Critical exponents and invariant measures for Kleinian groups

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Abstract

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1 Introduction

The goal of these three lectures will be to introduce you to critical exponents of Kleinian groups, with a special focus on their relation with a good invariant measure under the geodesic flow, the so-called *Bowen-Margulis measure*.

I will begin with a long introduction about discrete groups, critical exponents and geodesic flows. In a second part, I will define precisely the measures that will interest us, and the relations with critical exponents. In a third part, I will present (without proof) many classical or at least known results about critical exponents and the Bowen-Margulis measure. At last, I will provide proofs for a few important results, stating at which condition the Bowen-Margulis measure is ergodic, all classical notions of entropy coincide, and the Bowen-Margulis measure is the measure of maximal entropy of the geodesic flow.

1.1 Discrete groups

We denote by $\partial\mathbb{H}^n = S^{n-1}$ the visual boundary of $\mathbb{H}^n$. Recall that the isometries of $\mathbb{H}^n$ (except the identity) are of one of the three following types.

An isometry γ is loxodromic if it has two distinct fixed points $\gamma^\pm$ in $\partial\mathbb{H}^n$. In this case, one can find two disjoint half-balls $D^\pm$ centered respectively in γ^+ and γ^-, such that $\gamma^+(\mathbb{H}^n \setminus D^-) \subset D^+$ and $\gamma^-(\mathbb{H}^n \setminus D^+) \subset D^-$. Moreover, γ acts by translation along its axis (γ^-, γ^+).

An isometry p is parabolic if it fixes exactly one point p^+ in $\partial\mathbb{H}^n$. In this case, it stabilizes all *horospheres*, i.e. all spheres tangent to the boundary at the point p^+.

An isometry is elliptic in the other cases, where it fixes at least one point inside $\mathbb{H}^n$.

We consider a discrete subgroup Γ of the direct isometries of $\mathbb{H}^n$. Contrarily to the most known cases, we focus on the cases where $\Gamma\backslash\mathbb{H}^n$ is noncompact, or even Γ is infinitely generated. We also allow elliptic elements, so that $\Gamma\backslash\mathbb{H}^n$ is an orbifold rather than a manifold, but we will skip these technical details in the arguments that we will sketch.

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1.1.1 Examples of discrete groups

Let us recall the most classical examples to have in mind.

When Γ is the fundamental group of a compact hyperbolic manifold, one says that Γ is *cocompact*. When $n = 2$, it is easy to construct hyperbolic surfaces, by considering a $4g$-gone with right angles, whose sides are pairwise identified thanks to a hyperbolic isometry, to get a compact surface of genus g. **PICTURE**.

When $\Gamma\backslash\mathbb{H}^n$ has finite volume, one says that Γ is a *lattice*. Typical examples include cocompact groups and groups constructed by arithmetic ways, as the simplest $PSL(2,\mathbb{Z})$ or $SO(n,1,\mathbb{Z})$. **PICTURE fundamental domain $PSL(2,\mathbb{Z})$.**

Schottky groups are a very interesting class from a dynamical point of view, because they are free. Consider a finite (even) number of disjoint half-balls $D_i^{\pm}$ in $\mathbb{H}^n$ centeed in $\partial\mathbb{H}^n$, and isometries γ_i whose fixed points are the centers of $D_i^{\pm}$, sending $\mathbb{H}^n \setminus D_i^-$ inside D_i^+ and $\mathbb{H}^n \setminus D_i^+$ inside D_i^-.
Picture puntured torus and pair of pants
A classical argument, the Ping Pong argument, consisting to follow the orbit of a point $o \in \mathbb{H}^n \setminus \cup D_i^{\pm}$, allows to show that the group generated by the γ_i is discrete and free.

