Counting closed geodesics on noncompact hyperbolic surfaces II

Random Geometry seminar

Online, november 2020

Equidistribution of periodic orbits and counting

If $p \in \mathcal{P}$ is a periodic orbit, let μ_p be the Lebesgue measure on p.
Last week, we proved the following result :

Theorem : Let M be a compact neg. curved manifold. Then

$$\frac{hT}{e^{hT}} \sum_{p \in \mathcal{P}(T)} \frac{\mu_p}{\ell(p)} \to m_{BMS} \,.$$

Integrating the constant function equal to 1 led us to

Corollary : Let M be a compact negatively curved manifold. Then

$$\#\mathcal{P}(T) \sim \frac{e^{hT}}{hT} \,.$$

Ideas of the proof

The main ingredients in the proof were:

• Product structure and mixing

• Existence of a finite measure with product structure and mixing property, the Bowen-Margulis-Sullivan measure.

• The constant function 1 has compact support, so that it can be approximated by indicator functions of small boxes.

Our programm today :

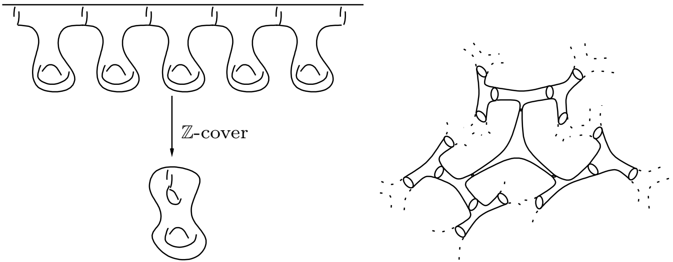
→ State the analogous result in the noncompact case.

→ Explain the construction of the Bowen-Margulis-Sullivan measure, and give a finiteness criterion.

→ Explain how to go from a convergence for continuous maps with compact support to continuous bounded maps. (The constant function equal to 1 is continuous and bounded!)

Noncompact manifolds

Important observation :



On a typical noncompact manifold, for T large enough, $\#\mathcal{P}(T) = +\infty$!!

Counting is done ...

A natural normalization

Choose an arbitrary compact set K of the surface, whose interior intersects some closed geodesic. Let $\mathcal{P}_K(T)$ be the set of closed geodesics of length T that intersect K.

Theorem (Sch.-Tapie 2019) : Let M be a neg. curved manifold, s.t. the Bowen-Margulis-Sullivan measure is finite. Then

$$\frac{hT}{e^{hT}} \sum_{p \in \mathcal{P}_K(T)} \frac{\mu_p}{\ell(p)} \to m_{BMS} ,$$

in the dual of bounded continuous functions.

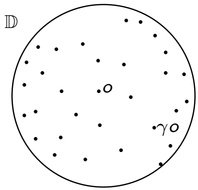
Corollary : Under the same assumptions, $\#\mathcal{P}_K(T) \sim \dfrac{e^{hT}}{hT}$.

Remark : Also true in the $CAT(-1)$-setting or with potentials and Gibbs measures in the Riemannian setting.

Sullivan's construction of the BMS-measure

A hyperbolic surface S is the quotient of $\mathbb{D}$ by a discrete group Γ. The Bowen-Margulis-Sullivan measure is constructed as a Γ-invariant measure on $T^1\mathbb{D}$ instead of a measure on T^1S. Recall the product structure $T^1\mathbb{D} \simeq S^1 \times S^1 \setminus \{Diag\} \times \mathbb{R}$. In these coordinates, $m_{BMS} \sim \nu \times \nu \times dt$, where ν is obtained as a limit of orbital measures

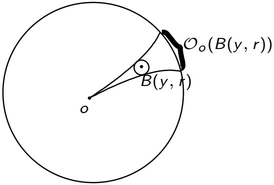
$$\nu = \lim_{s>\delta_\Gamma, s\to\delta_\Gamma} \frac{1}{P(s)} \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} \mathrm{Dirac}_{\gamma o} \,.$$



where δ_Γ is the exponential growth rate of the orbit Γo.

Construction of the BMS-measure (bis)

Theorem(Sullivan) The critical exponent δ_Γ coincides with the topological entropy of the geodesic flow.



