Asymptotic behavior of bifurcating autoregressive processes

B. Bercu B. de Saporta A. Gégout-Petit

Université de Bordeaux

Mathematical models for cell division IHP – 3 mars 2009

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Convergence

Outline

1 Introduction

- BAR model
- State of the art
- 2 Least square estimations
 - Generations filtration
 - Model and assumptions

3 Convergence

- Martingales
- Keystone result
- Laws of large numbers
- Central limit theorems

4 Further work

B. Bercu, B. de Saporta, A. Gégout-Petit

BAR model

Cell division

east square estimations

(1)

Convergence

Further work

◆□ > ◆□ > ◆三 > ◆三 > ・三 ・ つへぐ

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Introduction ••••

BAR model

Cell division

Further work

2 3

> < E Université de Bordeaux

< E.

A B >
 A B >
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A

B. Bercu, B. de Saporta, A. Gégout-Petit

BAR model

Cell division

east square estimations

Convergence

A B +
 A B +
 A

Further work



B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

3 1 4 3

Least square estimations

Convergence

A B +
 A B +
 A

• 3 >

Further work

BAR model

Cell division



B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

-

BAR model

Offspring



east square estimations

Convergence

Further work

Daughters of cell *n*: ■ 2*n* ■ 2*n* + 1

Quantitative characteristic of cell n (diameter, concentration of some molecule, ...)

 X_n

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Introduction 00000 BAR model Least square estimations

Convergence

Further work

Mathematical definition

Bifurcating auto regressive process BAR

$$\begin{cases} X_{2n} = a + b X_n + \varepsilon_{2n}, \\ X_{2n+1} = c + d X_n + \varepsilon_{2n+1}. \end{cases}$$

 X_1 initial cell

•
$$(\varepsilon_{2n}, \varepsilon_{2n+1})$$
 noise

Aim

Statistical inference on parameters a, b, c and d

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Convergence

State of the art

Stationary BAR

- Cowan, Staudte, Biometrics, 1986 : Introduction, biological motivation
- Huggins, Annals of Statistic, 1996 : MLE for large trees, order 1
- Huggins, Basawa, Journal of Applied Probability, 1999 and Australian Journal of Statistics, 2000 : MLE for large trees, order > 1
- Basawa, Zhou, Journal of Applied Probability, 2004 : BAR with exponentially distributed noise, and Journal of Time Series Analysis, 2005 : CLT for BAR

Main assumptions

iid noise, symmetry

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Image: A matrix

Further work

State of the art

Non stationary BAR

Guyon, Annals of Applied Probability, 2007 : Least-square estimation for gaussian BAR using Markov chains

Delmas, Marsalle, 2008 : previous talk!

Main assumptions

iid (gaussian) noise, no symmetry

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

State of the art

New results

east square estimations

Convergence

Further work

Our work

- noise: martingale difference
- rates of convergence
- martingales techniques

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Least square estimations

Convergence

Further work

Generations filtration

Generation



Generation 0 : $\mathbb{G}_0 = \{1\}$

Generation 1 : G₁ = {2,3}
Generation 2 : G₂ = {4,5,6,7}

Generation *n* :

$$\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$$

Tree up to generation *n*: $\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$

Size of generation n: $|\mathbb{G}_n| = 2^n$ Size of the tree up to generation n: $|\mathbb{T}_n| = 2^{n+1} - 1$

Image: A matrix

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Generations filtration

Generation



- Generation 0 : $\mathbb{G}_0 = \{1\}$
- Generation 1 : $\mathbb{G}_1 = \{2, 3\}$

Generation 2 : $\mathbb{G}_2 = \{4, 5, 6, 7\}$

Generation n :

 $\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$

Tree up to generation *n*: $\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$

Size of generation *n*: $|\mathbb{G}_n| = 2^n$ Size of the tree up to generation *n*: $|\mathbb{T}_n| = 2^{n+1} - 1$

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Generations filtration

Generation



- Generation 0 : $\mathbb{G}_0 = \{1\}$
- Generation 1 : $\mathbb{G}_1 = \{2,3\}$
- Generation 2 : $\mathbb{G}_2 = \{4, 5, 6, 7\}$

Generation n:

 $n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$

Tree up to generation *n*: $\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$

Size of generation n: $|\mathbb{G}_n| = 2^n$ Size of the tree up to generation n: $|\mathbb{T}_n| = 2^{n+1} - 1$

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Generations filtration

Generation



- Generation 0 : $\mathbb{G}_0 = \{1\}$
- Generation 1 : $\mathbb{G}_1 = \{2,3\}$
- Generation 2 : ℂ₂ = {4,5,6,7}
- Generation *n* :

