Maintenance optimization with piecewise deterministic Markov processes

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Outline

Maintenance

From corrective actions to preventive and condition-based interventions

Equipments

- with several components
- subject to random degradation and failures

Maintenance policy: sequence of interventions

- when ?
- what type: change or repair ?

Examples of maintenance policies

- change a component at failure
- repair or change a component every n months
- ...

Maintenance optimization

From corrective actions to preventive and condition-based interventions

Maintenance optimization problem: find some optimal balance between

- repairing/changing components too often
- do nothing and wait for the total failure of the system

Optimize some criterion

- minimize a cost: functioning, maintenance, ...
- maximize a reward: availability, ...

Maintenance optimization

From corrective actions to preventive and condition-based interventions

Maintenance optimization problem: find some optimal balance between

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- minimize a cost: functioning, maintenance, ...
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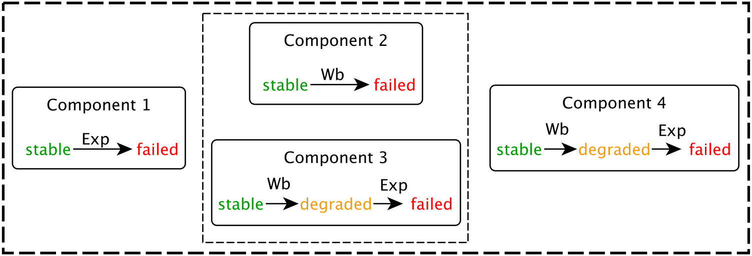
Equipment model

Typical model with 4 components

- Component 1: 2 states – stable $\xrightarrow{\text{Exponential}}$ failed $\mathbb{P}(T > t) = e^{-\lambda t}$
- Component 2: 2 states – stable $\xrightarrow{\text{Weibull}}$ failed $\mathbb{P}(T > t) = e^{-(t/\alpha)^{\beta}}$
- Components 3 and 4: 3 states

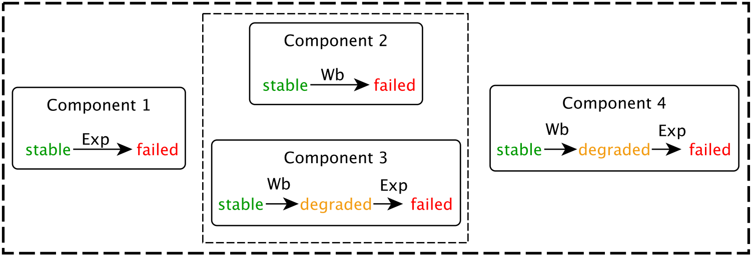
stable $\xrightarrow{\text{Weibull}}$ degraded $\xrightarrow{\text{Exponential}}$ failed



Maintenance operations

Possible maintenance operations

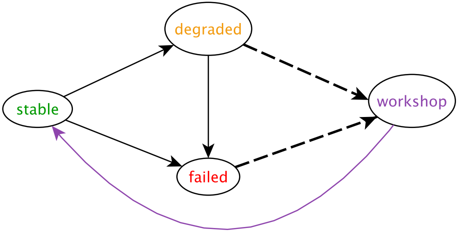
- All components, all states: do nothing
- Components 1 and 2, all states: change
- Components 3 and 4: change in all states, repair only in stable or degraded states



Global state of the equipment

The equipment is globally

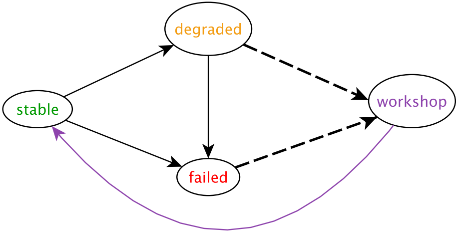
- stable if the 4 components are stable
- degraded if at leat one component is degraded and the others are stable or degraded
- failed if at least one component is failed
- in the workshop if there is an ongoing maintenance operation of change or repair



Criterion to optimize

Minimize the maintenance + unavailability costs

- unavailability cost proportional to time spend in failed state
- fixed cost for going to the workshop + repair < change costs



Our approach

- propose a general model for the evolution of the equipment state based on PDMPs
- formalize the maintenance problem as an impulse control problem for PDMPs
- derive a numerical scheme to approximate the value function (with error bounds)
- compute the approximate optimal maintenance cost
- propose a computable strategy close to optimality