The examples of *Fuchsian and quasi-fuchsian groups* will be discussed in detail in the lectures of Ursula Hamenstadt. Let us recall briefly how it works. Recall that the group of direct isometries of $\mathbb{H}^2$ (resp. $\mathbb{H}^3$) can be identified with $PSL(2,\mathbb{R})$ (resp. $PSL(2,\mathbb{C})$). Choosing a totally geodesic hyperplane $\mathbb{H}^2$ inside $\mathbb{H}^3$ allows to embed $PSL(2,\mathbb{R})$ inside $PSL(2,\mathbb{C})$ as the group of isometries of $\mathbb{H}^3$ preserving this hyperplane. If $\Gamma \subset PSL(2,\mathbb{R})$ is a discrete subgroup, such an embedding $\Phi : PSL(2,\mathbb{R}) \to PSL(2,\mathbb{C})$ allows to see Γ, or more precisely $\Phi(\Gamma)$, as a discrete subgroup of $PSL(2,\mathbb{C})$. The resulting discrete group is called a *Fuchsian group.* (A variant of the definition requires Γ to be a lattice in $PSL(2,\mathbb{R})$.
PICTURE
It is possible to deform Γ inside $PSL(2,\mathbb{C})$ through a Γ-equivariant quasi-conformal homeomorphism. Such a deformation of a fuchsian group is called a *quasi-fuchsian group.* **PICTURE**

Regular covers of compact manifolds are the easiest examples of manifolds with infinitely generated fundamental group.
PICTURE $\mathbb{Z}$-cover
Consider a compact hyperbolic surface $S_0 = \Gamma_0\backslash\mathbb{H}^2$, and a normal subgroup $\Gamma \lhd \Gamma_0$. The surface $S = \Gamma\backslash\mathbb{H}$ has a fundamental group Γ which is in general infinitely generated.
PICTURE tree-surface

1.1.2 Limit set

As Γ is a discrete group of isometries of $\mathbb{H}^n$, it acts properly discontinuously on $\mathbb{H}^n$ (i.e. for all compact sets $K \subset \mathbb{H}^n$, we have $\#\{\gamma \in \Gamma, \gamma K \cap K \neq \emptyset\} < \infty$), so that it has to accumulate somewhere in $\partial\mathbb{H}^n$. The limit set is the set $\Lambda(\Gamma) = \overline{\Gamma.x} \setminus \Gamma.x$ in $\partial\mathbb{H}^n$. It does not depend on x, and is also the smallest non-empty Γ-invariant set, as soon as it is infinite.

In the sequel, we will always assume that Γ is *nonelementary*, that is that Λ_Γ is infinite.

As an exercise, you can check that as soon as Γ contains two isometries that are hyperbolic or parabolic with disjoint sets of fixed points, then Γ is nonelementary. When Γ is an elementary discrete group and contains no elliptic ellements, then Γ is either generated by a single loxodromic isometry or a discrete group of parabolic isometries fixing a common point at infinity.

The *radial limit set* (or conical limit set) $\Lambda_{rad}(\Gamma)$ is the set of points $\xi \in \Lambda_\Gamma$ such that there exist infinitely many points $\gamma_i.o$ at bounded distance of the geodesic ray $[o\xi)$. **PICTURE**
Observe that $\Lambda_{rad}(\Gamma) = \cup_{N\in\mathbb{R}}\Lambda_{rad}^N(\Gamma)$, where $\Lambda_{rad}^N(\Gamma)$ is the set of points $\xi \in \Lambda_\Gamma$ such that there exist infinitely many points $\gamma_i.o$ at distance at most N of the geodesic ray $[o\xi)$.

A point $\xi \in \Lambda_\Gamma$ is said *parabolic* if there exists $p \in \Gamma$ parabolic such that $p\xi = \xi$. A point $\xi \in \Lambda_\Gamma$ is said to be a *bounded parabolic point* if the stabilizer Π_ξ of ξ in Γ acts cocompactly on $\Lambda_\Gamma \setminus \{\xi\}$. A group Γ is said to be *geometrically finite* when its limit set Λ_Γ consists only of radial limit points and bounded parabolic limit points. There are several characterizations of *geometrical finiteness*. We refer to [Bow] for details.

A geometrically finite group is finitely generated. In dimension 2, it is even equivalent. However, there are examples of finitely generated geometrically infinite groups. The most simple examples in dimension 3 are obtained by taking the suspension of a compact hyperbolic surface by a pseudo-Anosov diffeomorphism, see for example [O].

