\( \mu_1 = -0.2, \mu_2 = 0.2, \sigma = 0.15, \lambda = 2. \)
Previous Work

Results

- theoretical study of the value function
- theoretical study of detecting the change of drift
- numerical comparisons of strategies
  - well calibrated detection
  - miscalibrated detection
  - moving average

Conclusion

- Moving average strategy can overperform miscalibrated mathematical strategies
- Range of misspecifications for which this is true
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New Model

- Several changes of drift
  \((\xi_{2n+1}) \text{ iid Exp}(\lambda_1)\)
  \((\xi_{2n}) \text{ iid Exp}(\lambda_2)\)
  \(\tau_0 = 0, \tau_n = \xi_1 + \cdots + \xi_n\)

- Transaction costs
  - \(g_{01}\) buying cost
  - \(g_{10}\) selling cost

\[\mu(t) = \begin{cases} 
\mu_1 & \text{if } \tau_{2n} \leq t < \tau_{2n+1} \\
\mu_2 & \text{if } \tau_{2n+1} \leq t < \tau_{2n+2} 
\end{cases}\]

\[\mu_1 = -0.2, \mu_2 = 0.2, \]
\[\sigma = 0.15, \lambda_1 = \lambda_2 = 2.\]
Admissible strategies

Control process: $\pi_t \in \{0, 1\}$ proportion of the wealth invested in the risky asset

$$F^S_t = \sigma(S_u, u \leq t)$$

$\pi_t$ must be $F^S_t$-adapted

Problem

$$F^S_t \neq F^B_t = \sigma(B_u, u \leq t)$$

$\Rightarrow$ Change of framework
Filtering Theory

Optional Projection: \( F_t = \mathbb{P}(\mu(t) = \mu_1 \mid \mathcal{F}_t^S) \)

\[
\bar{B}_t = \frac{1}{\sigma} \left( \log \frac{S_t}{S_0} - \int_0^t (\mu_1 F_s + \mu_2 (1 - F_s) - \frac{\sigma^2}{2}) ds \right)
\]

Martinez, Rubenthaler, Tanré 2005

- \( \bar{B} \) is a \((\mathcal{F}^S)\) Brownian motion
- \( \mathcal{F}^S = \mathcal{F}^\bar{B} \)

\[
\frac{dS_t}{S_t} = (\mu_1 F_t + \mu_2 (1 - F_t)) dt + \sigma d\bar{B}_t
\]

Kurtz, Ocone 1988

\[
dF_t = (\lambda_1 F_t + \lambda_2 (1 - F_t)) dt + \frac{\mu_1 - \mu_2}{\sigma} F_t(1 - F_t) d\bar{B}_t
\]
New Framework

Control process: $\pi_t$

State: pair $(W_t, F_t)$

Dynamics:

$$\frac{dW^\pi_t}{W^\pi_t} = \left( \pi_t (\mu_1 F_t + \mu_2 (1 - F_t)) + (1 - \pi_t) r \right) dt + \pi_t \sigma d\bar{B}_t$$

$$- g_{01} \delta(\Delta \pi_t = 1) - g_{10} \delta(\Delta \pi_t = -1)$$

$$dF_t = \left( -\lambda_1 F_t + \lambda_2 (1 - F_t) \right) dt + \frac{\mu_1 - \mu_2}{\sigma} F_t (1 - F_t) d\bar{B}_t,$$

Criterion: expected utility of the terminal wealth

Utility: $U(x) = x^\alpha$, $\alpha \in ]0, 1[$
Continuity

Value Function

\[ V^0(t, x, f) = \sup_{\pi} \mathbb{E}[U(W_{\pi}^T) \mid \pi_{t^-} = 0, W_{t^-}^\pi = x, F_t = f] \]

\[ V^1(t, x, f) = \sup_{\pi} \mathbb{E}[U(W_{\pi}^T) \mid \pi_{t^-} = 1, W_{t^-}^\pi = x, F_t = f] \]