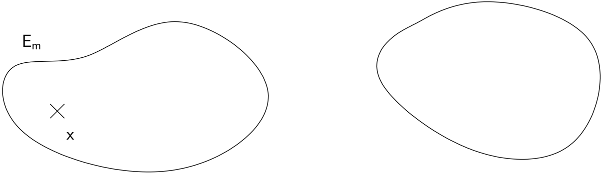
Outline

Piecewise deterministic Markov processes

[Davis 93] General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

Starting point

$$X_0 = (m, x)$$

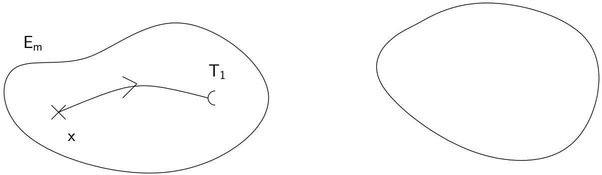


Piecewise deterministic Markov processes

[Davis 93] General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

X_t follows the deterministic flow until the first jump time $T_1 = S_1$

$$X_t = \big(m, \phi_m(x,t)\big), \quad \mathbb{P}_{(m,x)}(S_1 > t) = \mathrm{e}^{-\int_0^t \lambda_m\left(\phi_m(x,s)\right)ds}$$

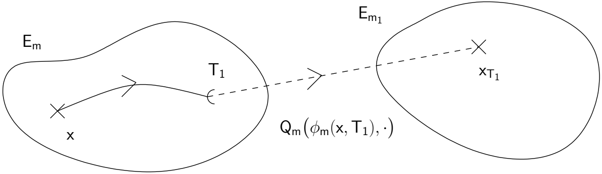


Piecewise deterministic Markov processes

[Davis 93] General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

Post-jump location (m_1, x_{T_1}) selected by the Markov kernel

$$Q_m(\phi_m(x, T_1), \cdot)$$

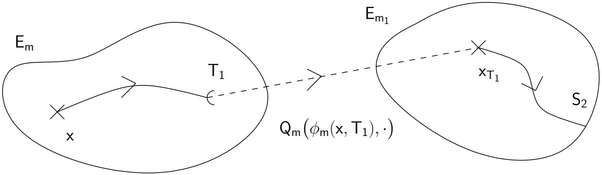


Piecewise deterministic Markov processes

[Davis 93] General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

X_t follows the flow until the next jump time $T_2 = T_1 + S_2$

$$X_{T_1+t} = \big(m_1, \phi_{m_1}(x_{T_1}, t)\big), \quad t < S_2$$



Piecewise deterministic Markov processes

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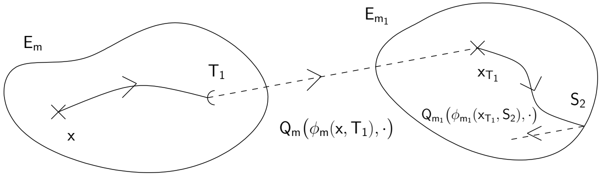
Post-jump location (m_2, x_{T_2}) selected by Markov kernel

$$Q_{m_1}\big(\phi_{m_1}(x_{T_1}, S_2), \cdot\big) \dots$$



Embedded Markov chain

$\{X_t\}$ strong Markov process [Davis 93]

Natural embedded Markov chain

- Z_0 starting point, $S_0 = 0$, $S_1 = T_1$
- Z_n new mode and location after n-th jump, $S_n = T_n - T_{n-1}$, time between two jumps

Proposition

(Z_n, S_n) is a discrete-time Markov chain
Only source of randomness of the PDMP

Examples of PDMPs

Applications of PDMPs

Engineering systems, operations research, management science, economics, internet traffic, dependability and safety, neurosciences, biology, ...

- mode: nominal, failures, breakdown, environment, number of individuals, response to a treatment, ...
- Euclidean variable: pressure, temperature, time, size, potential, protein level, ...

PDMP model of the equipment

- Euclidean variables: 5 time variables
  - functioning time of components 2, 3 and 4
  - calendar time
  - time spent in the workshop
- Discrete variables: 225 modes
  - state of the components / maintenance operations

Impulse control problem

Impulse control

Select

- intervention dates
- new starting point for the process at interventions

to minimize a cost function

- repair a component before failure
- change treatment before relapse
- ...

[CD 89], [Davis 93], [dSDZ 14], ...

