Numerical method for optimal stopping of piecewise deterministic Markov processes

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Outline

1. Piecewise deterministic Markov processes
   - Definition
   - Example

2. Optimal stopping

3. SDE’s

4. Numerical method
   - Theoretical results
   - Approximation of the value function
   - $\epsilon$-optimal stopping time

5. Numerical results
Definition of piecewise deterministic Markov processes

Davis (80’s)

General class of non-diffusion dynamic stochastic hybrid models: deterministic motion punctuated by random jumps.

Applications

Engineering systems, operations research, management science, economics, dependability and safety,…
Hybrid process $X_t = (m_t, y_t)$
- **discrete** mode $m_t \in \{1, 2, \ldots, p\}$
- **Euclidean** state variable $y_t \in \mathbb{R}^n$

Local characteristics for each mode $m$
- $E_m$ open subset of $\mathbb{R}^d$, $\partial E_m$ its boundary and $\overline{E}_m$ its closure
- **Flow** $\phi_m : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ deterministic motion between jumps, one-parameter group of homeomorphisms
- **Intensity** $\lambda_m : \overline{E}_m \to \mathbb{R}_+$ intensity of random jumps
- **Markov kernel** $Q_m$ on $(\overline{E}_m, \mathcal{B}(\overline{E}_m))$ selects the post-jump location

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**Numerical method for optimal stopping of piecewise deterministic Markov processes**
Two types of jumps

- $t^*(m, y)$ deterministic **exit time** when the process starts in mode $m$ at $y$:
  \[
  t^*(m, y) = \inf \{ t > 0 : \phi_m(y, t) \in \partial E_m \} 
  \]

- law of the first jump time $T_1$ starting from $(m, y)$
  \[
  \mathbb{P}_{(m, y)}(T_1 > t) = \begin{cases} 
  e^{-\int_0^t \lambda_m(\phi_m(y, s)) \, ds} & \text{if } t < t^*(m, y) \\
  0 & \text{if } t \geq t^*(m, y) 
  \end{cases}
  \]

**Remark**

$T_1$ has a density on $[0, t^*(m, y)]$ but has an **atom** at $t^*(m, y)$:

\[
\mathbb{P}_{(m, y)}(T_1 = t^*(m, y)) > 0 
\]
Iterative construction

Starting point

\[ X_0 = Z_0 = (m, y) \]
Iterative construction

$X_t$ follows the deterministic flow until the first jump time $T_1 = S_1$

$$X_t = (m, \phi_m(y, t)), \quad t < T_1$$
Iterative construction

Post-jump location $\mathbf{Z}_1 = (M_1, Y_1)$ selected by

$$Q_m(\phi_m(y, T_1), \cdot)$$
Iterative construction

$X_t$ follows the flow until the next jump time $T_2 = T_1 + S_2$

$X_{T_1 + t} = (M_1, \phi_{M_1}(Y_1, t)), \quad t < S_2$
Iterative construction

Post-jump location \( Z_2 = (M_2, Y_2) \) selected by

\[
Q_{M_1}(\phi_{M_1}(Y_1, S_2), \cdot) \ldots
\]
Embedded Markov chain

\( \{X_t\} \) strong Markov process (M.H.A. Davis)

**Natural embedded Markov chain**

- \( Z_0 \) starting point, \( S_0 = 0, S_1 = T_1 \)
- \( Z_n \) new mode and location after \( n \)-th jump, \( S_n = T_n - T_{n-1} \), time between two jumps

**Proposition**

\((Z_n, S_n)\) is a discrete-time Markov chain

Only source of randomness of the PDMP
Industrial example

Industrial collaboration: EADS Astrium

- crack propagation,
- material subject to corrosion and randomly exposed to different stressing ambiences.
Material subject to corrosion

Model

- $m_t \in \{1, 2, 3\}$ describes the ambience
- $d_t$ loss of thickness
- $\gamma_t$ duration of the initial protection against corrosion
- $\rho_t$ rate of corrosion

The process starts in ambience 1: $m_0 = 1$, $d_0 = 0mm$,

$$\gamma_0 \sim 1 - \exp\left(-\left[\frac{t}{\alpha}\right]\gamma\right), \quad \rho_0 \sim \mathcal{U}[\rho_1^1, \rho_1^2]$$
Dynamics

