Tail of the stationary solution of the stochastic equation

\[ Y_{n+1} = a_n Y_n + b_n \]

with Markovian coefficients

Queue de la solution stationnaire de l’équation \( Y_{n+1} = a_n Y_n + b_n \)

à coefficients markoviens

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Abstract

In this paper, we deal with the real stochastic difference equation \( Y_{n+1} = a_n Y_n + b_n \), \( n \in \mathbb{Z} \), where the sequence \((a_n)\) is a finite state space Markov chain. By means of the renewal theory, we give a precise description of the situation where the tail of its stationary solution exhibits power law behavior. To cite this article: B. de Saporta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

Résumé

On étudie la queue de la solution stationnaire de l’équation \( Y_{n+1} = a_n Y_n + b_n \), \( n \in \mathbb{Z} \), où \((a_n)\) est une chaîne de Markov à espace d’états fini. Par des méthodes de renouvellement, on donne une caractérisation détaillée du cas où la queue est polynomiale. Pour citer cet article : B. de Saporta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

1. Introduction

We study the following stochastic difference equation:

\[ Y_{n+1} = a_n Y_n + b_n, \quad n \in \mathbb{Z}, \tag{1} \]

where \((a_n)\) is a real, finite state space Markov chain, and \((b_n)\) is a sequence of real i.i.d. random variables. Random Equations of this type have many applications in stochastic modeling and statistics. Most of previously studied cases deal with i.i.d. coefficients \((a_n)\) : see [6], [7], [9] and [3]. For more recent work, see also [8]. Here we study the Markovian case. In statistical literature, it is called a Markov-switching
auto-regression, see [5] for interesting applications in econometrics. Such stochastic recursions are also a basic tool in queuing theory, see [1].

\section{Main theorems}

Assume that \((a_n, b_n)\) is stationary and ergodic, and that we have:

\[ \mathbb{E} \log |a_0| < 0, \quad \mathbb{E} \log^+ |b_0| < \infty. \]  

Then it is proved in [2] that Eq. (1) has a unique stationary solution \((Y_n)\), where

\[ Y_n = \sum_{k=0}^{\infty} a_{n-k} a_{n-2-k} \cdots a_{n-k-l} b_{n-k-l}, \quad n \in \mathbb{Z}. \]

To deal with the tail of \(Y_t\), we investigate the asymptotic behavior of \(\mathbb{P}(xY_1 > t)\), when \(t\) tends to infinity, and where \(x \in \{-1, 1\}\). We give two theorems, depending on the \(a_n\) being positive or not.

\begin{theorem}
Let \((a_n)\) be an irreducible, aperiodic, stationary Markov chain, with state space \(E = \{e_1, \ldots, e_p\} \subset \mathbb{R}^+\), transition matrix \(P = (p_{ij})\) and stationary law \(\nu\). Let \((b_n)\) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence \((a_n)\). If the following conditions are satisfied:

\begin{itemize}
  \item there is a \(\lambda > 0\) so that the matrix \(P_{\lambda} = \text{diag}(e_{\lambda}) P^t\) has spectral radius 1 (where \(P^t\) denotes the transpose of \(P\)),
  \item the log \(e_i\) are not integral multiples of a same number,
  \item there is a \(\delta > 0\) such that \(\mathbb{E}|b_0|^{\lambda+\delta} < \infty\),
\end{itemize}

then we have for \(x \in \{-1, 1\}\)

\[ t^x \mathbb{P}(xY_1 > t) \xrightarrow{t \to \infty} L(x), \]

where \(L(1) + L(-1)\) is positive. If \(b_0 \geq 0\), then \(L(-1) = 0\), and \(L(1) > 0\). If \(b_0 \leq 0\), then \(L(1) = 0\), and \(L(-1) > 0\).

\end{theorem}

\begin{theorem}
Let \((a_n)\) be an irreducible, aperiodic, stationary Markov chain, with state space \(E = \{e_1, \ldots, e_p\} \subset \mathbb{R}^+\) such that \(\{e_1, \ldots, e_l\} \subset \mathbb{R}^+\) and \(\{e_{l+1}, \ldots, e_p\} \subset \mathbb{R}^-\) for a 0 \(\leq l \leq p - 1\), transition matrix \(P = (p_{ij})\) and stationary law \(\nu\). Let \((b_n)\) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence \((a_n)\). If the following conditions are satisfied:

\begin{itemize}
  \item there is a \(\lambda > 0\) so that the matrix \(P_{\lambda} = \text{diag}(|e_{\lambda}|) P^t\) has spectral radius 1,
  \item the log \(|e_i|\) are not integral multiples of a same number,
  \item there is a \(\delta > 0\) such that \(\mathbb{E}|b_0|^{\lambda+\delta} < \infty\),
\end{itemize}

then we have, for \(x \in \{-1, 1\}\)

\[ t^x \mathbb{P}(xY_1 > t) \xrightarrow{t \to \infty} L(x), \]

where \(L(1) + L(-1)\) is positive. If in addition \(P^t\) is \(t\)-irreducible (see definition below) then \(L(1) = L(-1) > 0\).