Mathematical definition

Strategy $\mathcal{S} = (\tau_n, R_n)_{n\geq 1}$

- τ_n intervention times
- R_n new positions after intervention

Value function

$$\mathcal{J}^{\mathcal{S}}(x) = E_x^{\mathcal{S}}\left[\int_0^{\infty} e^{-\alpha s} f(Y_s)ds + \sum_{i=1}^{\infty} e^{-\alpha\tau_i} c(Y_{\tau_i}, Y_{\tau_i^+})\right]$$

$$\mathcal{V}(x) = \inf_{\mathcal{S}\in\mathbb{S}} \mathcal{J}^{\mathcal{S}}(x)$$

- f, c cost functions, α discount factor
- Y_t controlled process, $\mathbb{S}$ set of admissible strategies

Example of maintenance optimization

- τ_n: maintenance dates
- R_n: which components are to be changed/repaired

Value function

$$\mathcal{J}^{\mathcal{S}}(x) = E_x^{\mathcal{S}}\left[\int_0^\infty e^{-\alpha s} f(Y_s)ds + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i^+})\right]$$

$$\mathcal{V}(x) = \inf_{\mathcal{S}\in\mathbb{S}} \mathcal{J}^{\mathcal{S}}(x)$$

- f unavailability cost proportional to time spend in failed state
- c fixed cost for going to the workshop + repair < change costs
- $\alpha = 0$ (finite horizon)

Dynamic programming

Costa, Davis, 1988

For any function $g \geq$ cost of the no-impulse strategy

- $v_0 = g$
- $v_n = \mathcal{L}(v_{n-1})$

$$v_n(x) \xrightarrow[n\to\infty]{} \mathcal{V}(x)$$

de Saporta, Dufour 2012

Numerical scheme to compute an approximation of the value function

Dynamic programming operator

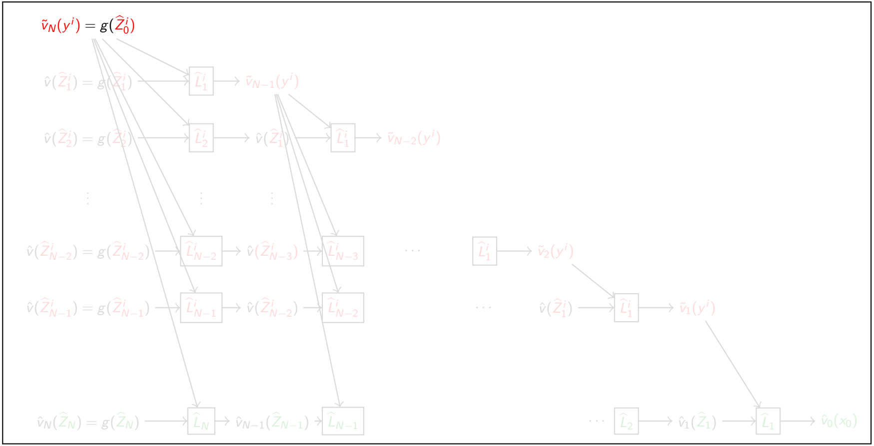
$$
\begin{aligned}
v_n(Z_n) &= \mathcal{L}(Mv_{n+1}, v_{n+1})(Z_n) \\
&= \Bigg(\inf_{t \leq t^*(Z_n)} \mathbb{E}\Big[F(Z_n, t) + \mathrm{e}^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) 1_{\{S_{n+1} < t \wedge t^*(Z_n)\}} \\
&\quad + \mathrm{e}^{-\alpha t \wedge t^*(Z_n)} Mv_{n+1}\big(\phi(Z_n, t \wedge t^*(Z_n))\big) \mathbb{1}_{\{S_{n+1} \geq t \wedge t^*(Z_n)\}} \mid Z_n\Big]\Bigg) \\
&\quad \wedge \mathbb{E}\Big[F(Z_n, t^*(Z_n)) + \mathrm{e}^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) \mid Z_n\Big]
\end{aligned}
$$

with

$$
\begin{aligned}
F(x,t) &= \int_0^{t \wedge t^*(x)} \mathrm{e}^{-\alpha s - \int_0^s \lambda(\phi(x,u))du} f\big(\phi(x,s)\big) ds \\
Mv_{n+1}(x) &= \inf_{y \in \mathbb{U}} \big\{ c(x,y) + v_{n+1}(y) \big\}
\end{aligned}
$$