Continuity

For all \(i \in \{0; 1\}, 0 \leq t \leq \hat{t} \leq T, x, \hat{x} > 0, 0 \leq f, \hat{f} \leq 1:\)

\[
|V^i(\hat{t}, \hat{x}, \hat{f}) - V^i(t, x, f)| \\
\leq C(1 + x^{\alpha^{-1}} + \hat{x}^{\alpha^{-1}})(|\hat{x} - x| + x(|\hat{f} - f| + |\hat{t} - t|^{1/2}))
\]
Continuity

Value Function

\[ V^0(t, x, f) = \sup_{\pi} \mathbb{E}[U(W^\pi_T) \mid \pi_{t-} = 0, W^\pi_{t-} = x, F_t = f] \]

\[ V^1(t, x, f) = \sup_{\pi} \mathbb{E}[U(W^\pi_T) \mid \pi_{t-} = 1, W^\pi_{t-} = x, F_t = f] \]

Continuity

For all \( i \in \{0; 1\}, 0 \leq t \leq \hat{t} \leq T, x, \hat{x} > 0, 0 \leq f, \hat{f} \leq 1: \)

\[ |V^i(\hat{t}, \hat{x}, \hat{f}) - V^i(t, x, f)| \leq C(1 + x^{\alpha-1} + \hat{x}^{\alpha-1})(|\hat{x} - x| + x(|\hat{f} - f| + |\hat{t} - t|^{1/2})) \]
Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and $x, f, i$:

$$V^i(s, x, f) = \sup_{\pi} \mathbb{E}[V^{\pi_{t^-}}(t, W^{s, x, f, \pi}_t, F^{s, f}_t)]$$

Proof:

$$J^i(s, x, f, \pi) = \mathbb{E}[U(W^{s, x, f, \pi}_T)] = \mathbb{E}[\mathbb{E}[U(W^{s, x, f, \pi}_T) | \mathcal{F}_s, t]]$$

$$= \mathbb{E}[J^{\pi_{t^-}}(t, W^{s, x, f, \pi}_t, F^{s, f}_t, \pi)]$$

$$\leq \mathbb{E}[V^{\pi_{t^-}}(t, W^{s, x, f, \pi}_t, F^{s, f}_t)]$$
Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and $x, f, i$:

$$V^i(s, x, f) = \sup_{\pi} \mathbb{E}[V^{\pi_t-}(t, W^{s,x,f}_t, F^{s,f}_t)]$$

Proof:

$$J^i(s, x, f, \pi) = \mathbb{E}[U(W^s_{T,x,f,\pi})] = \mathbb{E}[\mathbb{E}[U(W^s_{T,x,f,\pi}) | \mathcal{F}_{s,t}]]$$

$$= \mathbb{E}[J^{\pi_t-}(t, W^{s,x,f}_t, F^{s,f}_t, \pi)]$$

$$\leq \mathbb{E}[V^{\pi_t-}(t, W^{s,x,f}_t, F^{s,f}_t)]$$
Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and $x, f, i$:

$$V^i(s, x, f) = \sup_{\pi} \mathbb{E}[V^{\pi t-}(t, W^{s,x,f}_t, F^{s,f}_t)]$$

Proof:

$$J^i(s, x, f, \pi) = \mathbb{E}[U(W^s_t, x, f, \pi)] = \mathbb{E}[\mathbb{E}[U(W^s_t, x, f, \pi) | \mathcal{F}_s,t]]$$

$$= \mathbb{E}[J^{\pi t-}(t, W^{s,x,f}_t, \pi, F^{s,f}_t, \pi)]$$

$$\leq \mathbb{E}[V^{\pi t-}(t, W^{s,x,f}_t, F^{s,f}_t)]$$
Dynamic Programming Principle

For all $0 \leq s \leq t \leq T$ and $x, f, i$:

$$V^i(s, x, f) = \sup_{\pi} \mathbb{E}[V_{\pi}^{t-}(t, W_{t-}^{s, x, f, \pi}, F_{t}^{s, f})]$$