The last two hypotheses of these theorems are the same as in the i.i.d. case. In particular, the second one ascertains that the distribution of \(Y_1\) is non-lattice, and it is equivalent to requiring that the subgroup generated by the log \(e_i\) be dense in \(\mathbb{R}\). On the contrary, the first assumption comes from the Markovian dependence considered here. Indeed, we can prove that the spectral radius \(\rho(P_\lambda)\) can be computed from the formula \(\rho(P_\lambda) = \lim(\mathbb{E}|a_0| \cdots |a_{n-1}|)^{1/n}\). Therefore this assumption is a suitable substitute for the classical relation \(\mathbb{E}|a_0|^n = 1\) assumed in the i.i.d. case.
Note that the assumption of independence between the two sequences \((a_n)\) and \((b_n)\) can be avoided. Let \(F_n\) be the \(\sigma\)-field generated by \(a_0, \ldots, a_{n-1}\) and \(b_0, \ldots, b_{n-1}\). Then \((b_n)\) is only required to be a sequence of random variables such that \((a_n, b_n)\) be a stationary process, and \(b_{n-1}\) be independent of \(F_{n-1}\). We also need one more assumption, also assumed in the i.i.d. case: for all \(1 \leq i \leq p\), \(\mathbb{P}(b_0 + a_0 x = x \mid a_0 = e_i) < 1\).

The mapping \(\lambda \mapsto \log \rho(P_\lambda)\) being convex, its right-hand derivative in 0 being negative and as we have \(\rho(P_0) = \rho(P) = 1\), only two cases may occur.
- Either for all \(\lambda > 0\), \(\rho(P_\lambda) < 1\), in which case we can prove that \(\mathbb{E}|Y_1|^\lambda < \infty\) for all \(\lambda\), provided \(\mathbb{E}|b_0|^\lambda < \infty\), and therefore \(\mathbb{P}(|Y_1|^\lambda > t) = o(t^{-\lambda})\) for all \(\lambda\).
- Or there is a unique \(\lambda > 0\) so that \(\rho(P_\lambda) = 1\), this is the case we study here.

3. Sketch of the proof of Theorem 2.1

Similar theorems have already been proved in the i.i.d. multidimensional case: \(a_n\) are matrices and \(Y_n\) and \(b_n\) vectors. Renewal theory is used in [6] to prove a similar theorem when the \(a_n\) either have a density or are non-negative. Kesten’s results were extended in [9] to all i.i.d. random matrices satisfying similar assumptions as in our theorems. Finally in [3] a new specific implicit renewal theorem is proved and the same results as Kesten in the i.i.d. one-dimensional case are derived.

Here we follow the same steps as [9] and [3]. Our problem leads to a system of renewal equations of size \(p\), instead of a single renewal equation. We use a new renewal theorem given in [10] to get an asymptotic equivalent of \(\mathbb{P}(xY_1 > t)\), of the form \(L(x)t^{-\lambda}\). However the constants \(L(x)\) thus obtained are only non-negative.

The next step is to prove that \(L(1) + L(-1) > 0\). To do so, we extend the method given in [3] and [4]. First we prove the following lower bound:

\[
\mathbb{P}(|Y_1| > t) \geq C \mathbb{P}(\sup_n |a_0 \cdots a_{1-n}| > \frac{2t}{\varepsilon}),
\]

for a positive \(\varepsilon\) and a corresponding positive constant \(C\). And then we use a ladder height method, and again renewal theory to derive an accurate estimate of the right-hand side probability.

4. Sketch of the proof of Theorem 2.2

Now the sign of the products \(a_0 \cdots a_{1-n}\) is random. To be able to use the results of the positive case, we include this sign as a new dimension, and we derive a system of renewal equations of size \(2p\). Unfortunately, it is not necessarily irreducible, this is why we introduce a new definition.

**Definition 4.1** Let \(A = (a_{ij})\), \(i,j \leq p\), be a positive matrix, and \(1 \leq l \leq p - 1\) an integer. \(A\) is \(l\)-reducible if there is \((I, J)\) a non-trivial partition of \(\{1, \ldots, p\}\) such that:
- For all \(1 \leq i \leq l\), if \(i \in I\) then \(a_{ij} = 0 \ \forall j \in J\), if \(i \in J\) then \(a_{ij} = 0 \ \forall j \in I\).
- For all \(l + 1 \leq i \leq p\), if \(i \in I\) then \(a_{ij} = 0 \ \forall j \in I\), if \(i \in J\) then \(a_{ij} = 0 \ \forall j \in J\).

If \(A\) is not \(l\)-reducible, we say that \(A\) is \(l\)-irreducible.

If the matrix of our system is \(l\)-irreducible, then the proof runs the same as in the positive case, and in addition we know that both limits \(L(1)\) and \(L(-1)\) are equal, therefore they are both positive. If the
matrix is $l$-reducible, the system splits into two independent systems of size $p$, and for each of them the proof is the same as in the positive case. This time $L(1)$ and $L(-1)$ may be different.

References