Approximation scheme

Value function



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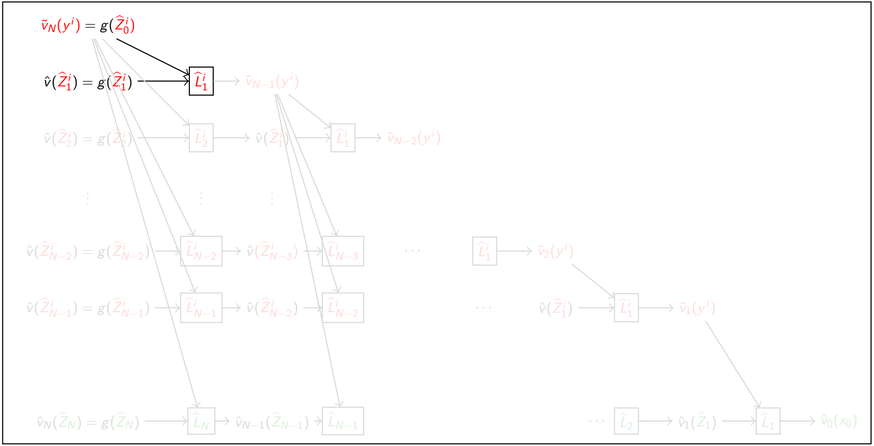

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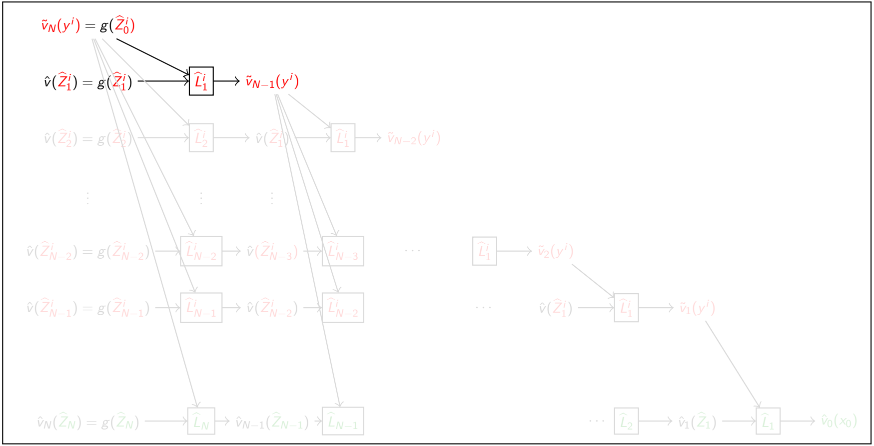

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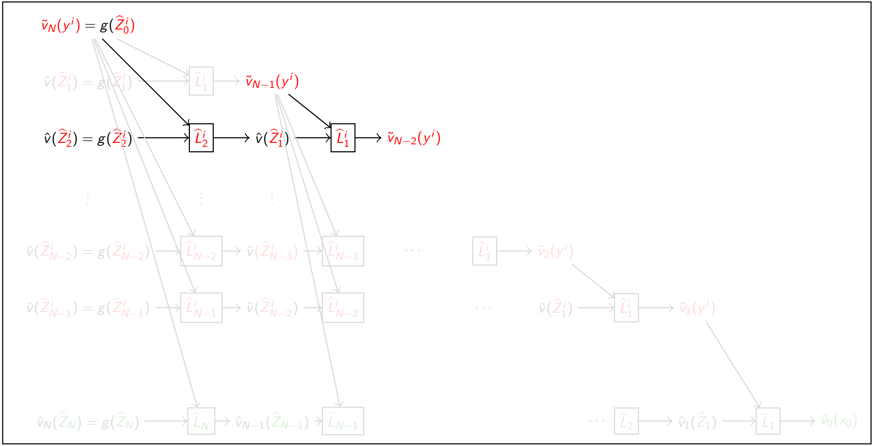

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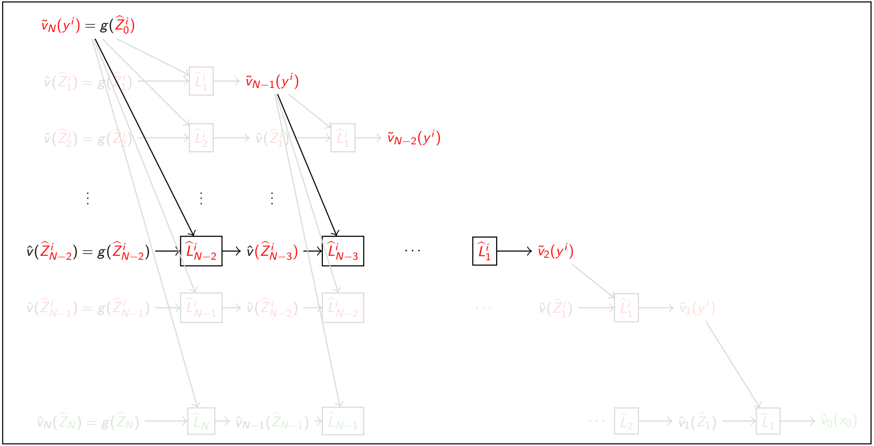