- Deterministic succession of ambiances: $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \cdots$
- Time spent in ambience $i \sim \text{Exp}(\lambda_i)$
- $\rho_t$ is constant on $[T_n, T_{n+1}[ \text{ and in ambience } i$, $\rho \sim U[\rho^1_i, \rho^2_i]$
- $\gamma_{T_{n+1}} = 0$ if $\gamma_{T_n} < S_{n+1}$, $\gamma_{T_{n+1}} = S_{n+1} - \gamma_{T_n}$ otherwise
- On $[T_n, T_{n+1}[$, in ambience $i$

$$d_t = \begin{cases} 0, & \text{if } t \leq \gamma_{T_n}, \\ \rho_{T_n}(t - (\gamma_{T_n} + c_i) + c_i \exp(-[t - \gamma_{T_n}] / c_i)), & \text{otherwise.} \end{cases}$$

The material is inefficient when the thickness loss is greater than $0.2\text{mm}$
Examples of trajectories for the loss of thickness $d_t$
Examples of trajectories for the loss of thickness $d_t$
Definition

- **Reward function** $g$
- **Time horizon** $N$-th jump $T_N$
- $\mathcal{M}_N$ set of all stopping times $\tau \leq T_N$

**Optimal stopping problem**

- compute the **value function**

$$V(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_{\tau})]$$

- find an $(\varepsilon)$-**optimal** stopping time $\tau^*$ that reaches $V(x)(-\varepsilon)$
References for PDMP’s

- O. Costa & M. Davis (88)
- O. Costa & F. Dufour (00)
- U. Gugerli (86)
- D. Gatarek (91)
- S. Lenhart & Y. Liao (85)
Application to maintenance optimization

- $X_t = (m_t, y_t)$ state of a machine/structure at time $t$
- $T_n$ failure of some components/changes of ambience

**Optimal stopping**

Find an optimal balance between

- changing the components too early/often
- no maintenance leading to a total breakdown
Our aim

Objective

Propose a numerical method

- to evaluate the value function
- to compute an $\varepsilon$-optimal stopping rule

with error bounds

Strategy

Adapt numerical procedures for optimal stopping of SDE’s
Numerical method for diffusion processes

Bally, Pagès, Pham, Printems 98–05

$Y_t$ continuous-time Markov diffusion process

1. time discretization (Euler scheme): $Y_k = Y_{k \Delta t}$ discrete-time Markov chain with continuous state space
2. quantization: replace $Y_k$ by a random variable $\hat{Y}_k$ taking values in a finite state space
3. replace the conditional expectations in the dynamic programming equation by finite sums

Assumptions + Lipschitz-continuous reward function $\implies$ convergence rate of the approximated value function to the original one
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Bally, Pagès, Pham, Printems 98–05

$Y_t$ continuous-time Markov diffusion process

1. time discretization (Euler scheme): $Y_k = Y_{k+1}$ discrete-time Markov chain with continuous state space

2. quantization: replace $Y_k$ by a random variable $\hat{Y}_k$ taking values in a finite state space

3. replace the conditional expectations in the dynamic programming equation by finite sums

Assumptions + Lipschitz-continuous reward function $\Rightarrow$ convergence rate of the approximated value function to the original one
Quantization

Quantization of a random variable $X \in L^p(\mathbb{R}^d)$

Approximate $X$ by $\hat{X}$ taking \textbf{finitely} many values such that $\|X - \hat{X}\|_p$ is \textbf{minimum}

- Find a finite weighted grid $\Gamma$ with $|\Gamma| = K$
- Set $\hat{X} = p_\Gamma(X)$ closest neighbour projection

Asymptotic properties

If $E[|X|^{p+\eta}] < +\infty$ for some $\eta > 0$ then

$$\lim_{K \to \infty} K^{p/d} \min_{|\Gamma| \leq K} \|X - \hat{X}_\Gamma\|_p = J_{p,d} \left( \int |h|^{d/(d+p)}(u)du \right),$$

where $P_x(du) = h(u)\lambda_d(du) + \nu$ with $\nu \perp \lambda_d$ and $J_{p,d}$ a constant.
There exist algorithms providing