Proof:

$$J^i(s, x, f, \pi) = \mathbb{E}[U(W_T^{s, x, f, \pi})] = \mathbb{E}[\mathbb{E}[U(W_T^{s, x, f, \pi}) | \mathcal{F}_{s,t}]]$$

$$= \mathbb{E}[J_{\pi}^{t-}(t, W_{t-}^{s, x, f, \pi}, F_{t}^{s, f}, \pi)]$$

$$\leq \mathbb{E}[V_{\pi}^{t-}(t, W_{t-}^{s, x, f, \pi}, F_{t}^{s, f})]$$
Fix \( \pi \) such that

\[
V = \sup_{\pi} \mathbb{E}[V_{\pi t-}^{t-}(t, W_{t-}^{s,x,f}, F_{t}^{s,f})] \leq \varepsilon + \mathbb{E}[V_{\pi t-}^{t-}(t, W_{t-}^{s,x,f}, F_{t}^{s,f})]
\]

\((B_p)_{p \in \mathbb{N}}\) partition of \([0; +\infty[ \times [0; 1]\) such that for all \(i, (x, f)\) and \((\hat{x}, \hat{f})\) in \(B_p\) and for all \(\pi\):

\[
|V^i(t, x, f) - V^i(t, \hat{x}, \hat{f})| \leq \varepsilon, \quad |J^i(t, x, f, \pi) - J^i(t, \hat{x}, \hat{f}, \pi)| \leq \varepsilon
\]

\[
V \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{\pi t-}^{t-}(t, W_{t-}^{s,x,f}, F_{t}^{s,f})1_{(W_{t-}^{s,x,f}, F_{t}^{s,f}) \in B_p}]
\]

fix \((x_p, f_p)\) in \(B_p\)

\[
V \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{\pi t-}^{t-}(t, x_p, f_p)1_{(W_{t-}^{s,x,f}, F_{t}^{s,f}) \in B_p}]
\]
Fix $\pi$ such that

$$ V = \sup_{\pi} \mathbb{E}[V_{t^-}(t, W^{s,x,f,\pi}, F_{t^-}^{s,f})] \leq \varepsilon + \mathbb{E}[V_{t^-}(t, W^{s,x,f,\pi}, F_{t^-}^{s,f})] $$

$(B_p)_{p \in \mathbb{N}}$ partition of $]0; +\infty[ \times [0; 1]$ such that for all $i$, $(x, f)$ and $(\hat{x}, \hat{f})$ in $B_p$ and for all $\pi$:

$$ |V^i(t, x, f) - V^i(t, \hat{x}, \hat{f})| \leq \varepsilon, \ |J^i(t, x, f, \pi) - J^i(t, \hat{x}, \hat{f}, \pi)| \leq \varepsilon $$

$$ V \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{t^-}(t, W^{s,x,f,\pi}, F_{t^-}^{s,f}) \mathbb{1}_{(W^{s,x,f,\pi}, F_{t^-}^{s,f}) \in B_p}] $$

fix $(x_p, f_p)$ in $B_p$

$$ V \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{t^-}(t, x_p, f_p) \mathbb{1}_{(W^{s,x,f,\pi}, F_{t^-}^{s,f}) \in B_p}] $$
Dynamic Programming Principle

Proof I

Fix $\pi$ such that

$$\mathcal{V} = \sup_{\pi} \mathbb{E}[V^{\pi t-}(t, W_{t-}^{s,x,f,\pi}, F_{t}^{s,f})] \leq \varepsilon + \mathbb{E}[V^{\pi t-}(t, W_{t-}^{s,x,f,\pi}, F_{t}^{s,f})]$$

$(\mathcal{B}_p)_{p \in \mathbb{N}}$ partition of $]0; +\infty[ \times [0; 1]$ such that for all $i$, $(x, f)$ and $(\hat{x}, \hat{f})$ in $\mathcal{B}_p$ and for all $\pi$:

$$|V^i(t, x, f) - V^i(t, \hat{x}, \hat{f})| \leq \varepsilon, \quad |J^i(t, x, f, \pi) - J^i(t, \hat{x}, \hat{f}, \pi)| \leq \varepsilon$$

$$\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi t-}(t, W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}]$$

fix $(x_p, f_p)$ in $\mathcal{B}_p$

$$\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi t-}(t, x_p, f_p) \mathbb{1}_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}]$$
Fix $\pi$ such that

$$
\mathcal{V} = \sup_{\pi} \mathbb{E}[V^\pi_{t^-}(t, W^s_{t^-}, x, f, \pi, F^s_t)] \leq \varepsilon + \mathbb{E}[V^\pi_{t^-}(t, W^s_{t^-}, x, f, \pi, F^s_t)]
$$