 $\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$

Tree up to generation *n*: $\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$

Size of generation *n*: $|\mathbb{G}_n| = 2^n$ Size of the tree up to generation *n*: $|\mathbb{T}_n| = 2^{n+1} - 1$

Least square estimations

Convergence

Further work

Generations filtration

Generation



- Generation 0 : $\mathbb{G}_0 = \{1\}$
- Generation 1 : $\mathbb{G}_1 = \{2,3\}$
- Generation 2 : ℂ₂ = {4, 5, 6, 7}
- Generation *n* :

$$\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$$

Tree up to generation *n*: $\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$

Size of generation n: $|\mathbb{G}_n| = 2^n$ Size of the tree up to generation n: $|\mathbb{T}_n| = 2^{n+1} - 1$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Least square estimations

Convergence

Further work

Generations filtration

Generation



- Generation 0 : $\mathbb{G}_0 = \{1\}$
- Generation 1 : **G**₁ = {2,3}
- Generation 2 : ℂ₂ = {4, 5, 6, 7}
- Generation *n* :

$$\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$$

Tree up to generation *n*: $\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k$

Size of generation *n*: $|\mathbb{G}_n| = 2^n$ Size of the tree up to generation *n*: $|\mathbb{T}_n| = 2^{n+1} - 1$

Generations filtration

Filtration



Least square estimations

Convergence

Further work

Definition

$$\mathcal{F}_0 = \sigma\{X_1\}$$

Information grows exponentially fast : $2 \times$ more cells in each generation

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Generations filtration

Filtration



Least square estimations

Convergence

Further work

Definition

$$\mathcal{F}_1 = \sigma\{X_1, X_2, X_3\}$$

Information grows exponentially fast : $2 \times$ more cells in each generation

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Generations filtration

Filtration



Least square estimations

Convergence

Further work

Definition

$$\mathcal{F}_2 = \sigma\{X_k \text{ with } k \in \mathbb{T}_2\}$$

Information grows exponentially fast : $2 \times$ more cells in each generation

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Generations filtration

Filtration



Least square estimations

Convergence

Further work

Definition

$$\mathcal{F}_n = \sigma\{X_k \text{ with } k \in \mathbb{T}_n\}$$

Information grows exponentially fast : $2 \times$ more cells in each generation

< • • • • • • •

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Generations filtration

Filtration



Least square estimations

Convergence

Further work

Definition

$$\mathcal{F}_n = \sigma\{X_k \text{ with } k \in \mathbb{T}_n\}$$

Information grows exponentially fast : $2 \times$ more cells in each generation

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

Least square estimations

Convergence

Further work

Model and assumptions

Our model

BAR process

$$\begin{cases} X_{2n} = a + b X_n + \varepsilon_{2n}, \\ X_{2n+1} = c + d X_n + \varepsilon_{2n+1}. \end{cases}$$

Assumptions

■
$$\mathbb{E}[X_1^8] < \infty$$

■ $0 < \max(|b|, |d|) < 1$
■ $|a| + |c| \neq 0$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

-

< ≥ > <

Least square estimations

Convergence

Further work

Model and assumptions

Assumptions on the noise

(H.1) $\forall n > 0$ and $\forall k \in \mathbb{G}_{n+1}$, $\mathbb{E}[\varepsilon_{k}^{2}|\mathcal{F}_{n}] = \sigma^{2} > 0$ $\mathbb{E}[\varepsilon_k | \mathcal{F}_n] = \mathbf{0}$ and a.s. • if $[k/2] \neq [\ell/2]$, ε_k and ε_ℓ independent conditionally to \mathcal{F}_n • if $[k/2] = [\ell/2]$, then for some $\rho < \sigma^2$

$$\sup_{n\geq 0} \sup_{k\in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] < \infty \qquad \text{a.s.}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

Least square estimations

Convergence

Further work

Model and assumptions

Assumptions on the noise

(H.1)
$$\forall n \ge 0 \text{ and } \forall k \in \mathbb{G}_{n+1},$$

 $\mathbb{E}[\varepsilon_k | \mathcal{F}_n] = 0 \quad \text{and} \quad \mathbb{E}[\varepsilon_k^2 | \mathcal{F}_n] = \sigma^2 > 0 \quad \text{a.s.}$
(H.2) $\forall n \ge 0 \text{ and } \forall k \ne \ell \in \mathbb{G}_{n+1},$
• if $[k/2] \ne [\ell/2], \varepsilon_k \text{ and } \varepsilon_\ell \text{ independent conditionally to } \mathcal{F}_n$
• if $[k/2] = [\ell/2], \text{ then for some } \rho < \sigma^2$
 $\mathbb{E}[\varepsilon_k \varepsilon_\ell | \mathcal{F}_n] = \rho \quad \text{a.s.}$
(H.3)

$$\sup_{n\geq 0} \sup_{k\in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] < \infty \qquad \text{ a.s.}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