- $\Gamma$
- law of $\hat{X}$
- transition probabilities for quantization of Markov chains

Example: $\mathcal{N}(0, I_2)$:
Algorithms

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Example: $\mathcal{N}(0, I_2)$:
Quantization technique have been developed recently in

- **numerical probability** Pages (98)
- **nonlinear filtering** Pages & Pham (05)
- **optimal stochastic control in finance** Bally & Pages (03); Pages & Pham & Printemps (05); Bally & Pages & Printemps (05)
Specificities of PDMP’s

- jumps at random times
- indicator functions in the dynamic programming equation

Solution

- use the embedded Markov chain \((Z_n, S_n)\)
- be careful with the time grids
Theoretical results

Iterative resolution

**Backward dynamic programming equation (U. Gugerli, 1986):**

- \( v_N = g \)
- \( v_n = L(v_{n+1}, g) \) for \( n \leq N - 1 \)

\[
v_0(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x [g(X_\tau)] = V(x)
\]

\[
L(w, g)(x) = \sup_{t \geq 0} \left[ \int_0^{t \wedge t^*(x)} \lambda Qw(\phi(x, s))e^{-\Lambda(x, s)} ds + g(\phi(x, t \wedge t^*(x)))e^{-\Lambda(x, t \wedge t^*(x))} \right] \\
\quad \vee \int_0^{t^*(x)} \lambda Qw(\phi(x, s))e^{-\Lambda(x, s)} ds + Qw(\phi(x, t^*(x)))e^{-\Lambda(x, t^*(x))}
\]
Theoretical results

Probabilistic interpretation

Backward dynamic programming equation

\[ v_N(Z_N) = g(Z_N) \]
\[ v_n(Z_n) = L(v_{n+1}, g)(Z_n) \text{ for } n \leq N - 1 \]

\[ v_0(Z_0) = \sup_{\tau \in M_N} \mathbb{E}_x[g(X_\tau)] \]

\[ v_n(Z_n) = L(v_{n+1}, g)(Z_n) \]
\[ = \sup_{u \leq t^*(Z_n)} \left\{ \mathbb{E}\left[v_{n+1}(Z_{n+1}) \mathbf{1}_{S_{n+1} < u} + g(\phi(Z_n, u)) \mathbf{1}_{S_{n+1} \geq u} \mid Z_n\right] \right\} \]
\[ \vee \mathbb{E}\left[v_{n+1}(Z_{n+1}) \mid Z_n\right] \]
Backward dynamic programming equation

- $v_N(Z_N) = g(Z_N)$
- $v_n(Z_n) = L(v_{n+1}, g)(Z_n)$ for $n \leq N - 1$

$$v_0(Z_0) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_{\tau})]$$

**Time**

$(Z_0, S_0) \rightarrow (Z_k, S_k) \rightarrow (Z_N, S_N)$

**Calculation**

$v_0(Z_0) \leftarrow v_k(Z_k) \rightarrow g(Z_N)$
Approximation of the value function

- $v_N(Z_N) = g(Z_N)$
- $v_n(Z_n) = L(v_{n+1}, g)(Z_n)$ for $n \leq N - 1$

\[
L(v_{n+1}, g)(Z_n) = \sup_{u \leq t^*(Z_n)} \left\{ \mathbb{E} \left[ v(Z_{n+1}) 1\{S_{n+1} < u\} + g(\phi(Z_n, u)) 1\{S_{n+1} \geq u\} \mid Z_n \right] \right\} \\
\lor \mathbb{E} [v(Z_{n+1}) \mid Z_n]
\]
Discretization

**Time discretization**

- \( v_N(Z_N) = g(Z_N) \)
- \( v_n(Z_n) = L(v_{n+1}, g)(Z_n) \) for \( n \leq N - 1 \)

\[
L_d(v_{n+1}, g)(Z_n) = \max_{u \in G(Z_n)} \left\{ \mathbb{E}\left[ v(Z_{n+1}) \mathbf{1}_{\{S_{n+1} \leq u\}} + g\left(\phi(Z_n, u)\right) \mathbf{1}_{\{S_{n+1} \geq u\}} \mid Z_n \right] \right\}
\]

\[\vee \mathbb{E}\left[ v(Z_{n+1}) \mid Z_n \right] \]
Approximation of the value function