$(\mathcal{B}_p)_{p \in \mathbb{N}}$ partition of $]0; +\infty[ \times [0; 1]$ such that for all $i$, $(x, f)$ and $(\hat{x}, \hat{f})$ in $\mathcal{B}_p$ and for all $\pi$:

$$
|V^i(t, x, f) - V^i(t, \hat{x}, \hat{f})| \leq \varepsilon, \quad |J^i(t, x, f, \pi) - J^i(t, \hat{x}, \hat{f}, \pi)| \leq \varepsilon
$$

$$
\mathcal{V} \leq \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^\pi_{t^-}(t, W^s_{t^-}, x, f, \pi, F^s_t) \mathbb{1}_{(W^s_{t^-}, x, f, \pi, F^s_t) \in \mathcal{B}_p}]
$$

fix $(x_p, f_p)$ in $\mathcal{B}_p$

$$
\mathcal{V} \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^\pi_{t^-}(t, x_p, f_p) \mathbb{1}_{(W^s_{t^-}, x, f, \pi, F^s_t) \in \mathcal{B}_p}]$$
Dynamic Programming Principle

Proof I

Fix \( \pi \) such that

\[ \mathcal{V} = \sup_{\pi} \mathbb{E}[V_{t^-}^{\pi}(t, W_{t^-}^{s,x,f,\pi}, F_{t^s}^{f})] \leq \epsilon + \mathbb{E}[V_{t^-}^{\pi}(t, W_{t^-}^{s,x,f,\pi}, F_{t^s}^{f})] \]

\( (B_p)_{p \in \mathbb{N}} \) partition of \( ]0; +\infty[ \times [0; 1] \) such that

for all \( i, (x, f) \) and \( (\hat{x}, \hat{f}) \) in \( B_p \) and for all \( \pi \):

\[ |V^i(t, x, f) - V^i(t, \hat{x}, \hat{f})| \leq \epsilon, \quad |J^i(t, x, f, \pi) - J^i(t, \hat{x}, \hat{f}, \pi)| \leq \epsilon \]

\[ \mathcal{V} \leq \epsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{t^-}^{\pi}(t, W_{t^-}^{s,x,f,\pi}, F_{t^s}^{f}) \mathbf{1}_{(W_{t^-}^{s,x,f,\pi}, F_{t^s}^{f}) \in B_p}] \]

fix \( (x_p, f_p) \) in \( B_p \)

\[ \mathcal{V} \leq \epsilon + \epsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{t^-}^{\pi}(t, x_p, f_p) \mathbf{1}_{(W_{t^-}^{s,x,f,\pi}, F_{t^s}^{f}) \in B_p}] \]
Dynamic Programming Principle
Proof II

\[ V \leq \epsilon + \epsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^p_{t-}(t, x_p, f_p) \mathbf{1}_{(W^s_{t-}, F^s_t) \in \mathcal{B}_p}] \]

For fixed \( p, i \), let \( \pi^{p,i} \) be a strategy on \([t, T]\) such that

\[ V^i(t, x_p, f_p) \leq \epsilon + J^i(t, x_p, f_p, \pi^{p,i}) \]

\[ V \leq \epsilon + \epsilon + \epsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^p_{t-}(t, x_p, f_p, \pi^{p,i}) \mathbf{1}_{(W^s_{t-}, F^s_t) \in \mathcal{B}_p}] \]

\[ \leq \epsilon + 2\epsilon + \epsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^p_{t-}(t, W^s_{t-}, F^s_t, \pi^{p,i}) \mathbf{1}_{(W^s_{t-}, F^s_t) \in \mathcal{B}_p}] \]
Dynamic Programming Principle
Proof II

$$V \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{t}^{\pi}(t, x_{p}, f_{p}) \mathbb{1}_{(W_{t}^{s,x,f,\pi}, F_{t}^{s,f}) \in B_{p}}]$$