(4) (3) (4) (4) (4)

Least square estimations

Convergence

Further work

Model and assumptions

Assumptions on the noise

(H.1)
$$\forall n \ge 0 \text{ and } \forall k \in \mathbb{G}_{n+1},$$

 $\mathbb{E}[\varepsilon_k | \mathcal{F}_n] = 0 \quad \text{and} \quad \mathbb{E}[\varepsilon_k^2 | \mathcal{F}_n] = \sigma^2 > 0 \quad \text{a.s.}$
(H.2) $\forall n \ge 0 \text{ and } \forall k \ne \ell \in \mathbb{G}_{n+1},$
• if $[k/2] \ne [\ell/2], \varepsilon_k \text{ and } \varepsilon_\ell \text{ independent conditionally to } \mathcal{F}_n$
• if $[k/2] = [\ell/2], \text{ then for some } \rho < \sigma^2$
 $\mathbb{E}[\varepsilon_k \varepsilon_\ell | \mathcal{F}_n] = \rho \quad \text{a.s.}$
(H.3)
 $\sup_{n \ge 0} \sup_{k \in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] < \infty \quad \text{a.s.}$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

(4) (3) (4) (4) (4)

Least square estimations

Convergence

Further work

Model and assumptions

Least square estimators

Estimator of $\theta = (a, b, c, d)^t$

$$\widehat{\theta}_n = \begin{pmatrix} \widehat{a}_n \\ \widehat{b}_n \\ \widehat{c}_n \\ \widehat{d}_n \end{pmatrix} = (\mathbf{I}_2 \otimes S_{n-1}^{-1}) \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} X_{2k} \\ X_k X_{2k} \\ X_{2k+1} \\ X_k X_{2k+1} \end{pmatrix}$$

with

$$S_n = \sum_{k\in\mathbb{T}_n} \left(egin{array}{cc} 1 & X_k \ X_k & X_k^2 \end{array}
ight)$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

Least square estimations

Convergence

Further work

Model and assumptions

Variance/covariance estimators

Estimator of conditional variance

$$\widehat{\sigma}_n^2 = \frac{1}{2|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} (\widehat{\varepsilon}_{2k}^2 + \widehat{\varepsilon}_{2k+1}^2)$$

Estimator of conditional covariance

$$\widehat{\rho}_{n} = \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \widehat{\varepsilon}_{2k} \widehat{\varepsilon}_{2k+1}$$

with, for all $k \in \mathbb{G}_n$

$$\begin{cases} \widehat{\varepsilon}_{2k} = X_{2k} - \widehat{a}_n - \widehat{b}_n X_k \\ \widehat{\varepsilon}_{2k+1} = X_{2k+1} - \widehat{c}_n - \widehat{d}_n X_k \end{cases}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

- E - - E

Least square estimations

Convergence

Further work

Martingales

Martingales

Definition

 (M_n) sequence of square integrable random variables adapted to the filtration \mathcal{F} is a martingale if

 $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$

Examples:

$$\sum_{k\in\mathbb{T}_{n-1}}\varepsilon_{2k}, \quad \sum_{k\in\mathbb{T}_{n-1}}\varepsilon_{2k+1}, \quad \sum_{k\in\mathbb{T}_{n-1}}X_k\varepsilon_{2k}, \quad \sum_{k\in\mathbb{T}_{n-1}}X_k\varepsilon_{2k+1},$$
$$\sum_{k\in\mathbb{T}_n}(\varepsilon_k^2-\sigma^2), \quad \sum_{k\in\mathbb{T}_{n-1}}(\varepsilon_{2k}\varepsilon_{2k+1}-\rho), \dots$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