Shadow lemma (Sullivan)

$$\nu(\mathcal{O}_o(B(\gamma o, 1)) \asymp e^{-\delta_\Gamma d(o,\gamma o)} = e^{-hd(o,\gamma o)}$$

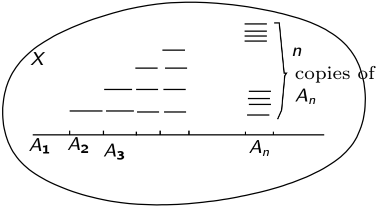
Equivalent to the local formulation seen last week :
$\mu_v^{ss}(A) = e^{ht}\mu_{g^t v}^{ss}(g^t A)$ and $\mu_v^{su}(A) = e^{-ht}\mu_{g^t v}^{su}(g^t A)$

Finiteness of the BMS-measure

For a general dynamical system $(X, \mathcal{B}, m, T)$, when the measure is invariant, ergodic and conservative, Kac lemma asserts that for a Borel set s.t. $0 < m(A) < \infty$

$$m(X) = \sum_{n \geq 1} n \times m(A_n)\,,$$

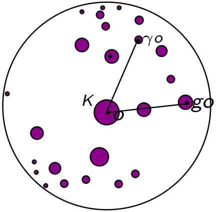
where A_n denotes the subset of A of points whose first return to A is after exactly n iterations of T.



Finiteness of the BMS-measure (bis)

Replace the Borel set A by a compact set with nonempty interior K. Define a subset Γ_K of Γ encoding the excursions outside K.

$$\Gamma_K = \{\gamma \in \Gamma,\, [o, \gamma o] \cap \Gamma K \subset K \cup \gamma K\}$$



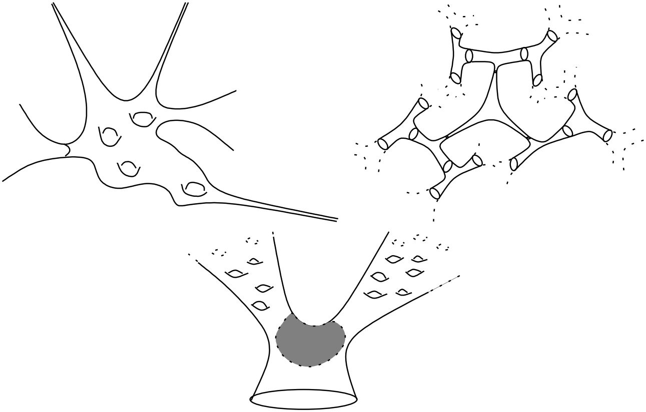
Pit-Sch. (2016) m_{BMS} is finite iff it is ergodic and conservative and

$$\sum_{\gamma \in \Gamma_K} d(o, \gamma o) e^{-\delta_\Gamma d(o, \gamma o)} < \infty\,.$$

Finiteness of the BMS-measure (ter)

Sch. Tapie (2018) If the exp. growth rate of Γ_K satisfies $\delta(\Gamma_K) < \delta_\Gamma$, then m_{BMS} is finite.

It leads to several nontrivial examples where the BMS measure is finite : geometrically finite manifolds, Ancona surfaces, Schottky products...



Equidistribution of periodic orbits

Last week's proof still works, but gives the following statement.

Theorem : Let M be a neg. curved manifold, s.t. the Bowen-Margulis-Sullivan measure is finite. Then

$$m_T := \frac{hT}{e^{hT}} \sum_{p \in \mathcal{P}(T)} \frac{\mu_p}{\ell(p)} \to m_{BMS} \,,$$

in the dual of continuous functions with compact support.

The proof of the main Theorem has two steps.

Step 1 : Show that $m_{T,K} := \frac{hT}{e^{hT}} \sum_{p \in \mathcal{P}_K(T)} \frac{\mu_p}{\ell(p)} \to m_{BMS}$ in the dual of continuous functions with compact support.

Step 2 : Show that the convergence holds in the dual of bounded continuous functions.