For fixed \( p, i \), let \( \pi_{p,i} \) be a strategy on \([t, T]\) such that

$$V^{i}(t, x_{p}, f_{p}) \leq \varepsilon + J^{i}(t, x_{p}, f_{p}, \pi_{p,i})$$

$$V \leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J_{t}^{\pi}(t, x_{p}, f_{p}, \pi_{p}, \pi_{t-}) \mathbb{1}_{(W_{t}^{s,x,f,\pi}, F_{t}^{s,f}) \in B_{p}}]$$

$$\leq \varepsilon + 2\varepsilon + \varepsilon$$

$$+ \sum_{p=0}^{\infty} \mathbb{E}[J_{t}^{\pi}(t, W_{t}^{s,x,f,\pi}, F_{t}^{s,f}, \pi_{p}, \pi_{t-}) \mathbb{1}_{(W_{t}^{s,x,f,\pi}, F_{t}^{s,f}) \in B_{p}}]$$
Dynamic Programming Principle
Proof II

\[ V \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V_{t-}^{\pi}(t, x_p, f_p) \mathbf{1}_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}] \]

For fixed \( p, i \), let \( \pi^{p,i} \) be a strategy on \([t, T]\) such that

\[ V^{i}(t, x_p, f_p) \leq \varepsilon + J^{i}(t, x_p, f_p, \pi^{p,i}) \]

\[ V \leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J_{t-}^{\pi}(t, x_p, f_p, \pi^{p,\pi_{t-}}) \mathbf{1}_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}] \]

\[ \leq \varepsilon + 2\varepsilon + \varepsilon \]

\[ + \sum_{p=0}^{\infty} \mathbb{E}[J_{t-}^{\pi}(t, W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}, \pi^{p,\pi_{t-}}) \mathbf{1}_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}] \]
Dynamic Programming Principle
Proof II

\[ V \leq \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[V^{\pi_t}(t, x_p, f_p) 1_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}] \]

For fixed \( p, i \), let \( \pi^{p,i} \) be a strategy on \([t, T]\) such that

\[ V^i(t, x_p, f_p) \leq \varepsilon + J^i(t, x_p, f_p, \pi^{p,i}) \]

\[ V \leq \varepsilon + \varepsilon + \varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_t}(t, x_p, f_p, \pi^{p,\pi_{t-}}) 1_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}] \]

\[ \leq \varepsilon + 2\varepsilon + \varepsilon \]

\[ + \sum_{p=0}^{\infty} \mathbb{E}[J^{\pi_{t-}}(t, W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}, \pi^{p,\pi_{t-}}) 1_{(W_{t-}^{s,x,f,\pi}, F_{t}^{s,f}) \in \mathcal{B}_p}] \]
Dynamic Programming Principle

Proof III

\[ V \leq 4\varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J^{\hat{\pi}_t} - (t, W_{t-}^{S,x,f,\hat{\pi}}, F_t^{S,f}, \pi P, \pi_t^p) \mathbf{1}(W_{t-}^{S,x,f,\hat{\pi}}, F_t^{S,f}) \in B_p] \]

Combining step

\[ \hat{\pi}_u = \begin{cases} \pi_u & \text{if } s \leq u < t \\ \pi P, \hat{\pi}_t^p & \text{if } u \geq t, \text{ and } (W_{t-}^{S,x,f,\hat{\pi}}, F_t^{S,f}) \in B_p \end{cases} \]

\[ V \leq 4\varepsilon + \mathbb{E}[J^{\hat{\pi}_t} - (t, W_{t-}^{S,x,f,\hat{\pi}}, F_t^{S,f}, \hat{\pi})] \]

\[ = 4\varepsilon + \mathbb{E}\left[ \mathbb{E}[U(W_T^{S,x,f,\hat{\pi}}) | F_s, t] \right] \]

\[ = 4\varepsilon + \mathbb{E}[U(W_T^{S,x,f,\hat{\pi}})] \]

\[ \leq 4\varepsilon + \mathbb{E}[V^i(s, x, f)] \]
Dynamic Programming Principle

Proof III

\[ V \leq 4\varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J_{\hat{\pi} t^+}(t, W_{s, x, f, \hat{\pi}, F_{s, f}, \pi p, \pi t^-}) \mathbb{1}_{(W_{s, x, f, \hat{\pi}, F_{s, f}) \in B_p} ]
\]

