Numerical method for optimal stopping of PDMP and maintenance optimization

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Outline

Introduction
  Piecewise deterministic Markov Processes
  Numerical methods for PDMP

Optimal stopping
  Problem formulation
  Discretization scheme
  Quantization
  Convergence

Application to maintenance optimization
  Example from Thales optronique
  Conclusion and perspectives
Piecewise deterministic Markov processes

Davis (80’s)

General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

Applications

Engineering systems, operations research, management science, economics, internet traffic, neurosciences, biology, dependability and safety...
Dynamics

Hybrid process $X_t = (m_t, y_t)$
- discrete mode $m_t \in \{1, 2, \ldots, p\}$
- Euclidean state variable $y_t \in \mathbb{R}^n$

Local characteristics for each mode $m$
- $E_m$ open subset of $\mathbb{R}^d$, $\partial E_m$ its boundary and $\overline{E}_m$ its closure
- Flow $\phi_m : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ deterministic motion between jumps, one-parameter group of homeomorphisms
- Intensity $\lambda_m : \overline{E}_m \rightarrow \mathbb{R}_+$ intensity of random jumps
- Markov kernel $Q_m$ on $(\overline{E}_m, \mathcal{B}(\overline{E}_m))$ selects post-jump location
Two types of jumps

- $t^*(m, y)$ deterministic exit time starting from $(m, y)$

$$t^*(m, y) = \inf\{t > 0 : \phi_m(y, t) \in \partial E_m\}$$

- law of the first jump time $T_1$ starting from $(m, y)$

$$\mathbb{P}_{(m,y)}(T_1 > t) = \begin{cases} e^{-\int_0^t \lambda_m(\phi_m(y,s)) \, ds} & \text{if } t < t^*(m, y) \\ 0 & \text{if } t \geq t^*(m, y) \end{cases}$$

Remark

$T_1$ has a density on $[0, t^*(m, y)]$ but has an atom at $t^*(m, y)$

$$\mathbb{P}_{(m,y)}(T_1 = t^*(m, y)) > 0$$
Iterative construction

Starting point

\[ X_0 = Z_0 = (m, y) \]
Iterative construction

$X_t$ follows the deterministic flow until the first jump time $T_1 = S_1$

$$X_t = (m, \phi_m(y, t)), \quad t < T_1$$
Iterative construction

Post-jump location $Z_1 = (M_1, Y_1)$ selected by the Markov kernel $Q_m(\phi_m(y, T_1), \cdot)$
Iterative construction

$X_t$ follows the flow until the next jump time $T_2 = T_1 + S_2$

$$X_{T_1 + t} = (M_1, \phi_{M_1}(Y_1, t)), \quad t < S_2$$
Iterative construction

Post-jump location \( Z_2 = (M_2, Y_2) \) selected by Markov kernel

\[
Q_{M_1}(\phi_{M_1}(Y_1, S_2), \cdot) \ldots
\]
Embedded Markov chain

\{X_t\} strong Markov process [Davis 93]

Natural embedded Markov chain

- $Z_0$ starting point, $T_0 = 0$, $S_0 = 0$
- $Z_n$ new mode and location after $n$-th jump
  $T_n$ date of $n$-th jump, $S_n = T_n - T_{n-1}$

Important property

$(Z_n, S_n)$ is a discrete-time Markov chain
Only source of randomness of the PDMP
Numerical methods for PDMP

Fact

- numerous application domains
- numerous theoretical results
  - [Davis 93], [Jacobsen 06], [Costa Dufour 13]
- processes easy to simulate if explicit flow
- very few dedicated numerical methods in the literature for optimal control
  - [Costa Davis 88, 89]
Aim of the talk

Propose a new numerical method
▶ adapted to the specificities of PDMPs
▶ with proofs and rate of convergence
▶ implementable in practice

To solve approximately the optimal stopping problem for PDMPs
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Problem setting

- **Reward function** $g$
- **Random time horizon**: $N$-th jump time $T_N$
- $\mathcal{M}_N$ set of all stopping times $\tau \leq T_N$

**Optimal stopping problem**

- **compute the value function**

\[ V(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)] \]

- **find an ($\varepsilon$-)optimal stopping time** $\tau^*$ that reaches $V(x)(-\varepsilon)$
Recursion for value functions

[Gugerli 1986]

Dynamic programming

- $v_N = g$
- $v_n = L(v_{n+1}, g)$ pour $n \leq N - 1$

$$v_0(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)] = V(x)$$

$$L(w, g)(x) = \sup_{u \leq t^*(Z_n)} \left\{ \mathbb{E}[w(Z_{n+1}) \mathbb{1}_{s_{n+1} < u} + g(\phi(Z_n, u)) \mathbb{1}_{s_{n+1} \geq u} \mid Z_n = x] \right\}$$

$\lor$ $\mathbb{E}[w(Z_{n+1}) \mid Z_n = x]$. 
Recursion for random variables

**Dynamic programming**

- $\nu_N(Z_N) = g(Z_N)$
- $\nu_n(Z_n) = L(\nu_{n+1}, g)(Z_n)$ pour $n \leq N - 1$

$$\nu_0(Z_0) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_{\tau})]$$

\[
\nu_n(Z_n) = L(\nu_{n+1}, g)(Z_n) \\
= \sup_{u \leq t^*(Z_n)} \left\{ \mathbb{E} \left[ \nu_{n+1}(Z_{n+1}) \mathbb{1}_{\{S_{n+1} < u\}} + g(\phi(Z_n, u)) \mathbb{1}_{\{S_{n+1} \geq u\}} \mid Z_n \right] \right\} \\
\lor \mathbb{E} \left[ \nu_{n+1}(Z_{n+1}) \mid Z_n \right]
\]
Strategy