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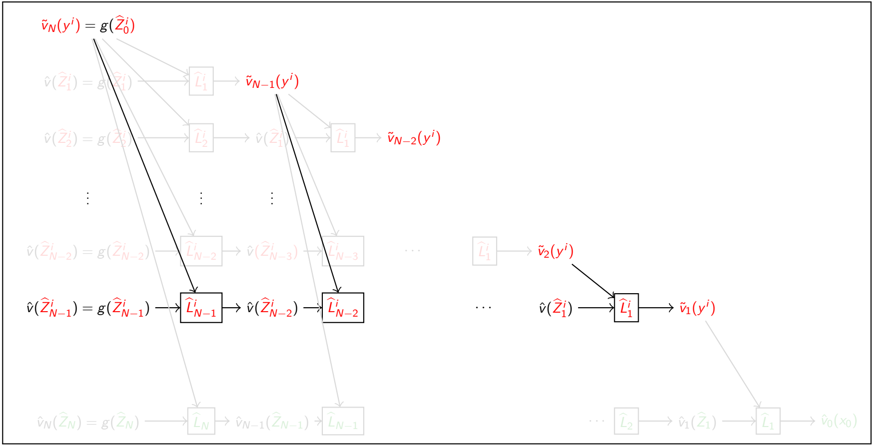

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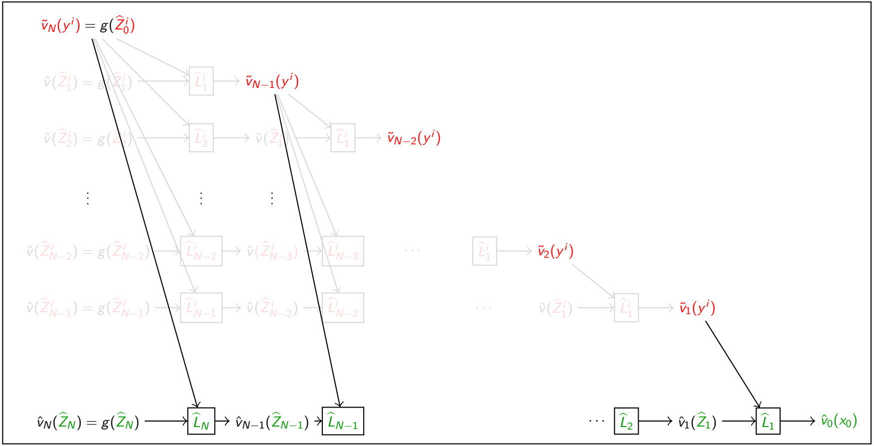

ϵ-optimal strategy

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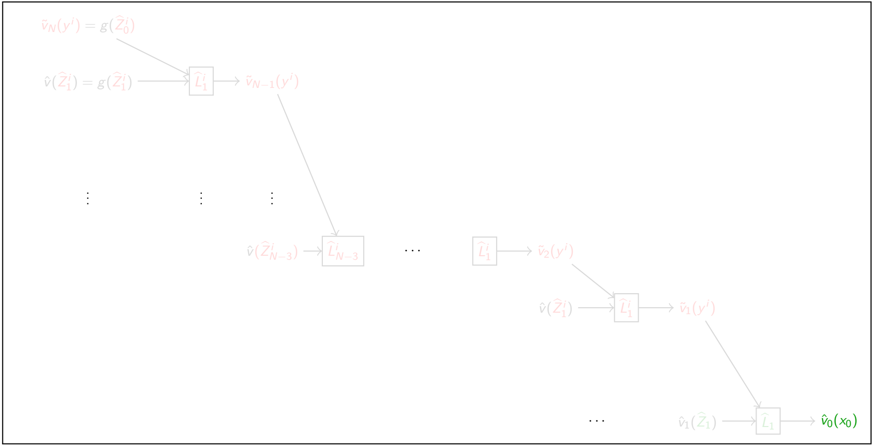