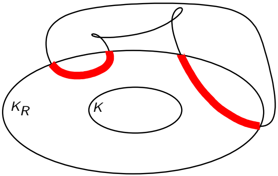
Step 1: Vague convergence of $m_{T,K}$ to m_{BMS}

We know that $m_T \to m_{BMS}$ and wish to prove that $m_{T,K} \to m_{BMS}$. We will prove that

$$m_{T,K} - m_T \to 0 \quad \text{in} \quad C_c(T^1 M)^*$$

Consider $\varphi \in C_c(T^1 M)$, $\mathrm{Supp}(\varphi) \subset K_R$. Thus,

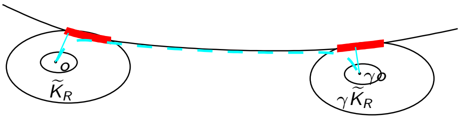
$$m_{T,K}(\varphi) - m_T(\varphi)) = m_{T,K}(\varphi) - m_{T,K_R}(\varphi) \leq \cdots$$



$$\cdots hTe^{-hT} \sum_{p \in \mathcal{P}_{K_R} \setminus \mathcal{P}_K(T-1,T)} \frac{\ell(p \cap K_R)}{\ell(p)} \simeq e^{-hT} \sum_{p \in \mathcal{P}_{K_R} \setminus \mathcal{P}_K(T-1,T)} \ell(p \cap K_R)$$

Step 1 : Vague convergence of $m_{T,K}$ to m_{BMS} (bis)

On the universal cover, lift these closed geodesics that intersect K_R and not K. For each piece of geodesic crossing $K_R \setminus K$, we get an element $\gamma \in \Gamma_{K_R}$.



Recall that $h = \delta_\Gamma$. We get a bound

$$e^{-hT} \sum_{p \in \mathcal{P}_{K_R} \setminus \mathcal{P}_K(T-1,T)} \ell(p \cap K_R) \leq \cdots$$

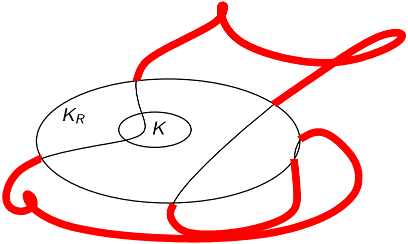
$$\cdots \leq \sum_{\gamma \in \Gamma_{K_R}, d(o,\gamma o) \simeq T} d(o,\gamma o) e^{-\delta_\Gamma d(o,\gamma o)} .$$

As m_{BMS} is finite, it is smaller than the reminder of a convergent series, so that it goes to 0 when $T \to \infty$.

Step 2 : Narrow convergence of $m_{T,K}$ to m_{BMS}

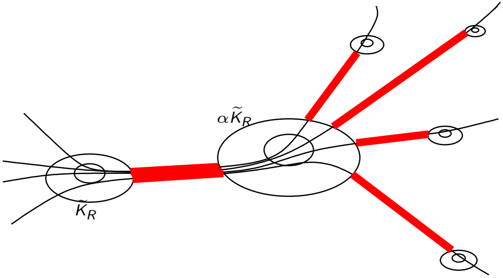
We know that $m_{T,K} \to m_{BMS}$ in $(C_c(T^1M))^*$. We will prove that the convergence holds in $(C_b(T^1M))^*$.
Enough to show nondivergence (tightness) : for all $\varepsilon > 0$, there exists a compact set K_R such that for all T large enough $m_{T,K}(K_R) \geq 1 - \varepsilon$.

$$m_{T,K}((K_R)^c) \leq e^{-hT} \sum_{p \in \mathcal{P}_K(T-1,T)} \ell(p \cap K_R^c)\,.$$



Step 2 : Narrow convergence of $m_{T,K}$ to m_{BMS}

As before, lift each geodesic in the above sum to the universal cover.



$$e^{-hT} \sum_{p\in\mathcal{P}_K(T-1,T)} \ell(p\cap K_R^c) \leq \cdots \leq \sum_{\alpha\in\Gamma_{K_R}, d(o,\alpha o)\geq 2R} d(o,\alpha o)e^{-\delta_\Gamma d(o,\alpha o)}$$

Counting closed geodesics on hyperbolic surfaces

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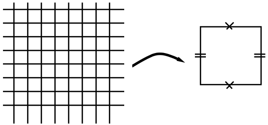
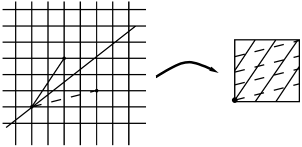
Counting geodesics on euclidean surfaces

The compact euclidean surface is the torus $\mathbb{R}^2/\mathbb{Z}^2$.