When the group Γ is cocompact, i.e. when $\Gamma\backslash\mathbb{H}^n$ is compact, it is easy to check that $\Lambda_\Gamma = \Lambda_{rad} = \partial\mathbb{H}^n$. When the group Γ is a lattice, i.e. when $\Gamma\backslash\mathbb{H}^n$ has finite volume, one can also show that $\Lambda_\Gamma = \partial\mathbb{H}^n$, but $\Lambda_\Gamma = \Lambda_{rad} \sqcup \Lambda_p$ and the latter is nonempty.

One says that Γ is *convex-cocompact* when the convex hull of Λ_Γ inside $\mathbb{H}^n$ has a cocompact neighbourhood (modulo Γ).

1.2 Geodesic flow

The geodesic flow $(g^t)_{t\in\mathbb{R}}$ acts on the unit tangent bundle of $\mathbb{H}^n$ as follows. Let $v \in T^1S$ be a vector, and denote by $(c_v(t))_{t\in\mathbb{R}}$ the unique geodesic such that $v = c_v'(0)$. Then $g^t(v) = c_v'(t)$. This geodesic flow commutes with the action of the isometries of $\mathbb{H}^n$ so that it is well defined on the unit tangent bundle T^1M of any quotient manifold $\Gamma\backslash\mathbb{H}^n$.

This flow is one of the most important dynamical systems since the birth of dynamical systems (Hadamard 1898, Hopf and Hedlund in the thirties, Anosov, Sinai to cite the Abel prize, ...) It is a hyperbolic flow. When the manifold is compact, it is the typical example of an Anosov flow. Its stochastic properties (transitivity, ergodicity, mixing, ...) come mainly from a property called the *local product structure*. **PICTURE**
If v and w are two vectors close one from another, one can glue the past of v and the future of w, meaning that there exists a vector u close from v and w, such that the geodesic (g^tu) is asymptotic to (g^tw) when $t \to +\infty$ and to (g^tv) when $t \to -\infty$. In other words, the future of a trajectory does not depend on the past of this trajectory, which means that the system is very chaotic. This property allows to prove ergodicity, mixing, ...

The *boundary at infinity* $\partial\mathbb{H}^n$ is defined as the set of equivalence classes of geodesic rays under the equivalence relation *staying at bounded distance*. The boundary at infinity of the hyperbolic space $\partial\mathbb{H}^n$ coincides with the visual boundary S^{n-1}.

A Busemann function is a map defined on $\partial\mathbb{H}^n \times \mathbb{H}^n \times \mathbb{H}^n$ by

$$\beta_\xi(x,y) = \lim_{t\to+\infty} d(x,\xi_x(t)) - d(y,\xi_x(t)) = \text{"}d(x,\xi) - d(y,\xi)\text{"}\,,$$

where $\xi_x(t)$ is the geodesic ray from x to ξ.

The so-called *Hopf coordinates* are given by the following homeomorphism from $T^1\mathbb{H}^n$ to $\partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \{Diagonal\} \times \mathbb{R}$.

$$v \in T^1\mathbb{H}^n \mapsto (v^-, v^+, \beta_{v^+}(o, \pi(v)))\,.$$

We shorten the notation by writing $\partial^2\mathbb{H}^n = \partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \{Diagonal\}$. These coordinates are very convenient in order to use the product structure describe above. They have good properties. First, the geodesic flow acts by translation on the third factor, by

$$g^t(v^-, v^+, s) = (v^-, v^+, s+t)\,.$$

In addition, the action by isometries of an isometry γ of $\mathbb{H}^n$ can be written as :

$$\gamma(v^-, v^+, s) = (\gamma.v^-, \gamma.v^+, s + \beta_{v^+}(\gamma^{-1}.o, o))\,.$$

Moreover, they allow to enlight the product structure of the dynamics (see above), which is at the origine of the chaos. In other words, one sees and uses easily thanks to these coordinates the fact that the past is independent of the future, or that the phase space is (homeomorphic to) a product space.

The dynamics of the geodesic flow is interesting on the unit tangent bundle T^1M of a quotient manifold $M = \Gamma\backslash\mathbb{H}^n$. The point is that the Hopf coordinates descend to T^1M in the sense that T^1M is homeomorphic to the quotient $\Gamma\backslash(\partial^2\mathbb{H}^n \times \mathbb{R})$.