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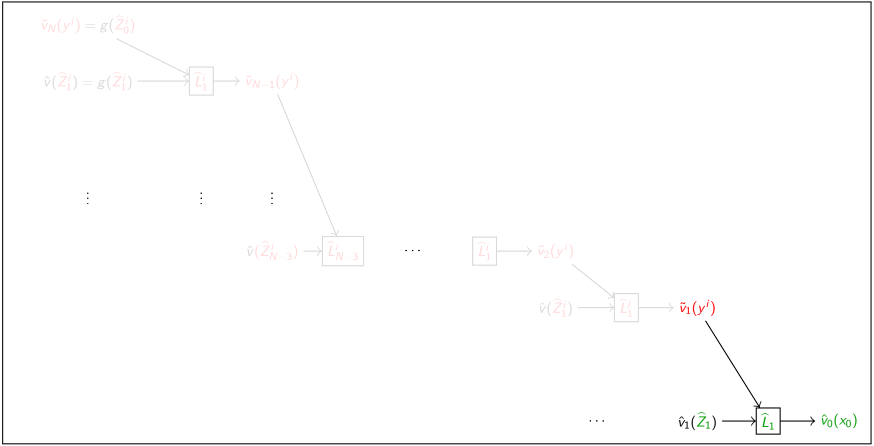

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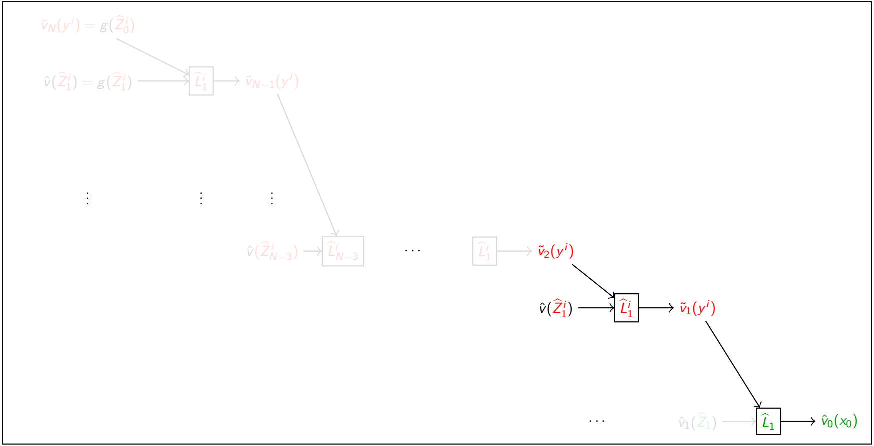

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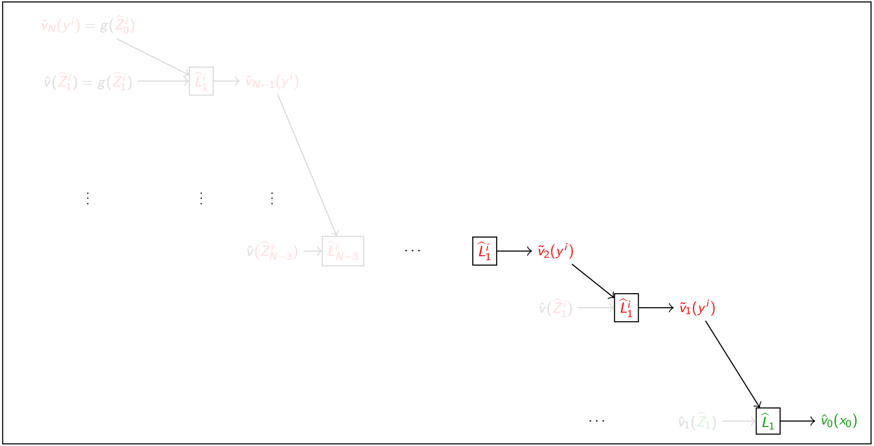

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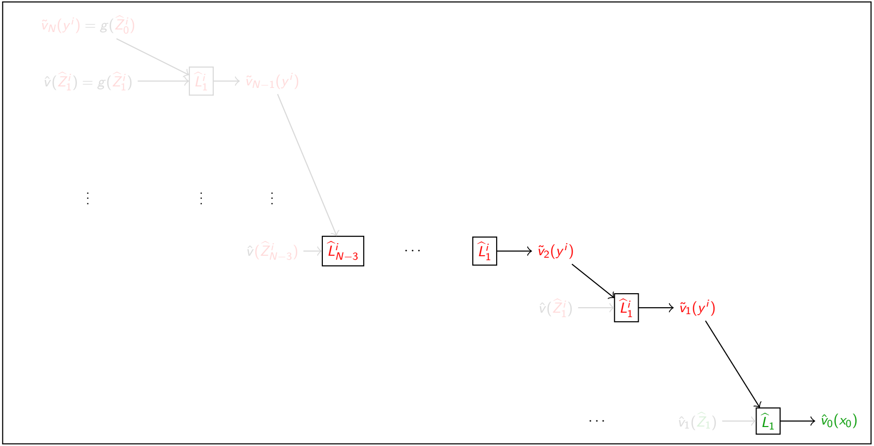