Combining step

\[ \hat{\pi} u = \begin{cases} 
\pi u & \text{if } s \leq u < t \\
\pi^p, \hat{\pi} t^- & \text{if } u \geq t, \text{ and } (W_{s, x, f, \hat{\pi}, F_{s, f}) \in B_p 
\end{cases} \]

\[ V \leq 4\varepsilon + \mathbb{E}[J_{\hat{\pi} t^+}(t, W_{s, x, f, \hat{\pi}, F_{s, f}, \hat{\pi})] \]
\[ = 4\varepsilon + \mathbb{E}[\mathbb{E}[U(W_{s, x, f, \hat{\pi}) | F_s, t))]
\[ = 4\varepsilon + \mathbb{E}[U(W_{s, x, f, \hat{\pi})]
\[ \leq 4\varepsilon + \mathbb{E}[V^i(s, x, f)] \]
Dynamic Programming Principle
Proof III

\[ V \leq 4\varepsilon + \sum_{p=0}^{\infty} \mathbb{E}[J_{t^-}^{\pi_{t^-}}(t, W_{t^-}^{s,x,f,\pi}, F_{t}^{s,f}, \pi_{p,\pi_{t^-}}) \mathbb{1}(W_{t^-}^{s,x,f,\pi}, F_{t}^{s,f}) \in B_p] \]

Combining step

\[ \hat{\pi}_u = \begin{cases} 
\pi_u & \text{if } s \leq u < t \\
\pi_{p,\hat{\pi}_{t^-}} & \text{if } u \geq t, \text{ and } (W_{t^-}^{s,x,f,\hat{\pi}}, F_{t}^{s,f}) \in B_p 
\end{cases} \]

\[ V \leq 4\varepsilon + \mathbb{E}[J_{t^-}^{\hat{\pi}_{t^-}}(t, W_{t^-}^{s,x,f,\hat{\pi}}, F_{t}^{s,f,\hat{\pi}})] \]

\[ = 4\varepsilon + \mathbb{E}[\mathbb{E}[U(W_{T}^{s,x,f,\hat{\pi}}) | \mathcal{F}_s,t]]] \]

\[ = 4\varepsilon + \mathbb{E}[U(W_{T}^{s,x,f,\hat{\pi}})] \]

\[ \leq 4\varepsilon + \mathbb{E}[V^i(s, x, f)] \]
Hamilton Jacobi Bellman Equations

\[
\begin{align*}
\min \left\{ - \frac{\partial \varphi^0}{\partial t} - L^0 \varphi^0; \quad \varphi^0(t, x, f) - \varphi^1(t, x(1 - g_{01}), f) \right\} &= 0 \\
\min \left\{ - \frac{\partial \varphi^1}{\partial t} - L^1 \varphi^1; \quad \varphi^1(t, x, f) - \varphi^0(t, x(1 - g_{10}), f) \right\} &= 0
\end{align*}
\]

\[
\begin{align*}
L^0 \varphi(t, x, f) &= x r \frac{\partial \varphi}{\partial x}(t, x, f) + \left( -\lambda_1 f + \lambda_2 (1 - f) \right) \frac{\partial \varphi}{\partial f}(t, x, f) + \\
&\quad \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 \frac{\partial^2 \varphi}{\partial f^2}(t, x, f)
\end{align*}
\]

\[
\begin{align*}
L^1 \varphi(t, x, f) &= x (\mu_1 f + \mu_2 (1 - f)) \frac{\partial \varphi}{\partial x}(t, x, f) + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 \varphi}{\partial x^2}(t, x, f) \\
&\quad + \left( -\lambda_1 f + \lambda_2 (1 - f) \right) \frac{\partial \varphi}{\partial f}(t, x, f) + x (\mu_1 - \mu_2) f (1 - f) \frac{\partial^2 \varphi}{\partial x \partial f}(t, x, f) \\
&\quad + \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 \frac{\partial^2 \varphi}{\partial f^2}(t, x, f)
\end{align*}
\]
Viscosity Solutions

\[
(P) \quad F(t, x, v(t, x), D_t v(t, x), Dv(t, x), D^2 v(t, x)) = 0
\]

