Counting closed geodesics on noncompact hyperbolic surfaces

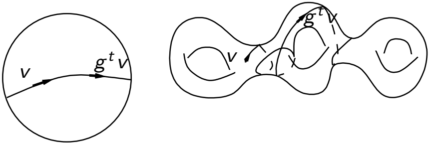
Big surfaces seminar

Online, 20 may 2021

Periodic geodesics on hyperbolic surfaces

S is a hyperbolic surface, T^1S its unit tangent bundle.

Each unit tangent vector $v \in T^1S$ determines a geodesic.



It defines the geodesic flow $(g^t)_{t\in\mathbb{R}}$ on T^1S.
(Oriented) closed geodesic of length T on S $<->$ periodic orbit of period T for (g^t) on T^1S.

$$\mathcal{P}(T) = \{\text{closed geodesics of length } \leq T\}$$

What can we say about $\mathcal{P}(T)$?

Some history

Hadamard (1898) : Infinitely many periodic orbits $\#\mathcal{P}(T) \to \infty$.
Morse (1920') : Periodic orbits are dense in T^1S
Further study :
Hopf, Hedlund (30'), Anosov, Sinai (50-60'),

Huber (59) S compact hyperbolic surface. Then

$$\#\mathcal{P}(T) \sim \frac{e^T}{T} \quad \text{when} \quad T \to \infty\,.$$

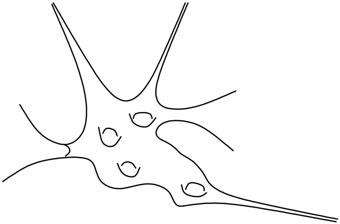
Margulis (1964) : Anosov flows.

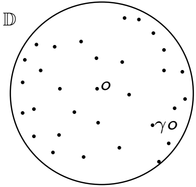
Bowen, Bowen-Ruelle (70'), Parry-Pollicott (90') : hyperbolic flows.

Noncompact manifolds - simple case

When the surface $S = \mathbb{D}/\Gamma$ is geometrically finite:
Margulis method, works of Sullivan (80'), Rudolph (90') give

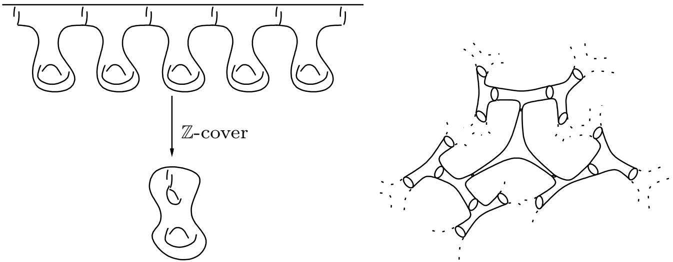
$$\#\mathcal{P}(T) \sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma T} \quad \text{when} \quad T \to \infty \,,$$





where δ_Γ is the exponential growth rate of the orbits of Γ.

Noncompact surfaces - Important observation



On a typical noncompact manifold, for T large enough, $\#\mathcal{P}(T) = +\infty$!!

Counting is done ...

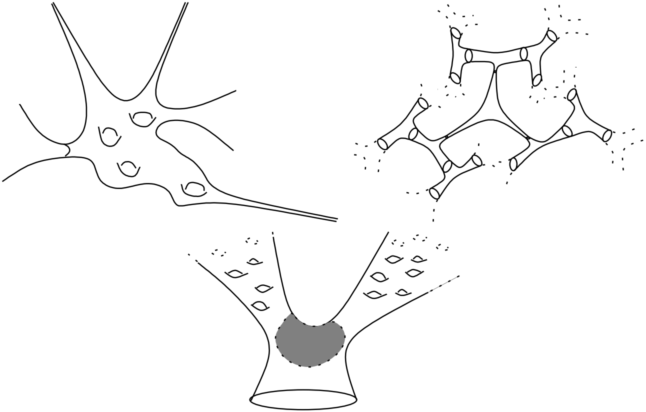
A natural normalization

Choose an arbitrary compact set K of the surface, whose interior intersects some closed geodesic. Let $\mathcal{P}_K(T)$ be the set of closed geodesics of length T that intersect K.