Its geodesics are the straight lines.



Observe that the slope of the geodesics remains constant.

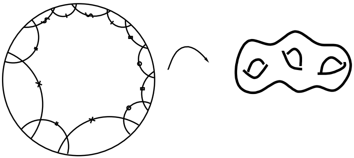
- Rational slope $\rightarrow$ closed geodesic
- Irrational slope $\rightarrow$ geodesic dense in the surface.

$$\mathcal{P}(T) = \{\text{closed geodesics of length} \leq T\}\,.$$

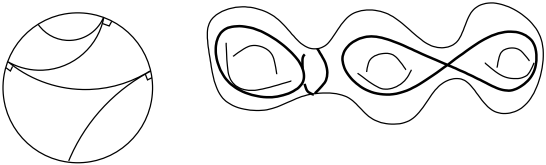
$$\#\mathcal{P}(T) \sim \pi T^2\,.$$

Hyperbolic surfaces

Build compact hyperbolic surfaces from the hyperbolic disk $\mathbb{D}$ as tori with many holes. Geodesics of $\mathbb{D}$ are circles and diameters.



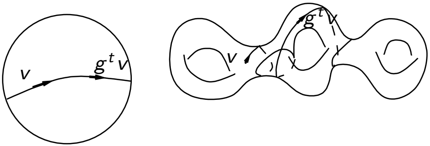
On a compact surface $S = \mathbb{D}/\Gamma$, geodesics have many different behaviours.



- closed geodesics
- geodesics dense in S and even in T^1S
- Any behaviour that you can imagine exists.

Geodesic flow

Each unit tangent vector $v \in T^1S$ determines a geodesic.



It defines the geodesic flow $(g^t)_{t\in\mathbb{R}}$ on T^1S.
Closed geodesic of length T on $S <-> $ periodic orbit of period T for (g^t) on T^1S.

$$\mathcal{P}(T) = \{\text{closed geodesics of length } \le T\}$$

What can we say about $\mathcal{P}(T)$?

Some history

Hadamard (1898) : Infinitely many periodic orbits $\#\mathcal{P}(T) \to \infty$.
Morse (1920') : Periodic orbits are dense in $T^1 S$
Further study : Hopf, Hedlund (30'), Anosov, Sinai (50-60'),
Huber (59) S compact hyperbolic surface. Then

$$\#\mathcal{P}(T) \sim \frac{e^T}{T} \quad \text{when} \quad T \to \infty\,.$$

Margulis (1964) : Anosov flows. Let M be a compact negatively curved manifold, and (g^t) its geodesic flow. Then

$$\#\mathcal{P}(T) \sim \frac{e^{hT}}{hT} \quad \text{when} \quad T \to \infty\,,$$

with h the topological entropy of the geodesic flow.
Bowen, Bowen-Ruelle (70') : Equidistribution of periodic orbits of hyperbolic flows.
Parry-Pollicott (1990) : Hyperbolic flows. Different method.
Since 2000 : Dolgopyat ... Tsuji-Zhang (2020). Exponential mixing of the geodesic flow → error term in the above estimates.

Margulis asymptotic

Our goal today : discuss the proof of

Margulis (1964) : Anosov flows. Let M be a compact negatively curved manifold, and (g^t) its geodesic flow. Then

$$\#\mathcal{P}(T) \sim \frac{e^{hT}}{hT} \quad \text{when} \quad T \to \infty\,,$$

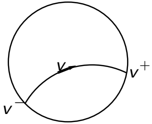
with h the topological entropy of the geodesic flow.

Occasion to discuss the following points,:
→ the product structure of the geodesic flow,
→ the mixing property.

Goal next week : discuss an extension of this result to geodesic flows of noncompact negatively curved manifolds (Schapira-Tapie 2019).

The product structure of the geodesic flow

(Global) product structure on $T^1\mathbb{D}$:



$$T^1\mathbb{D} \simeq S^1 \times S^1 \times \mathbb{R} \qquad \text{(Hopf coordinates)}\,.$$

Also true for general negatively curved manifolds.

Any asymptotic past $\xi \in S^1$ can be connected with any asymptotic future $\eta \in S^1$. Past and future are independent.