**Definition**

- \( v \) is a *viscosity sub-solution* of \((P)\) if

  \[
  F(\bar{t}, \bar{x}, v(\bar{t}, \bar{x}), D_t \varphi(\bar{t}, \bar{x}), D\varphi(\bar{t}, \bar{x}), D^2 \varphi(\bar{t}, \bar{x})) \leq 0
  \]

  for all \((\bar{t}, \bar{x})\) and all functions \(\varphi \in C^{1,2}\) such that \((\bar{t}, \bar{x})\) is a local maximum of \(v - \varphi\)

- \( v \) is a *viscosity super-solution* of \((P)\) if

  \[
  F(\bar{t}, \bar{x}, v(\bar{t}, \bar{x}), D_t \varphi(\bar{t}, \bar{x}), D\varphi(\bar{t}, \bar{x}), D^2 \varphi(\bar{t}, \bar{x})) \geq 0
  \]

  for all \((\bar{t}, \bar{x})\) and all functions \(\varphi \in C^{1,2}\) such that \((\bar{t}, \bar{x})\) is a local minimum of \(v - \varphi\)
Identification of the Value Function

\( \mathcal{V}_\alpha \) : set of continuous functions \( \varphi \) on \([0; T] \times [0; +\infty] \times [0; 1]\)
satisfying \( \varphi(t, 0, f) = 0 \) and

\[
\sup_{[0; T] \times [0; +\infty] \times [0; 1]} \frac{|\varphi(t, x, f) - \varphi(t, \hat{x}, \hat{f})|}{(1 + x^{\alpha-1} + \hat{x}^{\alpha-1})(|x - \hat{x}| + x|f - \hat{f}|)} < \infty.
\]

**Theorem**

\( (\mathcal{V}^0, \mathcal{V}^1) \) is the unique viscosity solution of HJB equation in
\( \mathcal{V}_\alpha \times \mathcal{V}_\alpha \) satisfying

\[
\mathcal{V}^0(T, x, f) = \mathcal{V}^1(T, x, f) = U(x) = x^\alpha
\]
Outline

1. Introduction to Stochastic Control
2. Motivation
3. Framework
4. Study of the Value Function
   - Dynamic Programming Principle
   - Hamilton Jacobi Bellman Equations
5. Numerical Results
   - Value Function
   - An Efficient Strategy
   - Comparison of Strategies
**Discretization Scheme**

**Dependence on** \( x \):

\[
V^i(t, x, f) = \sup_{\pi} \mathbb{E}[U(W^t_{T,x,f}, \pi)] = x^\alpha V^i(t, 1, f)
\]

**Numerical Scheme**

- \( \hat{V}^0(T, f) = \hat{V}^1(T, f) = 1 \)
- With the PDE part in HJB, compute \( \overline{V}^0(t, \cdot) \) and \( \overline{V}^1(t, \cdot) \) from \( \hat{V}^0(t + dt, \cdot) \) and \( \hat{V}^1(t + dt, \cdot) \)
- Comparison
  - if \( \overline{V}^0(t, f) \geq (1 - g_{01})^\alpha \overline{V}^1(t, f) \), set \( \hat{V}^0(t, f) = \overline{V}^0(t, f) \)
  - otherwise \( \hat{V}^0(t, f) = (1 - g_{01})^\alpha \overline{V}^1(t, f) \)
  - if \( \overline{V}^1(t, f) \geq (1 - g_{10})^\alpha \overline{V}^0(t, f) \), set \( \hat{V}^1(t, f) = \overline{V}^1(t, f) \)
  - otherwise \( \hat{V}^1(t, f) = (1 - g_{10})^\alpha \overline{V}^0(t, f) \)
Discretization Scheme

Dependence on $x$:

$$V^i(t, x, f) = \sup_{\pi} \mathbb{E}[U(W_t^{t,x,f,\pi})] = x^\alpha V^i(t, 1, f)$$

