

# Numerical method for optimal stopping of piecewise deterministic Markov processes

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OSA – 23 june 2009

# Outline

- 1 Piecewise deterministic Markov processes
  - Definition
  - Example
- 2 Optimal stopping
- 3 Numerical method
  - Strategy
  - Approximation of the value function
  - $\epsilon$ -optimal stopping time
- 4 Numerical results

# Definition of piecewise deterministic Markov processes

## Davis (80's)

General class of **non-diffusion** dynamic stochastic models:  
**deterministic** motion punctuated by **random** jumps.

## Applications

Engineering systems, operations research, management science, economics. . .

## Examples

Queuing systems, investment planning, stochastic scheduling, target tracking, insurance analysis, optimal exploitation of resources, . . .

# Dynamics

$E$  open subset of  $\mathbb{R}^n$ ,  $\partial E$  its boundary and  $\bar{E}$  its closure

## Local characteristics

- **Flow**  $\phi: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  deterministic motion between jumps, one-parameter group of homeomorphisms
- **Intensity**  $\lambda: \bar{E} \rightarrow \mathbb{R}_+$  intensity of random jumps
- **Markov kernel**  $Q$  on  $(\bar{E}, \mathcal{B}(\bar{E}))$  selects the post-jump location

## Two types of jumps

- $t^*(x)$  deterministic **exit time** when the process starts in  $x$ :

$$t^*(x) = \inf\{t > 0 : \phi(x, t) \in \partial E\}$$

- law of the first jump time  $T_1$  starting from  $x$

$$\mathbb{P}_x(T_1 > t) = \begin{cases} e^{-\int_0^t \lambda(\phi(x,s)) ds} & \text{if } t < t^*(x) \\ 0 & \text{if } t \geq t^*(x) \end{cases}$$

### Remark

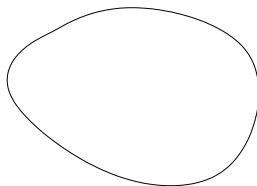
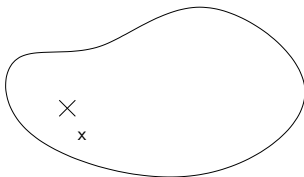
$T_1$  has a density on  $[0, t^*(x)[$  but has an **atom** at  $t^*(x)$ :

$$\mathbb{P}_x(T_1 = t^*(x)) > 0$$

# Iterative construction

Starting point :

$$X_0 = Z_0 = x$$

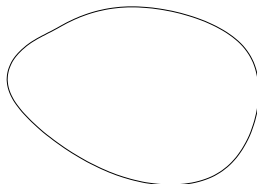
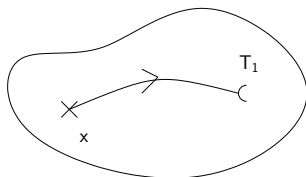


# Iterative construction

$X_t$  follows the deterministic flow until the first jump time  $T_1 = S_1$

:

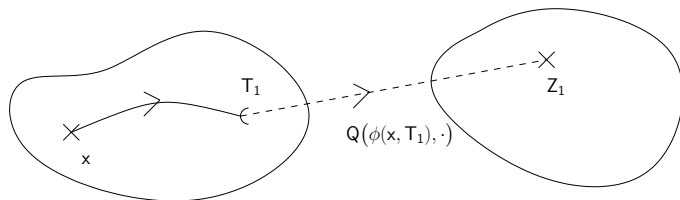
$$X_t = \phi(x, t), \quad t < T_1$$



# Iterative construction

Post-jump location  $Z_1$  selected by

$$Q(\phi(x, T_1), \cdot)$$

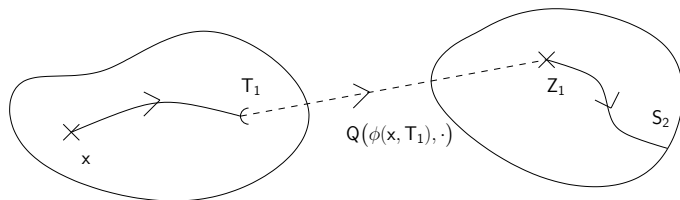




# Iterative construction

$X_t$  follows the flow until the next jump time  $T_2 = T_1 + S_2$  :