Consequence : Topological mixing on T^1M. For any open sets A, B, there exists $T > 0$ s.t. for all $t \geq T$,

$$g^t A \cap B \neq \emptyset\,.$$

Product structure of the geodesic flow (II)

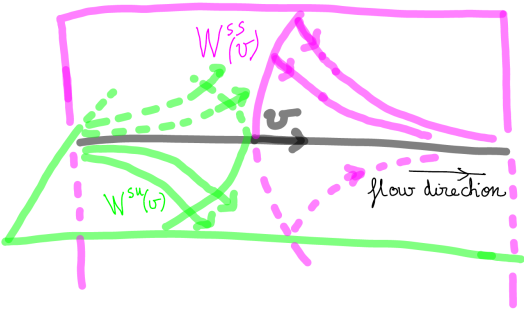
For hyperbolic flows, there exists a local product structure.

Strong stable manifold

$W^{ss}(v) = \{w \in T^1M, d(g^t v, g^t w) \to 0 \text{ when } t \to \infty\}$

Strong unstable manifold

$W^{su}(v) = \{w \in T^1M, d(g^t v, g^t w) \to 0 \text{ when } t \to -\infty\}$



The Bowen-Margulis-Sullivan measure

An invariant measure is a (probability) measure on T^1M s.t. $\mu(g^tA) = \mu(A)$ for all Borel sets A and $t \in \mathbb{R}$.

Bowen-Margulis-Sullivan measure : dynamically the most relevant invariant measure on T^1M.

Three different constructions :

$\rightarrow$ (Bowen) limit of periodic measures

$\rightarrow$ (Margulis) limit of averages on stable/ unstable manifolds

$\rightarrow$ (Sullivan) geometric construction through the boundary

In the Hopf coordinates on $T^1\mathbb{D}$, $m_{BMS} \sim \nu \times \nu \times dt$

In the local product structure on T^1M, $m_{BMS} \sim \mu_v^{ss} \times \mu_v^{su} \times dt$, and $v \rightarrow \mu_v^{ss}$, $v \rightarrow \mu_v^{su}$ vary continuously.

Moreover, $\mu_v^{ss}(A) = e^{ht}\mu_{g^tv}^{ss}(g^tA)$ and $\mu_v^{su}(A) = e^{-ht}\mu_{g^tv}^{su}(g^tA)$

Mixing of the Bowen-Margulis-Sullivan measure

The BMS measure is mixing: For any Borel sets $A, B \subset T^1M$,

$$m_{BMS}(g^{-t}B \cap A) \to m_{BMS}(A)m^{BMS}(B)\,.$$

Past and future become asymptotically independent.

Equivalent statement

$$\frac{m_{BMS}(g^{-t}B \cap A)}{m_{BMS}(B)} \to m_{BMS}(A)\,.$$

Equidistribution of periodic orbits

If $p \in \mathcal{P}$ is a periodic orbit, let μ_p be the Lebesgue measure on p. We will prove the following result.

Theorem : Let M be a compact neg. curved manifold. Then

$$\frac{hT}{e^{hT}} \sum_{p \in \mathcal{P}(T)} \frac{\mu_p}{\ell(p)} \to m_{BMS} \,.$$

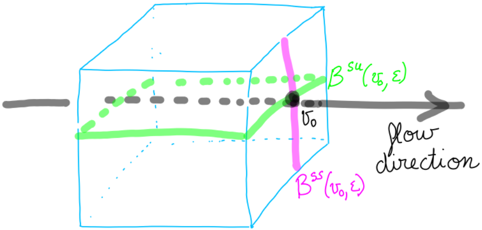
Integrating the constant function equal to 1 leads to

Corollary : Let M be a compact negatively curved manifold. Then

$$\#\mathcal{P}(T) \sim \frac{e^{hT}}{hT} \,.$$

Proof of the equidistribution theorem

We will use small boxes $B = \cup_{w \in B^{ss}(v_0,\varepsilon)} \cup_{|\tau|\leq\varepsilon} g^\tau B^{su}(w,\varepsilon)$.