A classical result due to Eberlein [?] says that the *nonwandering set of the geodesic flow* on T^1M coincides in these coordinates with the set of vectors pointing towards and backwards to the limit set. Recall first that a vector $v \in T^1M$ is nonwandering if all neighbourhoods V of v return infinitely many times near v, in the sense that $\int_0^\infty \mathbf{1}_{V\cap g^tV\neq\emptyset}\,dt = +\infty$.

Let us denote by Ω the nonwandering set of the geodesic flow on T^1M. Eberlein proved that $\Omega = \{v \in T^1M\,,\ v^\pm \in \Lambda_\Gamma\}$.

When M is compact, $\Lambda_\Gamma = \partial\mathbb{H}^n$, and $\Omega = T^1M$ is compact. When M is convex-cocompact but not compact, then Ω is a proper compact invariant subset of T^1M. In other cases, Ω is noncompact, which makes the dynamical study more difficult. When M has finite volume, then $\Omega = T^1M$ has finite volume. More generally, when Γ is of the first kind, i.e. $\Lambda_\Gamma = \partial\mathbb{H}^n$, then $\Omega = T^1M$.

Observe that a Radon measure m on T^1M invariant under the geodesic flow lifts to a Radon measure $\tilde{m}$ on $T^1\mathbb{H}^n$ which is invariant under both actions of the geodesic flow and the group Γ. Recall that $T^1\mathbb{H}^n \simeq \partial^2\mathbb{H}^n \times \mathbb{R}$, and the geodesic flow acts by translation on $\mathbb{R}$. Therefore, the measure $\tilde{m}$ induces a *geodesic current* μ on $\partial\mathbb{H}^n$, that is a Γ-invariant Radon measure on $\partial^2\mathbb{H}^n$. Moreover, the correspondance between m and its associated geodesic current is $1-1$ and onto.

This correspondance in mind, it is natural to try to construct Γ-invariant measures on $\partial\mathbb{H}^n$, consider their product on $\partial^2\mathbb{H}^n$, and build in this way geodesic currents, and therefore invariant measures under the geodesic flow on T^1M. Unfortunately, a classical exercise shows that, as soon as Γ is nonelementary, it has no nontrivial invariant Radon measure on $\partial\mathbb{H}^n$. However, this strategy is not bad, as we will see later. The Patterson-Sullivan construction of the measure of maximal entropy of the geodesic flow uses the above idea in a refined way. They construct a quasi-invariant measure on the limit set, consider its product with itself, which turns out to be almost a geodesic current, and modify it in such a way that they obtain a geodesic current.

1.3 General Picture

The general philosophy is that there is a deep relation between the action of the group Γ on its limit set and the action of the geodesic flow (g^t) on T^1M. The *critical exponent* is at the same time the exponential growth rate of the group, the Hausdorff dimension of the (radial) limit set, and the topological entropy of the geodesic flow.

The Hausdorff measure of the limit set allows to build the measure of maximal entropy of the geodesic flow.

Many such properties are true for wider classes of groups than cocompact groups or lattices.

2 The construction of the Bowen-Margulis measure

2.1 Critical exponents

2.1.1 Poincaré series

Let Γ be a Kleinian group, i.e. a discrete group of isometries of $\mathbb{H}^n$. Assume that Γ is infinite.

Choose the point $o = (0, \dots, 0, 1)$ as an origin in $\mathbb{H}^n$. The *Poincaré series* $P(s)$ is defined for $s \in \mathbb{R}$ by

$$P(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma.o)} .$$

More generally, define

$$P(x, s) = \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma.o)} .$$

As the γ's are isometries, all these series diverge or converge simultaneously. Let $\delta_\Gamma \in \mathbb{R}_+ \cup \{+\infty\}$ be their *critical exponent*. The series diverge for $s < \delta_\Gamma$ and converge for $s > \delta_\Gamma$. One does not know a priori what happens when $s = \delta_\Gamma$. It turns out to be of the greatest importance. The group Γ is said to be *divergent* if $P(\delta_\Gamma) = +\infty$ and *convergent* otherwise.