ヘロット (雪) (目) (

Least square estimations

Convergence

Further work

Martingales

Main martingale

Estimator and martingale

$$(\widehat{\theta}_n - \theta) = (\mathbf{I}_2 \otimes S_{n-1}^{-1}) \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} \varepsilon_{2k} \\ X_k \varepsilon_{2k} \\ \varepsilon_{2k+1} \\ X_k \varepsilon_{2k+1} \end{pmatrix} = \Sigma_{n-1}^{-1} M_n$$

with $\Sigma_n = I_2 \otimes S_n$ and $M_n \mathcal{F}$ -martingale

$$M_n = \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} \varepsilon_{2k} \\ X_k \varepsilon_{2k} \\ \varepsilon_{2k+1} \\ X_k \varepsilon_{2k+1} \end{pmatrix}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux

ヘロット (雪) (目) (

east square estimations

Convergence

Martingales

Martingale convergence results

 (M_n) scalar \mathcal{F} -martingale bounded in L^2 $\Delta M_{n+1} = M_{n+1} - M_n$ Increasing process $< M >_n = \sum_{k=0}^n \mathbb{E}[(\Delta M_{n+1})^2 | \mathcal{F}_n]$

Convergence of L^2 martingales

If $\lim_{n\to\infty} \langle M \rangle_n = +\infty$, then $\frac{M_n}{\langle M \rangle_n} \to 0$ a.s. + moment conditions then $\left(\frac{M_n}{\langle M \rangle_n}\right)^2 = \mathcal{O}(\frac{\log(\langle M \rangle_n)}{\langle M \rangle_n})$ a.s.

Similar results for vector-valued martingales. Here $\langle M \rangle_n = l_2 \otimes S_n$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

イロト イヨト イヨト イヨト

Least square estimations

Convergence

Further work

Keystone result

LLN for the noise

Laws of large numbers

If (H.1), (H.2) and (H.3) hold

$$\lim_{n \to +\infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} \varepsilon_k = 0 \quad \text{a.s.}$$
$$\lim_{n \to +\infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} \varepsilon_k^2 = \sigma^2 \quad \text{a.s.}$$
$$\lim_{n \to +\infty} \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \varepsilon_{2k} \varepsilon_{2k+1} = \rho \quad \text{a.s.}$$

Direct application of LLN for scalar martingales

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Keystone result

Asymptotic behavior of the increasing process

$$S_n = \sum_{k\in\mathbb{T}_n} \left(\begin{array}{cc} 1 & X_k \\ X_k & X_k^2 \end{array} \right)$$

Limit of S_n

If (H.1), (H.2) and (H.3) hold

$$\lim_{n\to\infty}\frac{S_n}{|\mathbb{T}_n|}=L \qquad \text{a.s.}$$

where *L* symmetric positive definite explicit matrix

n

LLN for the noise and recurrence equation defining the BAR process

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Least square estimations 000000 Convergence

Further work

Laws of large numbers

Law of large numbers for θ

Theorem

If (H.1), (H.2) and (H.3) hold $\hat{\theta}_n$ converges a.s. to θ with rate

$$\| \widehat{\theta}_n - \theta \|^2 = \mathcal{O}\left(\frac{\log |\mathbb{T}_{n-1}|}{|\mathbb{T}_{n-1}|}\right)$$
 a.s.

Quadratic strong law

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\mathbb{T}_{k-1}| (\widehat{\theta}_k - \theta)^t \Lambda(\widehat{\theta}_k - \theta) = 4\sigma^2 \qquad \text{a.s}$$

with
$$\Lambda = I_2 \otimes L$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Least square estimations

Convergence

Further work

Laws of large numbers

Vector-valued martingales

$$(\widehat{\theta}_n - \theta) = \sum_{n=1}^{-1} M_n$$

LLN for vector-valued martingales of the form $M_n = \sum_{k=1}^n \Psi_{k-1}\xi_k$, with fixed-sized (Ψ_k) and (ξ_k)

Problem

here the size of (Ψ_k) and (ξ_k) doubles at each generation

Solution

rewrite the LLN proof in this case

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

Least square estimations

Convergence

Further work

Laws of large numbers

Vector-valued martingales

$$(\widehat{\theta}_n - \theta) = \sum_{n=1}^{-1} M_n$$

LLN for vector-valued martingales of the form $M_n = \sum_{k=1}^n \Psi_{k-1}\xi_k$, with fixed-sized (Ψ_k) and (ξ_k)

Problem

here the size of (Ψ_k) and (ξ_k) doubles at each generation

Solution

rewrite the LLN proof in this case

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

ヘロット (雪) (目) (

-east square estimations 000000 Convergence

Further work

Laws of large numbers

LLN for conditional variance

$$\sigma_n^2 = \frac{1}{2|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} (\varepsilon_{2k}^2 + \varepsilon_{2k+1}^2)$$