Enough to prove that for any such small box B,

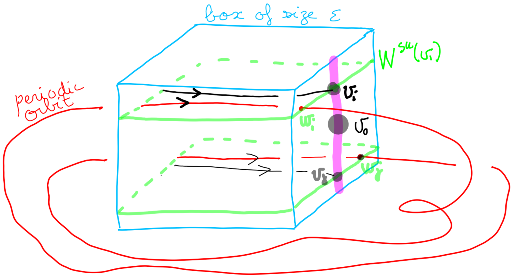
$$\frac{hT}{e^{hT}} \sum_{p\in\mathcal{P}(T)} \frac{\ell(p\cap B)}{\ell(p)} \to m_{BMS}(B)\,.$$

Replacing $\mathcal{P}(T)$ by $\mathcal{P}(T-\varepsilon, T) = \mathcal{P}(T)\setminus\mathcal{P}(T-\varepsilon)$, it is equivalent to show that

$$e^{-hT} \sum_{p\in\mathcal{P}(T-\varepsilon,T)} \ell(p\cap B) \to \varepsilon m_{BMS}(B)\,.$$

Proof of the equidistribution theorem

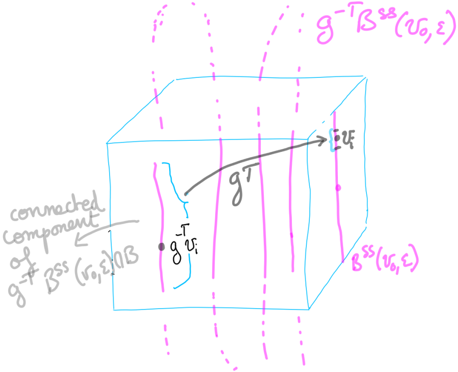
$e^{-hT}\sum_{p\in\mathcal{P}(T-\varepsilon,T)}\ell(p\cap B)$ equals $\varepsilon e^{-hT}\times$ nber of crossings of B by a periodic orbit.



Each piece (of size ε) of a periodic orbit of length $\simeq T$ which crosses B leads to a vector $v_i \in B^{ss}(v_0,\varepsilon)$, and a periodic vector $w_i \in B^{su}(v_i,\varepsilon)$, so that $g^{-T}v_i \simeq g^{-T}w_i = w_i$, thus to a piece of $g^{-T}B^{ss}(v_0,\varepsilon)$ which comes back inside B.

Proof of the equidistribution theorem

Conversely (fixed point argument), each connected component of $g^{-T}B^{ss}(v_0,\varepsilon)\cap B$ leads to a periodic orbit of length T crossing B.
Therefore, the quantity to estimate is roughly
$\varepsilon e^{-hT}\times$ nber of connected components of $g^{-T}B^{ss}[v_0,\varepsilon)\cap B$.



It is also roughly $\varepsilon e^{-hT}\dfrac{\mu^{ss}(g^{-T}B^{ss}(v_0,\varepsilon))\cap B}{\mu^{ss}(B^{ss}(v_0,\varepsilon))}$

Proof of the equidistribution theorem

Recall that m_{BMS} is mixing :

$$m_{BMS}(g^{-t}A \cap B) \to m_{BMS}(A) \times m_{BMS}(B)\,.$$

The product structure of $m_{BMS} \sim \mu^{ss} \times \mu^{su} \times dt$ leads easily to another equivalent formulation of mixing of m^{BMS} :

$$\mu^{ss}(g^{-T}B^{ss}(v_0,\varepsilon) \cap B) \sim e^{hT}\mu^{ss}(B^{ss}(v_0,\varepsilon)) \times m_{BMS}(B)\,.$$

Thus,

$$e^{-hT}\frac{\mu^{ss}(g^{-T}B^{ss}(v_0,\varepsilon) \cap B)}{\mu^{ss}(B^{ss}(v_0,\varepsilon))} \to m_{BMS}(B)\,.$$

It concludes the proof.

Conclusion

Two main ingredients in the proof:
• Product structure and mixing
• Existence of a finite measure with product structure and mixing property

lead to

→ Equidistribution of periodic measures towards m_{BMS}
→ Counting of periodic orbits

Remark : The argument of equidistribution of periodic measures is proven on indicator functions of boxes. An approximation argument allows to get convergence for integrals of continuous compactly supported functions. And the constant function equal to 1 is compactly supported only when M is compact.