If $a_n := \#\{\gamma \in \Gamma, n \le d(o, \gamma.o) < n + 1\}$, observe that the series $P(s)$ behaves exactly as the series $\sum_{n\in\mathbb{N}} a_n e^{-sn}$, so that

$$\delta_\Gamma = \limsup_{n\to+\infty} \frac{1}{n} \log \#\{\gamma \in \Gamma, n \le d(o, \gamma.o) < n + 1\} .$$

2.1.2 Computations

First, observe that if $\Gamma' < \Gamma$ is a subgroup of Γ, then $\delta_{\Gamma'} \le \delta_\Gamma$.

Let us give some examples. If $\Gamma =< h >$ is generated by a single hyperbolic isometry, then $d(o, h^n.o) \sim |n|d(o, h.o)$ so that $\delta_{<h>} = 0$.

If $\Gamma =< p >$ is generated by a single parabolic isometry, then $d(o, p^n.o) \sim 2 \log n$ so that $\delta_{<p>} = \frac{1}{2}$.

More generally, if Π is a parabolic group generated by k independent parabolic isometries, i.e. Π is isomorphic to $\mathbb{Z}^k$, then $\delta_\Pi = \frac{k}{2}$.

If Γ contains a free subgroup generated by two hyperbolic isometries g and h, then $\delta_\Gamma \ge \delta_{<g,h>} > 0$. Indeed, if an element $\gamma \in< g, h >$ has a word length $|n|$ in g and h, then $d(o, \gamma.o) \le |n| \max\{d(o, g.o), d(o, h.o)\}$ so that $\delta_{<g,h>} \ge \frac{\log 3}{\max\{d(o,g.o),d(o,h.o)\}}$.

Proposition 2.1 *When Γ is a lattice in $\mathbb{H}^n$, then $\delta_\Gamma = n-1$. In general, $\delta_\Gamma \le n-1$.*

Proof : Let us prove it only in the case where $\Gamma\backslash\mathbb{H}^n$ is compact. As Γ is discrete, let $r = \frac{1}{2}\inf_{\gamma\in\Gamma} d(o,\gamma.o) > 0$. Let D be the diameter of $M = \Gamma\backslash\mathbb{H}^n$. For all $R > 0$, we have

$$\cup_{\gamma\in\Gamma,\gamma.o\in B(o,R)} B(\gamma.o, r) \subset B(o, R+r) \leq \cup_{\gamma\in\gamma,\gamma.o\in B(o,R+r)} B(\gamma.o, D)\,.$$

Observe that the left inclusion is true without any assumption on the group Γ (other than being discrete).

We deduce from these inclusions, and from the geometric fact that $\mathrm{vol}(B(o,R)) \sim c.e^{(n-1).R}$ that $n-1 \leq \delta_\Gamma \leq n-1$, the first inequality being true without any compactness assumption on Γ.

We admit the result when Γ is a noncocompact lattice. **COMPLETER REFERENCE?** $\square$

2.1.3 Properties of critical exponents

Notice also that when $\delta_\Gamma > 0$, then

$$\delta_\Gamma = \limsup_{n\to+\infty} \frac{1}{n} \log \#\{\gamma\in\Gamma,\, d(o,\gamma.o) \leq n\}\,.$$

For cultural purposes, let us add some complements on the critical exponents.

Theorem 2.2 (Roblin) *When Γ is nonelementary, then*

$$\delta_\Gamma = \lim_{n\to+\infty} \frac{1}{n} \log \#\{\gamma\in\Gamma,\, n \leq d(o,\gamma.o) < n+1\}\,.$$

This was proven by Roblin [**?**] using conformal densities (i.e. measures) on the limit set, and later by Peigné [**?**] by subadditivity arguments.

Moreover, the critical exponent is related to the first eigenvalue of the Laplacian, i.e. the Hodge de Rahm operator;

Theorem 2.3 (Sullivan) *When $\delta_\Gamma > \frac{n-1}{2}$, then the first eigenvalue of the Laplacian satisfies $\lambda_0 = \delta_\Gamma(n-1-\delta_\Gamma)$.*

2.2 The Patterson-Sullivan construction

This construction, now classical, is due to Patterson [**?**] in the case of surfaces and has been extended by Sullivan [**?**] to higher dimensional hyperbolic spaces. The construction extends also to variable negative curvature [**?**].