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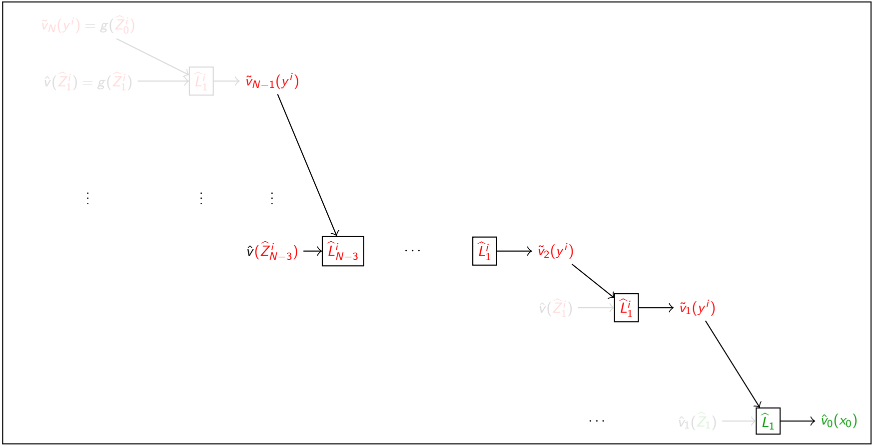

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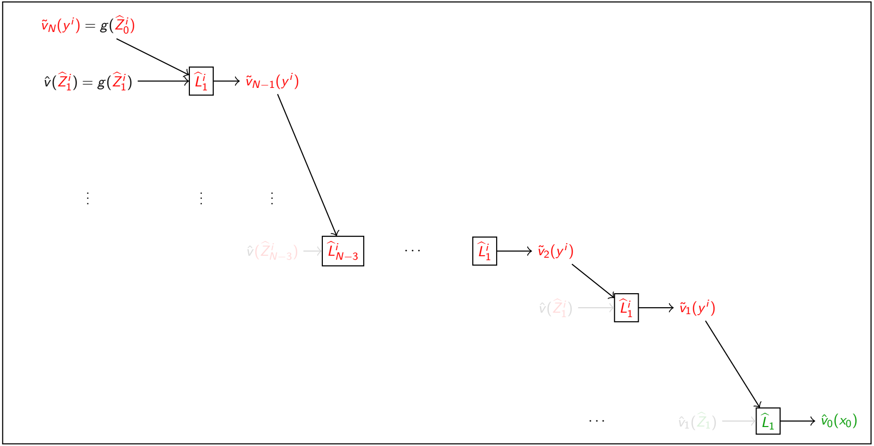

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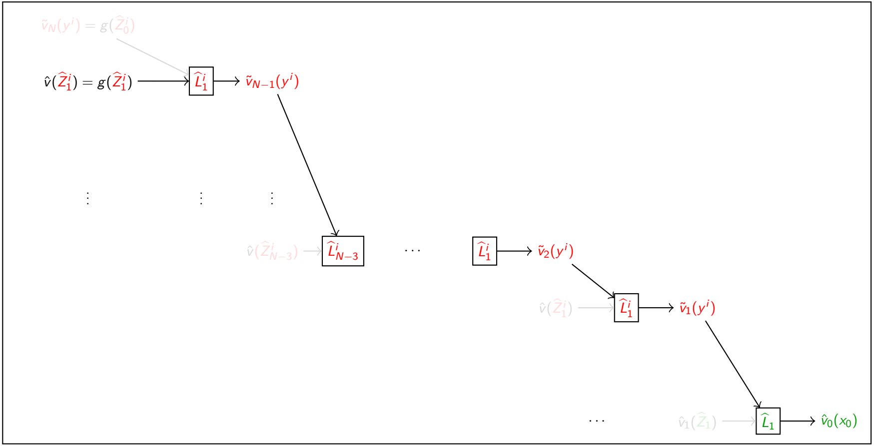

Outline

Parameters to tune

- Number of points in the control grid (underlying continuous model)
- Number of point in the quantization grids
- Approximation horizon N such that $v_N(x) - \mathcal{V}(x)$ small enough
- bounding function g
- Time discretization step for inf