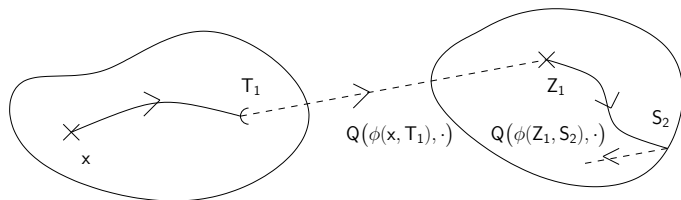
$$X_{T_1+t} = \phi(Z_1, t), \quad t < S_2$$



# Iterative construction

Post-jump location  $Z_2$  selected by

$$Q(\phi(Z_1, S_2), \cdot) \dots$$



# Embedded Markov chain

- $Z_0$  starting point,  $S_0 = 0$ ,  $S_1 = T_1$
- $Z_n$  new location after  $n$ -th jump,  $S_n = T_n - T_{n-1}$ , time between two jumps

## Proposition

$(Z_n, S_n)$  is a discrete-time Markov chain on  $E \times [0, +\infty[$   
Only source of randomness of the PDMP

# Simple example

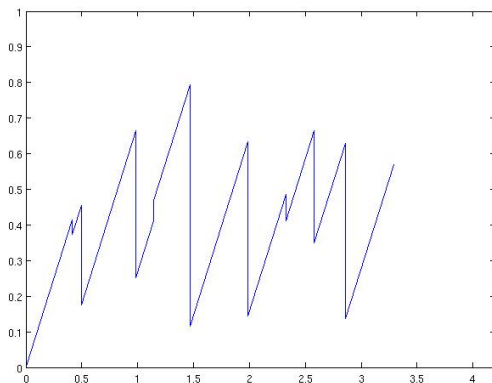
- object moving on  $[0; 1[$  with constant speed  $v$

## Local characteristics

- $\phi(x, t) = x + vt$
- $\lambda(x) = \beta x^\alpha, \beta > 0, \alpha \geq 1$  : as the object comes closer to 1 the probability to jump increases
- $Q(x, \cdot)$  uniform law on  $[0; 1/2]$

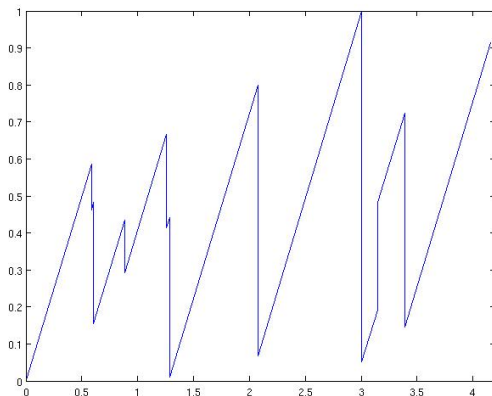
# Trajectories

Examples of trajectories for  $X_0 = 0$ ,  $\nu = 1$ ,  $\alpha = 1$ ,  $\beta = 3$  up to the 10-th jump



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# Definition

- Cost function  $g$
- Time horizon  $N$ -th jump  $T_N$
- $\mathcal{M}_N$  set of all stopping times  $\tau \leq T_N$

## Optimal stopping problem

- compute the value function

$$V(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)]$$

- find an optimal stopping time  $\tau^*$  that reaches  $V(x)$

# Example of application

- $X_t$  state of a machine at time  $t$
- $T_n$  failure of some components

## Optimal stopping

Find an optimal **balance** between

- changing the components too early/often
- avoiding a total breakdown of the machine