Discretization

Quantization

\[ v_N(Z_N) = g(Z_N) \]

\[ v_n(Z_n) = L(v_{n+1}, g)(Z_n) \text{ for } n \leq N - 1 \]

\[
\hat{L}_d(v_{n+1}, g)(Z_n) = \max_{u \in G(Z_n)} \left\{ \mathbb{E} \left[ v(\hat{Z}_{n+1}) \mathbb{1}_{\{\hat{S}_{n+1} < u\}} + g(\phi(Z_n, u)) \mathbb{1}_{\{\hat{S}_{n+1} \geq u\}} \mid \hat{Z}_n \right] \right\} \\
\lor \mathbb{E}[v(Z_{n+1}) \mid \hat{Z}_n]
\]
Approximation of the value function

- $\hat{v}_N(\hat{Z}_N) = g(\hat{Z}_N)$
- $\hat{v}_n(\hat{Z}_n) = \hat{L}_d(\hat{v}_{n+1}, g)(\hat{Z}_n)$ for $n \leq N - 1$

\[
\hat{L}_d(\nu_{n+1}, g)(Z_n) = \max_{u \in G(Z_n)} \left\{ \mathbb{E}\left[ \nu(\hat{Z}_{n+1}) 1_{\{\hat{S}_{n+1} < u\}} + g(\phi(Z_n, u)) 1_{\{\hat{S}_{n+1} \geq u\}} \mid \hat{Z}_n \right] \right\} \lor \mathbb{E}[\nu(Z_{n+1}) \mid \hat{Z}_n]
\]
Approximation of the value function

Convergence rate

Theorem

Lipschitz assumptions on $\phi$, $\lambda$, $Q$, $t^*$ and $g$

$$|v_0(x) - \hat{v}_0(x)| \leq C \sqrt{EQ}$$

$C$ explicit constant,

$EQ$ quantization error

$\sqrt{\cdot}$ due to the indicator functions
**Dealing with indicator functions**

Set $\eta$ s.t. $\forall s \in G(\hat{Z}_0)$, $s + \eta < t^*(\hat{Z}_0)$

$$\left\| \max_{u \in G(x)} \mathbb{E}_{Z_0} \left[ |1\{S_1 < u\} - 1\{\hat{S}_1 < u\}| \right] \right\|_2 \leq \frac{1}{\eta} \|S_1 - \hat{S}_1\|_2 + C\eta$$

**Easy cases:**

![Diagram of indicator functions with $S_1$ and $\hat{S}_1$](image)

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Dealing with indicator functions

Set $\eta$ s.t. $\forall s \in G(\hat{Z}_0)$, $s + \eta < t^*(\hat{Z}_0)$

$$\left\| \max_{u \in G(x)} E_{Z_0} \left[ |1_{\{S_1 < u\}} - 1_{\{\hat{S}_1 < u\}}| \right] \right\|_2 \leq \frac{1}{\eta} \|S_1 - \hat{S}_1\|_2 + C \eta$$

Easy cases:

```
\begin{tikzpicture}
  \draw (0,0) -- (6,0);
  \draw (0,-1) -- (6,-1);
  \node at (0,0) {$u$};
  \node at (3,0) {$t^*$};
  \node at (0,-1) {$S_1$};
  \node at (3,-1) {$\hat{S}_1$};
\end{tikzpicture}
```
Approximation of the value function

$S_1$ and $\hat{S}_1$ are either side of $u$

$$\left| 1 \{ S_1 < u \} - 1 \{ \hat{S}_1 < u \} \right| \leq 1 \{ |S_1 - u| \leq \eta \} + 1 \{ |S_1 - \hat{S}_1| > \eta \}$$

$$\mathbb{E}_{Z_0} \left[ 1 \{ u - \eta \leq S_1 \leq u + \eta \} \right] = \int_{u - \eta}^{u + \eta} \lambda(\phi(Z_0, u)) du \leq C \eta$$
$S_1$ and $\hat{S}_1$ are either side of $u$

$$|1\{s_1 < u\} - 1\{\hat{s}_1 < u\}| \leq 1\{|s_1 - u| \leq \eta\} + 1\{|s_1 - \hat{s}_1| > \eta\}$$

$$\mathbb{E}_{Z_0}[1\{u - \eta \leq s_1 \leq u + \eta\}] = \mathbb{E}_{Z_0}[1\{u - \eta \leq s_1 \leq u + \eta\}] = \int_{u - \eta}^{u + \eta} \lambda(\phi(Z_0, u)) du \leq C\eta$$
Optimal stopping time: $\tau^*$

$$\mathbb{E}_x[g(X_{\tau^*})] = v_0(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)]$$

Existence?