Let Γ be a nonelementary discrete group of isometries of $\mathbb{H}^n$, and $x \in \mathbb{H}^n$ a point. Let $s > \delta_\Gamma$. Define

$$\nu_x^s = \frac{1}{P(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma.o)} \delta_{\gamma.o}\,,$$

where δ_y denotes the Dirac mass at the point y. Notice that ν_x^s is a finite measure, and ν_o^s is a probability measure.

As $\mathbb{H}^n \cup \partial\mathbb{H}^n$ is compact, we can choose a subsequence $s_n \to \delta_\Gamma$ such that $\nu_o^{s_n}$ weakly converges to some probability measure ν_o on $\mathbb{H}^n \cup \partial\mathbb{H}^n$.

If Γ is divergent, then one easily checks that the measure $\nu_o^{s_n}(K)$ of any compact subset $K \subset \mathbb{H}^n$ goes to 0, so that the limit measure ν_o is supported on $\partial\mathbb{H}^n$, and in fact on Λ_Γ by construction.

If Γ is convergent, following Patterson's trick, one modifies the Poincaré series by adding a weight:

$$\tilde{P}(s) = \sum_{\gamma\in\Gamma} h(d(x,\gamma.o))e^{-sd(x,\gamma.o)}\delta_{\gamma.o}\,,$$

where h is a slowly growing function allowing the series to diverge at $s = \delta_\Gamma$ without changing its critical exponent.

In both cases, one obtains a measure ν_o supported on Λ_Γ, which satisfies for all $\gamma \in \Gamma$ **CHECK FORMULA**

$$\frac{d\gamma_*\nu_o}{d\nu_o}(\xi) = \exp(-\delta_\Gamma\beta_\xi(o,\gamma^{-1}.o))\,.$$

Now, for all $x \in \mathbb{H}^n$, one considers a subsequence s_{n_k} of s_n (depending a priori of the point x) such that $\nu_x^{s_{n_k}}$ converges to ν_x.

Finally, we get a δ_Γ-conformal density on the boundary, that is a family of finite conformal measures $(\nu_x)_{x\in\mathbb{H}^n}$ on $\partial\mathbb{H}^n$, that satisfy

1. for all $x \in \mathbb{H}^n$ and $\gamma \in \Gamma$, $\gamma_*\nu_x = \nu_{\gamma.x}$,
2. for all $x, y \in \mathbb{H}^n$,
$$\frac{d\nu_x}{d\nu_y}(\xi) = \exp(-\delta_\Gamma\beta_\xi(x,y)\,.$$
CHECK FORMULA

The above construction provides a δ_Γ conformal measure supported on Λ_Γ.

But the family of visual (angular) measures (λ_x) on each unit sphere in $T_x^1\mathbb{H}^n$ provides a $(n-1)$-conformal measure supported on the full boundary $\partial\mathbb{H}^n$. This family is interesting when $\Lambda_\Gamma = \partial\mathbb{H}^n$, and less in other cases.

2.3 Sullivan's Shadow lemm

A *shadow* $\mathcal{O}_x(B(y,R))$ is, viewed from the point x, the shadow made by the ball $B(y,R)$ when looking at the boundary at infinity. In other words,

$$\mathcal{O}_x(B(y,R)) = \{\xi \in \partial\mathbb{H}^n,\ [x,\xi) \cap B(y,R) \neq \emptyset\}\,,$$

where $[x,\xi)$ denotes the geodesic ray from x to ξ.

Theorem 2.4 (Sullivan's Shadow lemma) *Let $(\nu_x)_{x\in\mathbb{H}^n}$ be a δ-conformal measure, i.e. a measure satisfying* conformal *(2.2). Assume that it has support Λ_Γ. For all $x \in \mathbb{H}^n$ there exists $r_0 > 0$ such that for all $r > r_0$ there exists a constant $C_{r,x} > 0$ such that for all $\gamma \in \Gamma$,*

$$\frac{1}{C_{r,x}}e^{-\delta d(x,\gamma x)} \leq \nu_x(\mathcal{O}_x(B(\gamma.x,r)) \leq C_{r,x}e^{-\delta d(x,\gamma x)}\,.$$