Step 1: Exact simulation of the PDMP

Implementation of an exact simulator for reference strategies to serve as benchmark

	1	3	5
intervention date	never	1 day failed	1 day degraded or failed
C1 failed	nothing	change	change
C4 degraded	nothing	repair	repair
C4 failed	nothing	change	change
C2 failed and C3 stable	nothing	change 2+3	change 2+3
C2 failed and C3 degraded	nothing	change 2+3	change 2+3
C2 stable and C3 degraded	nothing	repair 3	repair 3
C2 stable and C3 failed	nothing	change 2+3	change 2+3
Mean cost	19680	11114	8359

Step 2 : Discretisation of the control set $\mathbb{U}$

Control set $\mathbb{U}(x)$: possible points to restart from after an intervention from state x. For the numerical computation, must be

- finite
- the same at any point

For the equipment model, the control set is

- infinite
- point dependent as some actions are forbidden in some modes

Solution

- discretize the control set
- manage the point dependency with infinite costs

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Step 2 : Discretisation of the control set $\mathbb{U}$

Finite control set $\mathbb{U}$

$\Longrightarrow$ discretize the functioning times at interventions

$\Longrightarrow$ project the real times on the grid feasibly

Compromise between precision and computation time

Tests on strategy 5

Grid	Number of points	relative error
$3 \times 3 \times 3 \times 5$	419	0.1458
$4 \times 4 \times 4 \times 5$	627	0.1331
$5 \times 5 \times 5 \times 5$	1055	0.1235
$3 \times 3 \times 3 \times 11$	788	0.0962
$4 \times 4 \times 4 \times 11$	1219	0.0819
$5 \times 5 \times 5 \times 11$	1855	0.0730
$6 \times 6 \times 6 \times 11$	2790	0.0672
$7 \times 7 \times 7 \times 11$	3570	0.0634
$8 \times 8 \times 8 \times 11$	4647	0.0604
$3 \times 3 \times 3 \times 21$	1403	0.0775
$4 \times 4 \times 4 \times 21$	2195	0.0626
$5 \times 5 \times 5 \times 21$	3423	0.0534
$6 \times 6 \times 6 \times 21$	4900	0.0436
$7 \times 7 \times 7 \times 21$	6489	0.0384
$8 \times 8 \times 8 \times 21$	8399	0.0350

Step 3: Discretizing the embedded Markov chain

- calibration on reference strategies

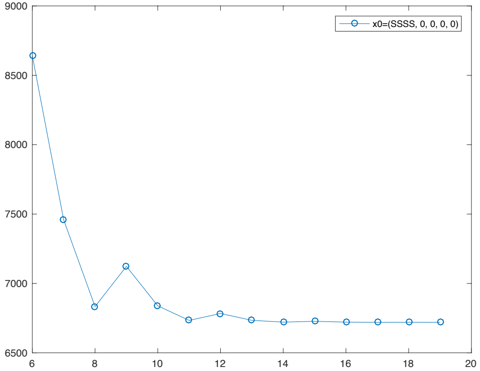
Compromise between precision and computation time

Number of points	Strategy 1	Strategy 3	Strategy 5
50	19680	11075	8326
100	19680	11134	8367
200	19680	11104	8361
400	19680	11124	8366
1000	19680	11109	8355
Exact cost	19680	11114	8359

Step 4: Calibrating N the number of allowed jumps + interventions

Horizon N (number of iterations)

- 5 for Strategy 1
- up to 30 for Strategy 3 (mean 6)
- up to 25 for Strategy 5 (mean 6)



Step 5: Approximation of the value function

Strategy 1	Strategy 3	Strategy 5	Approx. Value function
19680	11114	8359	6720

- relative gain of 19.6% vs Strategy 5
- numerical validation of the algorithm with various starting points: consistent results

Step 6: Optimally controlled trajectories

Strategy 1	Strategy 3	Strategy 5	Approx. Value function	Optimally controlled traj.
19680	11114	8359	6720	6735

- relative gain of 19.6% vs Strategy 5
- numerical validation of the algorithm with various starting points: consistent results

Outline

Conclusion

Numerical method to approximate the value function

- rigorously validated
- with general error bounds
- numerical demanding but viable in low dimensional examples

Work in progress

- validation of the optimal strategy
- reference strategy 6 ?
- sensitivity analysis

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References

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[P 98] G. Pagès *A space quantization method for numerical integration*
[PPP 04] G. Pagès, H. Pham, J. Printems *An optimal Markovian quantization algorithm for multi-dimensional stochastic control problems*