Theorem

If (H.1), (H.2) and (H.3) hold, $\hat{\sigma}_n^2$ converges a.s. to σ^2

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k \in \mathbb{T}_{n-1}} (\widehat{\varepsilon}_{2k} - \varepsilon_{2k})^2 + (\widehat{\varepsilon}_{2k+1} - \varepsilon_{2k+1})^2 = 4\sigma^2 \quad \text{a.s.}$$
$$\lim_{n \to \infty} \frac{|\mathbb{T}_n|}{n} (\widehat{\sigma}_n^2 - \sigma_n^2) = 4\sigma^2 \quad \text{a.s.}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

east square estimations 000000 Convergence

Further work

Laws of large numbers

LLN for conditional covariance

$$\rho_n = \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \varepsilon_{2k} \varepsilon_{2k+1}$$

Theorem

If (H.1), (H.2) and (H.3) hold, $\hat{\rho}_n$ converges a.s. to ρ

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k\in\mathbb{T}_{n-1}}(\widehat{\varepsilon}_{2k}-\varepsilon_{2k})(\widehat{\varepsilon}_{2k+1}-\varepsilon_{2k+1}) = 2\rho \quad \text{a.s.}$$
$$\lim_{n\to\infty}\frac{|\mathbb{T}_n|}{n}(\widehat{\rho}_n-\rho_n) = 4\rho \quad \text{a.s.}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Least square estimations

Convergence

Further work

Central limit theorems

Additional assumptions

(H.4) $\forall n \ge 0 \text{ and } \forall k \in \mathbb{G}_{n+1},$ $\mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] = \tau^4 \quad \text{a.s.}$ and $\forall k \ne \ell \in \mathbb{G}_{n+1}$ with $[k/2] = [\ell/2]$ et pour $\nu^2 < \tau^4$ $\mathbb{E}[\varepsilon_k^2 \varepsilon_\ell^2 | \mathcal{F}_n] = \nu^2 \quad \text{a.s.}$ (H.5)

$$\sup_{n\geq 0} \sup_{k\in\mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^8 | \mathcal{F}_n] < \infty \qquad \text{a.s.}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Université de Bordeaux

CLT

Central limit theorems

ast square estimations

Convergence

Further work

Theorem

If (H.1) – (H.5) hold

$$\begin{split} \sqrt{|\mathbb{T}_{n-1}|} (\widehat{\theta}_n - \theta) & \stackrel{\mathcal{L}}{\longrightarrow} & \mathcal{N}(\mathbf{0}, \Gamma \otimes L^{-1}) \\ \sqrt{|\mathbb{T}_{n-1}|} (\widehat{\sigma}_n^2 - \sigma^2) & \stackrel{\mathcal{L}}{\longrightarrow} & \mathcal{N}\left(\mathbf{0}, \frac{\tau^4 - 2\sigma^4 + \nu^2}{2}\right) \\ \sqrt{|\mathbb{T}_{n-1}|} (\widehat{\rho}_n - \rho) & \stackrel{\mathcal{L}}{\longrightarrow} & \mathcal{N}(\mathbf{0}, \nu^2 - \rho^2) \end{split}$$

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Versité de Bordeaux

<ロ> <同> <同> < 同> < 同>

Least square estimations

Convergence

Further work

Central limit theorems

Sketch of the proof

Lindeberg conditions does not hold

- \blacksquare CLT for martingale differences not valid for filtration $\mathcal F$
- ⇒ New pair-wise filtration G CLT for martingale differences valid

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Central limit theorems

Sketch of the proof

- Lindeberg conditions does not hold
- \blacksquare CLT for martingale differences not valid for filtration $\mathcal F$
- New pair-wise filtration G
 CLT for martingale differences valid

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Central limit theorems

Sketch of the proof

- Lindeberg conditions does not hold
- **CLT** for martingale differences not valid for filtration \mathcal{F}
- New pair-wise filtration G CLT for martingale differences valid

Université de Bordeaux

B. Bercu, B. de Saporta, A. Gégout-Petit

Least square estimations

Convergence

Further work

Study BAR with random coefficients b and d

$$\begin{cases} X_{2n} = a + b X_n + \varepsilon_{2n}, \\ X_{2n+1} = c + d X_n + \varepsilon_{2n+1}. \end{cases}$$

Questions

Biological interpretation?

B. Bercu, B. de Saporta, A. Gégout-Petit

Asymptotic behavior of BAR

Least square estimations

Convergence

< 口 > < 🗗

Further work

The end



B. Bercu, B. de Saporta, A. Gégout-Petit

Université de Bordeaux