Proof : Using both properties of a δ-conformal measure and the triangular inequality in a triangle with vertices $\gamma^{-1}.x$, x, and some $\xi \in \mathcal{O}_{\gamma^{-1}x}(B(x,r))$, we have

$$\nu_x(\mathcal{O}_x(B(\gamma.x,r)) = \nu_{\gamma^{-1}x}(\mathcal{O}_{\gamma^{-1}x}(B(x,r)) \asymp e^{-\delta d(x,\gamma^{-1}x)}\nu_x(\mathcal{O}_{\gamma^{-1}x}(B(x,r))$$

The right term is bounded from above by the mass of ν_x. To get the lower bound, the point is to observe that $\inf_{y\in\mathbb{H}^n\cup\partial\mathbb{H}^n} \nu_x(\mathcal{O}_y(B(x,r))$ is positive as soon as r is not too small, so that for all $y \in \mathbb{H}^n \cup \partial\mathbb{H}^n$, the shadow viewed from y of $B(x,r)$ intersects the limit set. $\square$

This Shadow lemm has been generalized by several people, in the case of geometrically finite groupes with cusps, replacing $\gamma.x$ by a point in a cusp (see Stratmann-Velani [**?**], Schapira [**?**], Hersonsky-Paulin [**?**]).

Let us mention an easy corollary of the Shadow Lemma.

Corollary 2.5 *A δ-conformal measure has no atoms in the radial limit set.*

Proof : Indeed, a point $\xi \in \Lambda_{rad}(\Gamma)$ belongs to infinitely many shadows $\mathcal{O}_o(B(\gamma_n.o,R))$, for some $R>0$, with $\gamma_n.o$ converging to ξ. The conclusion follows from the Shadow Lemma. $\square$

2.4 The Patterson-Sullivan construction of the Bowen-Margulis measure

Thanks to the Hopf coordinates $v=(v^-,v^+,t)$, we define on $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ a Radon measure invariant under both actions of the geodesic flow and the group Γ by the formula

$$d\tilde{m}_{BM}(v) = e^{\delta_\Gamma \beta_{v^+}(o,\pi(v))+\delta_\Gamma\beta_{v^-}(o,\pi(v))}\, d\nu_o(v^-)\, d\nu_o(v^+)\, dt\,.$$

The reader can easily check the invariance properties. Moreover, the above formula does not depend on the choice of the point o, by the conformality properties of $(\nu_x)_{x\in\mathbb{H}^n}$.

When the measure ν_o has no atoms, the measure $\tilde{m}_{BM}$ gives zero mass to the diagonal of $\partial\mathbb{H}^n \times \partial\mathbb{H}^n$ so that this measure induces on the quotient on T^1M a Radon measure, which is invariant under the geodesic flow, and which will be called the *Bowen-Margulis measure*. In fact, before this Patterson-Sullivan construction, Bowen and Margulis provided, in the compact case, two alternative constructions of this measure by using averages over periodic orbits for the first, and averages on leaves of the strong stable foliation for the second.

By abuse of notation, we will call this measure *the* Bowen-Margulis measure, before any justification (which will come later) of the fact that the construction produces a unique measure!

3 More about critical exponents

The main objects have been introduced, so that we can state some results that we wish to emphasize, without proofs.

3.1 Critical exponents and dimension

Notice that, by definition, a vector $v \in T^1M$ admits a lift $\tilde{v} \in T^1\mathbb{H}^n$ such that v^+ belongs to the radial limit set if and only if the geodesic ray $(g^tv)_{t\geq 0}$ returns infinitely often in a compact set. The famous result of Bishop-Jones below states that, in some sense, the critical exponent is the size of the set of vectors whose geodesic returns infinitely often in a compact set.

Theorem 3.1 (Bishop-Jones [?]) *Let Γ be a nonelementary Kleinian group. Then δ_Γ is the Hausdorff dimension of the radial limit set.*

3.2 Critical exponents and subgroups

We already noticed that if Γ is a subgroup of Γ_0, $\delta_\Gamma \leq \delta_{\Gamma_0}$. But at which condition this inequality becomes an equality ?

For example, if $\Gamma \lhd \Gamma_0$, then $\Lambda_\Gamma = \Lambda_{\Gamma_0}$ but there is no reason that $\Lambda_{rad}(\Gamma)$ coincides with $\Lambda_{rad}(\Gamma_0)$. Think for example to the case where $\mathbb{H}^n/\Gamma_0$ is compact, and $\mathbb{H}^n/\Gamma$ is an infinite regular cover. They do not have the same recurrent vectors of course.