$\epsilon$-optimal stopping time: $\hat{\tau}$

$$v_0(x) - \epsilon \leq \mathbb{E}_x[g(X_{\hat{\tau}})] \leq v_0(x)$$
Optimal stopping time: $\tau^*$

$$\mathbb{E}_x[g(X_{\tau^*})] = v_0(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_{\tau})]$$

Existence?

$\epsilon$-optimal stopping time: $\hat{\tau}$

$$v_0(x) - \epsilon \leq \mathbb{E}_x[g(X_{\hat{\tau}})] \leq v_0(x)$$
Proposition of a computable stopping rule $\hat{\tau}$

- **explicit** iterative construction
- no extra computation

true stopping time in $\mathcal{M}_N$
Proposition of a computable stopping rule $\hat{\tau}$

- explicit iterative construction
- no extra computation
- true stopping time in $\mathcal{M}_N$
Optimality

Theorem

Same assumptions

\[ |v_0(x) - \mathbb{E}_x[g(X_{\hat{\tau}})]| \leq C_1 EV + C_2 \sqrt{EQ} \]

\(C_1, C_2\) explicit constants

EV value function error

EQ quantization error

Provides another approximation of the value function via Monte Carlo simulations
Material subject to corrosion

Model

- $m_t \in \{1, 2, 3\}$ describes the ambience
- $d_t$ loss of thickness
- $\gamma_t$ duration of the initial protection against corrosion
- $\rho_t$ rate of corrosion

The process starts in ambience 1: $m_0 = 1$, $d_0 = 0$ mm,

$$\gamma_0 \sim 1 - \exp\left(-\left[\frac{t}{\alpha}\right]^{\gamma}\right), \quad \rho_0 \sim \mathcal{U}[\rho_1, \rho_2]$$
Dynamics

- Deterministic succession of ambiances: $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \cdots$
- Time spent in ambience $i \sim \text{Exp}(\lambda_i)$
- $\rho_t$ is constant on $[T_n, T_{n+1}]$ and in ambience $i$, $\rho \sim \mathcal{U}[\rho_i^1, \rho_i^2]$
- $\gamma_{T_{n+1}} = 0$ if $\gamma_{T_n} < S_{n+1}$, $\gamma_{T_{n+1}} = S_{n+1} - \gamma_{T_n}$ otherwise
- On $[T_n, T_{n+1}]$, in ambience $i$

$$d_t = \begin{cases} 0, & \text{if } t \leq \gamma_{T_n}, \\ \rho_{T_n} \left( t - (\gamma_{T_n} + c_i) + c_i \exp\left(-\frac{t - \gamma_{T_i}}{c_i}\right) \right), & \text{otherwise}. \end{cases}$$

The material is inefficient when the thickness loss is greater than 0.2mm
Reward function

Reward function $g$ depends only on the loss of thickness

- Early maintenances are penalized
- The material is inefficient when the loss is greater than 0.2mm
Iterative stopping rule
Iterative stopping rule
Iterative stopping rule
Iterative stopping rule
Iterative stopping rule

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Iterative stopping rule
Iterative stopping rule
Iterative stopping rule
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Iterative stopping rule
Iterative stopping rule
Iterative stopping rule

![Graph showing iterative stopping rule with x-axis labeled as $10^5$ and y-axis values ranging from 0 to 0.2. The graph illustrates the numerical method for optimal stopping of piecewise deterministic Markov processes.]
Iterative stopping rule

Numerical method for optimal stopping of piecewise deterministic Markov processes

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## Results

<table>
<thead>
<tr>
<th>Number of points in the quantization grids</th>
<th>Approximated value function</th>
<th>Monte Carlo approximated value function</th>
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<td>0.94</td>
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<td>2.70</td>
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<tr>
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</tbody>
</table>

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