# Iterative resolution

Gugerli 1986 :  $V(x) = v_0(x)$

## Backward dynamic programming

- $v_N = g$
- $v_k = L(v_{k+1}, g)$  for  $k \leq N - 1$

$$\begin{aligned} L(v, g)(x) &= \sup_{u \leq t^*(x)} \left\{ \mathbb{E} \left[ v(Z_1) \mathbf{1}_{\{S_1 < u\}} + g(\phi(x, u)) \mathbf{1}_{\{S_1 \geq u\}} \mid Z_0 = x \right] \right\} \\ &\quad \vee \mathbb{E} [v(Z_1) \mid Z_0 = x] \end{aligned}$$

# Our aim

## Objective

Propose a **numerical method**

- to evaluate the **value function**
- to compute an optimal **stopping rule**

with **error bounds**

# Numerical method for diffusion processes

Bally, Pagès, Pham, Printems 98–05

$Y_t$  continuous-time diffusion process

- 1 **time discretization** (Euler scheme) :  $Y_k = Y_{k\Delta t}$  discrete-time Markov chain
- 2 **quantization** : replace  $Y_k$  by a random variable  $\hat{Y}_k$  taking values in a **finite** state space
- 3 replace the **conditional expectations** by finite sums

Assumptions + **Lipschitz**-continuous cost function  $\implies$   
**convergence rate** of the approximated value function to the original one

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# Quantization

## Quantization of a random variable $X$

Find a random variable  $\hat{X}$  such that

- $\hat{X} \in \Gamma$  with  $|\Gamma| = m$
- $\|X - \hat{X}\|_2$  is **minimum**

## Algorithms

There exist algorithms providing

- $\Gamma$
- **law** of  $\hat{X}$
- **transition probabilities** for quantization of Markov chains

# Specificities of PDMP's

$$L(v, g)(x) = \sup_{u \leq t^*(x)} \left\{ \mathbb{E} \left[ v(Z_1) \mathbf{1}_{\{S_1 < u\}} + g(\phi(x, u)) \mathbf{1}_{\{S_1 \geq u\}} \mid Z_0 = x \right] \right\} \\ \vee \mathbb{E} [v(Z_1) \mid Z_0 = x]$$

- jumps at random times
- indicator functions
- supremum

## Solution

- use the embedded Markov chain  $(Z_n, S_n)$
- be careful with the time grids



# Approximation of the value function

$$\widehat{V}(x) = \widehat{v}_0(x) \text{ with } \widehat{v}_N = g, \text{ and } \widehat{v}_k = \widehat{L}_d(\widehat{v}_{k+1}, g)$$

## Discretized operator

$$L(v, g)(x)$$

$$= \sup_{u \leq t^*(x)} \left\{ \mathbb{E} \left[ v(Z_1) \mathbf{1}_{\{S_1 < u\}} + g(\phi(x, u)) \mathbf{1}_{\{S_1 \geq u\}} \mid Z_0 = x \right] \right\}$$

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- 1 discretization of  $[0, t^*(x)[$ , transformation of **sup** into **max**
- 2  $(\widehat{Z}_n, \widehat{S}_n)$  quantization of  $(Z_n, S_n)$ , transformation of the conditional expectations into **finite sums**

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## Discretized operator

$$L_d(v, g)(x)$$

$$= \max_{u \in G(x)} \left\{ \mathbb{E} \left[ v(Z_1) \mathbf{1}_{\{S_1 < u\}} + g(\phi(x, u)) \mathbf{1}_{\{S_1 \geq u\}} \mid Z_0 = x \right] \right\}$$

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$$\vee \mathbb{E} [v(Z_1) \mid \widehat{Z}_0 = x]$$