Numerical Scheme

1. $\hat{V}^0(T, f) = \hat{V}^1(T, f) = 1$
2. With the PDE part in HJB, compute $\overline{V}^0(t, \cdot)$ and $\overline{V}^1(t, \cdot)$ from $\hat{V}^0(t + dt, \cdot)$ and $\hat{V}^1(t + dt, \cdot)$
3. Comparison
   - if $\overline{V}^0(t, f) \geq (1 - g_{01})^\alpha \overline{V}^1(t, f)$, set $\hat{V}^0(t, f) = \overline{V}^0(t, f)$
   - otherwise $\hat{V}^0(t, f) = (1 - g_{01})^\alpha \overline{V}^1(t, f)$
   - if $\overline{V}^1(t, f) \geq (1 - g_{10})^\alpha \overline{V}^0(t, f)$, set $\hat{V}^1(t, f) = \overline{V}^1(t, f)$
   - otherwise $\hat{V}^1(t, f) = (1 - g_{10})^\alpha \overline{V}^0(t, f)$
Optimal Portfolio Allocation

Benoîte de Saporta

Introduction
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Value Function
Dynamic Programming Principle
Hamilton Jacobi Bellman Equations
Numerical Results

Discretization Scheme

Dependence on $x$:

$$V^i(t, x, f) = \sup_\pi \mathbb{E}[U(W^t_{t+x}, f, \pi)] = x^\alpha V^i(t, 1, f)$$

Numerical Scheme

- $\hat{V}^0(T, f) = \hat{V}^1(T, f) = 1$

- With the PDE part in HJB, compute $\overline{V}^0(t, \cdot)$ et $\overline{V}^1(t, \cdot)$ from $\hat{V}^0(t + dt, \cdot)$ and $\hat{V}^1(t + dt, \cdot)$

- Comparison
  - if $\overline{V}^0(t, f) \geq (1 - g_{01})^\alpha \overline{V}^1(t, f)$, set $\hat{V}^0(t, f) = \overline{V}^0(t, f)$
  - otherwise $\hat{V}^0(t, f) = (1 - g_{01})^\alpha \overline{V}^1(t, f)$
  - if $\overline{V}^1(t, f) \geq (1 - g_{10})^\alpha \overline{V}^0(t, f)$, set $\hat{V}^1(t, f) = \overline{V}^1(t, f)$
  - otherwise $\hat{V}^1(t, f) = (1 - g_{10})^\alpha \overline{V}^0(t, f)$
Value Function $V^0$

**Shape**

Parameters: $T = 3$, $\mu_2 = -\mu_1 = 0.2$, $\lambda_1 = \lambda_2 = 2$, $\sigma = 0.15$, $g_{01} = g_{10} = 0.001$
Value Function $V^0$

Regularity I

Transaction costs $g_{01} = g_{10} = 0.01$

Zoom between $t = 2.5$ and $t = 3 = T$
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Value Function $V^0$
Regularity II

Section at $f = 0.05$

Sections at $t = 2.90, t = 2.91, t = 2.92, t = 2.93,$
**Efficient Strategy**

- Compute $\hat{V}^0$, $\hat{V}^1$
- Estimate $\hat{F}_t$ from the stock
- Compare $\hat{V}^0(t, \hat{F}_t)$ et $\hat{V}^1(t, \hat{F}_t)$:
  - buy if $\hat{V}^0(t, \hat{F}_t) = (1 - g_{01})^\alpha \hat{V}^1(t, \hat{F}_t)$
  - sell if $\hat{V}^1(t, \hat{F}_t) = (1 - g_{10})^\alpha \hat{V}^0(t, \hat{F}_t)$

$\mu_1 = -0.2$, $\mu_2 = 0.2$, $\sigma = 0.15$, $\lambda_1 = 2$, $\lambda_2 = 2$, $T = 3$
Efficient Strategy vs Value Function

- Computation of the value function:
  - time discretization step $10^{-6}$
  - space discretization step $10^{-3}$

- $10^5$ Monte Carlo simulations of the Efficient Strategy

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<tr>
<td>Strategy</td>
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<td>1.039</td>
<td>1.038</td>
<td>1.037</td>
<td>1.036</td>
</tr>
</tbody>
</table>
Miscalibrated Efficient Strategy vs Moving Average

miscalibrated parameters:
\[ \mu_1 = -1.8, \mu_2 = 1.8, \sigma = 0.15, \]
\[ \lambda_1 = 4, \lambda_2 = 4 \]

Real parameters:
\[ \mu_1 = -0.2, \mu_2 = 0.2, \sigma = 0.15, \lambda_1 = 2, \lambda_2 = 2, T = 3, \delta = 0.8 \]

100000 Monte Carlo Simulations