- discretize the chain \((Z_n, S_n)\) using quantization
- replace \((Z_n, S_n)\) by its approximation \((\hat{Z}_n, \hat{S}_n)\) in \(L\) → computable approximation
- study convergence, derive error bounds
  - indicator functions in the dynamic programming equation → be careful with the time grids
Quantization

Quantization of a random variable $X \in L^p(\mathbb{R}^d)$

Approximate $X$ by $\hat{X}$ taking finitely many values such that
$\|X - \hat{X}\|_p$ is minimum

- Find a finite weighted grid $\Gamma$ with $|\Gamma| = K$
- Set $\hat{X} = p_{\Gamma}(X)$ closest neighbor projection

Asymptotic properties

If $E[|X|^{p+\eta}] < +\infty$ for some $\eta > 0$ then

$$\lim_{K \to \infty} K^{1/d} \min_{|\Gamma| \leq K} \|X - \hat{X}_{\Gamma}\|_p = C$$
Algorithms

There exist algorithms providing

- $\Gamma$
- law of $\hat{X}$
- transition probabilities for quantization of Markov chains

Example: $\mathcal{N}(0, I_2)$:
Algorithms

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Example: $\mathcal{N}(0, I_2)$:
Optimal stopping

Quantization

Grids construction

Model $\rightarrow$ simulator of trajectories $\rightarrow$ grids
Grids construction

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Model $\rightarrow$ simulator of trajectories $\rightarrow$ grids
# Assets and drawbacks of quantization

## Assets
- A simulator of the target law is enough to build the grids
- Automatic construction of grids
- Convergence rate for $\mathbb{E}[|f(X) - f(\hat{X})|]$ if $f$ Lipschitz

## Drawbacks
- Computation time
- Curse of dimension
- Open questions of convergence of the algorithms
**Discretization**

**Approximation of the value function**

- $\hat{v}_N(\hat{Z}_N) = g(\hat{Z}_N)$
- $\hat{v}_n(\hat{Z}_n) = \hat{L}^n_d(\hat{v}_{n+1}, g)(\hat{Z}_n)$ for $n \leq N - 1$

\[
\hat{L}^n_d(v_{n+1}, g)(\hat{Z}_n) = \max_{u \in G(\hat{Z}_n)} \left\{ \mathbb{E}\left[ v(\hat{Z}_{n+1}) \mathbb{1}_{\{\hat{S}_{n+1} < u\}} + g(\phi \hat{Z}_n, u) \mathbb{1}_{\{\hat{S}_{n+1} \geq u\}} \mid \hat{Z}_n \right] \right\} \lor \mathbb{E}[v(\hat{Z}_{n+1}) \mid \hat{Z}_n]
\]
Convergence


**Theorem**

Lipschitz regularity assumptions on $\phi$, $\lambda$, $Q$, $t^*$ and $g$

$$|v_0(x) - \hat{v}_0(x)| \leq C\sqrt{EQ}$$

$C$ explicit constant,

$EQ$ quantization error

$\sqrt{\cdot}$ due to the indicator functions
Theorem

Same assumptions

\[ |v_0(x) - \mathbb{E}_x[g(X_{\hat{\tau}})]| \leq C_1 EV + C_2 \sqrt{EQ} \]

\(C_1, C_2\) explicit constants, \(EV\) value function error, \(EQ\) quantization error

Provides another approximation of the value function via Monte Carlo simulations
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Application to maintenance optimization

- $X_t = (m_t, y_t)$ state of a machine at time $t$
- $T_n$ failure of some components

Maintenance optimization

Find an optimal balance between
- changing the components too early/often
- do nothing until total breakdown
Industrial problem from Thales optronique

Compute an optimized maintenance date for an equipment subject to different kinds of failures

Air conditioning unit

- State 1: stable state
- State 2: degraded ball bearing
- State 3: failed electrovalve
- State 4: electronic failure
- State 5: failed ball bearing
**PDMP model**

**Transition rates**

- degraded ball bearing and failed electrovalve: Weibull distributions $\Rightarrow$ time dependent intensity
- electronic and ball bearing failures: exponential distribution

![State Transition Diagram]

- State 1: stable state
- State 2: degraded ball bearing
- State 3: failed electrovalve
- State 4: electronic failure
- State 5: failed ball bearing

**PDMP model**

- discrete mode $m_t \in \{1, 2, 3, 4, 5\}$
- Euclidean state variable $y_t = t$ working time
Trajectories without maintenance

1: stable state
2: degraded ball bearing
3: failed electrovalve
4: electronic failure
5: failed ball bearing
Reward function

\[ g(m, t) = \frac{t}{p(m)} \]

- \( p(1) = 6 \) price of maintenance in stable mode
- \( p(2) = 6 \) price of maintenance in degraded ball bearing mode
- \( p(3) = 5 \) price of repair of electrovalve failure
- \( p(4) = 3.5 \) price of repair of electronic failure
- \( p(5) = 12 \) price of repair of ball bearing failure

Maintenance optimisation

- Better to start a maintenance in mode 2 than wait for total failure mode 5
- Failure modes 3 and 4 cost less than maintenance

Average performance without maintenance: 342.72
Trajectories with maintenance

Without maintenance

Average performance: 342.72

With maintenance

Average performance: 592.47
Conclusion and perspectives

Assets and drawbacks of the numerical method

▶ practical method
▶ computation time on line/off line
▶ curse of dimension

Perspectives

▶ Optimal policy for impulse control Thales Optronique
▶ Numerical methods for MDP Airbus, DCNS
Thank you