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# Convergence rate

## Theorem

Lipschitz assumptions on  $\phi$ ,  $\lambda$ ,  $Q$ ,  $t^*$  and  $g$

$$|V(x) - \hat{V}(x)| \leq C\sqrt{EQ}$$

$C$  explicit constant,  
 $EQ$  quantization error

$\sqrt{\cdot}$  due to the indicator functions

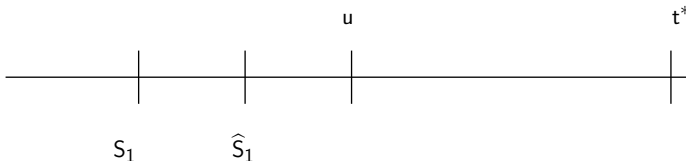
# Indicator functions

## Dealing with indicator functions

Set  $\eta$  s.t.  $\forall s \in G(\hat{Z}_n), s + \eta < t^*(\hat{Z}_n)$

$$\left\| \max_{u \in G(x)} \mathbf{E}_x \left[ \left| \mathbf{1}_{\{S_1 < u\}} - \mathbf{1}_{\{\hat{S}_1 < u\}} \right| \right] \right\|_2 \leq \frac{1}{\eta} \|S_1 - \hat{S}_1\|_2 + C\eta$$

Easy cases:



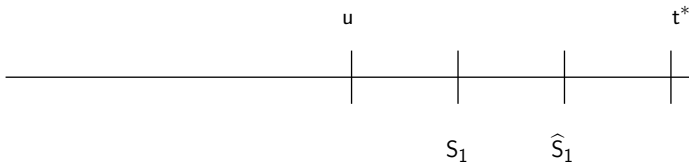
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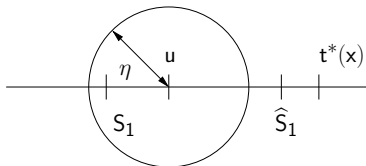
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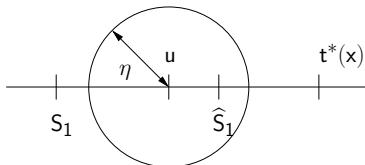
Approximation of the value function

 $S_1$  and  $\widehat{S}_1$  are either side of  $u$ 

$$|\mathbf{1}_{\{S_1 < u\}} - \mathbf{1}_{\{\widehat{S}_1 < u\}}| \leq \mathbf{1}_{\{|S_1 - u| \leq \eta\}} + \mathbf{1}_{\{|S_1 - \widehat{S}_1| > \eta\}}$$

$$\begin{aligned} \mathbf{E}_x[\mathbf{1}_{\{u-\eta \leq S_1 \leq u+\eta\}}] &= \mathbf{E}[\mathbf{1}_{\{u-\eta \leq S_1 \leq u+\eta\}}] \\ &= \int_{u-\eta}^{u+\eta} \lambda(\phi(x, u)) du \leq C\eta \end{aligned}$$

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# ε-optimal stopping time

## Optimal stopping time

$$\mathbb{E}_x[g(X_{\tau^*})] = V(x) = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_x[g(X_\tau)]$$

Existence?

## ε-optimal stopping time

$$V(x) - \epsilon \leq \mathbb{E}_x[g(X_\tau)] \leq V(x)$$

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# Stopping rule

## Proposition of a computable stopping rule $\hat{\tau}$

- explicit iterative construction
- no extra computation
- true stopping time in  $\mathcal{M}_N$

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- **explicit** iterative construction
- no extra computation
- **true stopping time** in  $\mathcal{M}_N$

# Optimality

## Theorem

Same assumptions

$$|V(x) - \mathbb{E}_x[g(X_{\hat{\tau}})]| \leq C_1 EV + C_2 \sqrt{EQ}$$

$C_1, C_2$  explicit constants

$EV$  value function error

$EQ$  quantization error

Provides another approximation of the value function via **Monte Carlo** simulations