Theorem (Sch.-Tapie 2019) : Let S be a hyperbolic surface whose Bowen-Margulis-Sullivan measure is finite. Then

$$\#\mathcal{P}_K(T) \sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma T} .$$

Wide classes of examples



The strategy of the proof

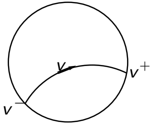
Classical strategy (hide some details due to noncompactness).
Relies on the product structure and mixing property of the Bowen-Margulis measure.
Babillot (01) : this measure is mixing as soon as it is finite.
Pit-Schapira (16), Schapira-Tapie (19) : this measure is finite as soon as the entropy at infinity is strictly smaller than the entropy :

$$\delta_\Gamma^\infty < \delta_\Gamma$$

Our goal now : explain what are the Bowen-Margulis measure, entropy, entropy at infinity, and give some examples where the strict inequality holds.

Sullivan's construction of the BM-measure

We consider a hyperbolic surface $S = \mathbb{D}/\Gamma$, with Γ a discrete group. The product structure $T^1\mathbb{D} \simeq S^1 \times S^1 \setminus \{Diag\} \times \mathbb{R}$ is a key tool.



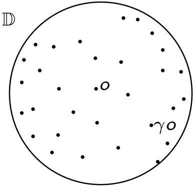
The Bowen-Margulis measure is constructed as a Γ-invariant measure on $T^1\mathbb{D}$ instead of a measure on $T^1S = T^1\mathbb{D}/\Gamma$.

In these coordinates, the measure is constructed as a product $m_{BMS} \sim \nu \times \nu \times dt$, where ν is obtained as a limit of orbital measures.

Sullivan's construction of the BM-measure (II)

The BM measure is constructed as a product $m_{BMS} \sim \nu \times \nu \times dt$, where ν is obtained as a limit of orbital measures :

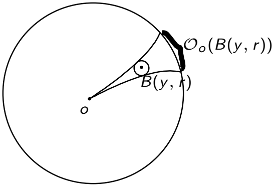
$$\nu = \lim_{s>\delta_\Gamma, s\to\delta_\Gamma} \frac{1}{P(s)} \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} \mathrm{Dirac}_{\gamma o} \, .$$



where δ_Γ is the exponential growth rate of the orbit Γo.

Construction of the BMS-measure (III)

Sullivan : The critical exponent δ_Γ coincides with the topological entropy of the geodesic flow.



Shadow lemma (Sullivan)

$$\nu(\mathcal{O}_o(B(\gamma o, 1)) \asymp e^{-\delta_\Gamma d(o,\gamma o)}$$

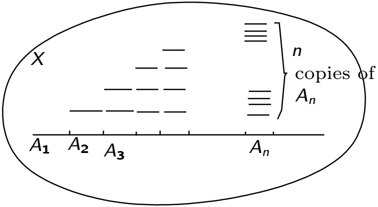
Bishop-Jones The critical exponent is also the Hausdorff dimension of (a subset of) the limit set $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o \subset S^1$.

Finiteness of the BMS-measure

For a general dynamical system $(X, \mathcal{B}, m, T)$, when the measure is invariant, ergodic and conservative, Kac lemma asserts that for a Borel set s.t. $0 < m(A) < \infty$

$$m(X) = \sum_{n \geq 1} n \times m(A_n),$$

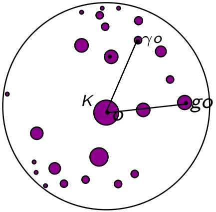
where A_n denotes the subset of A of points whose first return to A is after exactly n iterations of T.



Finiteness of the BMS-measure (bis)

Replace the Borel set A by a compact set with nonempty interior K. Define a subset Γ_K of Γ encoding the excursions outside K.

$$\Gamma_K = \{\gamma \in \Gamma,\ [o, \gamma o] \cap \Gamma K \subset K \cup \gamma K\}$$



Pit-Sch. (2016) m_{BMS} is finite iff it is ergodic and conservative and

$$\sum_{\gamma \in \Gamma_K} d(o, \gamma o) e^{-\delta_\Gamma d(o, \gamma o)} < \infty\,.$$

Finiteness of the BMS-measure (ter)

Sch. Tapie (2018) If the exp. growth rate of Γ_K satisfies $\delta(\Gamma_K) < \delta_\Gamma$, then m_{BMS} is finite.

It leads to several nontrivial examples where the BMS measure is finite : geometrically finite manifolds, Ancona surfaces, Schottky products...