# Example

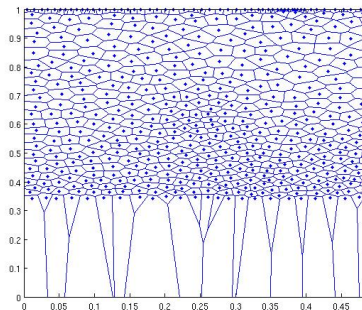
- object moving on  $[0; 1[$  with constant speed = 1

## Characteristics

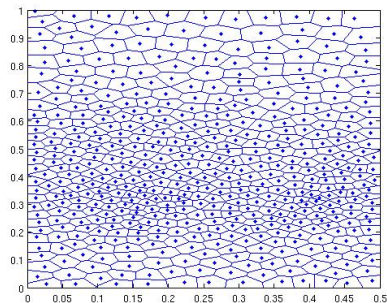
- $\phi(x, t) = x + t$
- $\lambda(x) = 3x$  : as the object comes closer to 1 the jump probability increases
- $Q(x, \cdot)$  uniform distribution on  $[0; 1/2]$
- horizon :  $N = 10$  jumps
- starting point  $x = 0$

# Quantization grids 500 points

$(Z_1, S_1)$

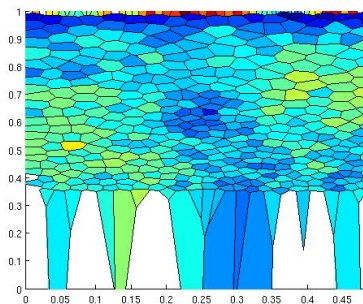


$(Z_2, S_2)$

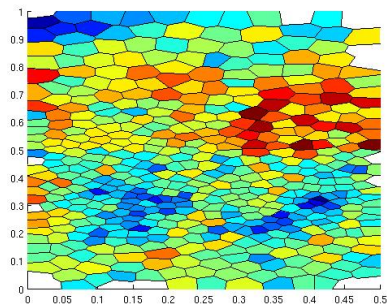


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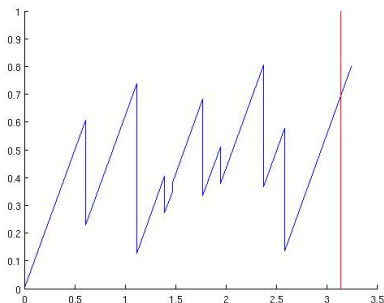
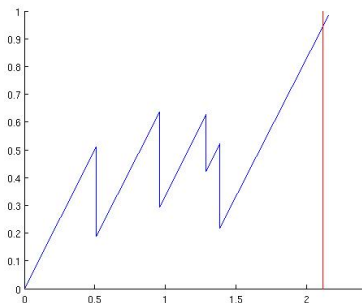
$(Z_2, S_2)$





# Stopping rule

## Examples of stopped trajectories



# Evaluations of the value function

$V$  unknown

$$\mathbb{E}_x[g(X_{\hat{\tau}})] \leq V(x) \leq \mathbb{E}\left[\sup_{0 \leq t \leq T_N} g(X_t)\right] = 0.9878$$

## Numerical results

$Pt$	$\hat{V}_0$	$\mathbb{E}_x[g(X_{\hat{\tau}})]$	$B_1$	$B_2$	$B_3$
10	0.7760	0.8173	0.1705	74.64	897.0
50	0.8298	0.8785	0.1093	43.36	511.5
100	0.8242	0.8785	0.1028	34.15	400.3
500	0.8432	0.8899	0.0989	21.03	243.1
900	0.8514	0.8968	0.0910	17.98	206.9

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$$B_1 = \mathbb{E}[\sup_{0 \leq t \leq T_N} g(X_t)] - \mathbb{E}_x[g(X_{\hat{\tau}})]$$

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## Numerical results

$B_2 =$  upper bound for  $|V_0 - \hat{V}_0|$

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## Numerical results

$B_3 =$  upper bound for  $|V_0 - \mathbb{E}_x[g(X_{\hat{\tau}})]|$

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