Using a tricky variant of the shadow lemm, applied to all $y = \gamma x$ for $\gamma \in N(\Gamma)$, Roblin answers partially the question.

Theorem 3.2 (Roblin) *If $\Gamma \lhd \Gamma_0$ then*

$$\frac{\delta_{\Gamma_0}}{2} \leq \delta_\Gamma \leq \delta_{\Gamma_0}$$

When the quotient Γ_0/Γ is amenable, then the answer can be precised. The reader who does not know the definition of amenability can skip this result.

Theorem 3.3 (Roblin, Brooks) *Let $\Gamma \lhd \Gamma_0$ be a normal subgroup.*

1. *If Γ_0/Γ is amenable then $\delta_\Gamma = \delta_{\Gamma_0}$,*
2. *If $\delta_\Gamma = \delta_{\Gamma_0}$ and Γ_0 is a lattice or a convex-cocompact group with $\delta_{\Gamma_0} > n/2$ then Γ_0/Γ is amenable.*

The first part is due to Roblin. The second part was proven by Brooks through harmonic analysis, which leads unfortunately to assumptions that are likely not to be optimal.

Let us conclude this paragraph with a result guaranteeing the strict inequality.

Theorem 3.4 (Dal'bo-Otal-Peigné) *If Γ_0 contains a divergent subgroup Γ such that Λ_Γ is strictly included in Λ_{Γ_0} then Γ_0 is divergent and $\delta_{\Gamma_0} > \delta_\Gamma$.*

We will not provide any proof of all these results.

3.3 A little bit more about critical exponents

We already mentioned that if Γ is a lattice, then $\delta_\Gamma = n - 1$. In the case of finite volume lattices, it follows from Bishop-Jones theorem for example.

If Γ is convex-cocompact but not cocompact, then $0 < \delta_\Gamma < n - 1$. **Add REFERENCE** When Γ is geometrically finite with cusps, and not a lattice, then $\frac{k}{2} \leq \delta_\Gamma < n - 1$, where k is the maximal rank of its parabolic subgroups.

Theorem 3.5 (Reference??) *When Γ is a finitely generated geometrically infinite group, then $\delta_\Gamma = 2$. Moreover, either $\Lambda_\Gamma = S^2$ or Λ_Γ is of Lebesgue measure zero.*

Theorem 3.6 (Doyle, Philipps-Sarnak) *Let $n \geq 3$. Then the supremum of the critical exponents of the Schottky subgroups of $Isom(\mathbb{H}^n)$ is bounded from above by a constant $c_n < n - 1$.*

This theorem was first proven by Philipps-Sarnak [**?**] for $n \geq 4$, and later by Doyle [**?**] for $n = 3$, using harmonic analysis.

Theorem 3.7 (Otal-Peigné) *The critical exponent δ_Γ is the topological entropy of the geodesic flow.*

4 The Hopf-Tsuji-Sullivan dichotomy, the variational principle, and counting estimates

We will see that when the Bowen-Margulis measure is finite, it is ergodic and conservative, and has wonderful ergodic properties.

We begin with some good criteria to check ergodicity or finiteness

4.1 Ergodicity, finiteness

There is an important dichotomy between the case where Γ is divergent, i.e. the series $\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)}$ diverges at the critical exponent, and the case where Γ is convergent, when the series converges at $s = \delta_\Gamma$.

For the reader with probabilistic background, the theorem below is a version of Borel-Cantelli lemm adapted to a dynamical system-the geodesic flow - instead of a sequence of independent random variables, the independence being replaced by the product structure of the geodesic flow.

Theorem 4.1 (The Hopf Dichotomy - divergent case) *The following assertions are equivalent.*

1. *The group Γ is divergent*
2. *The Patterson-Sullivan measure ν_o gives full measure to the radial limit set: $\nu_o(\Lambda_{rad}) = 1$*
3. *The geodesic flow is ergodic and conservative w.r.t the Bowen-Margulis measure m_{BM}, i.e. for m_{BM}-almopst all $v \in T^1M$, there exists a neighbourhood K of v such that $\int_0^\infty \mathbf{1}_K(g^t v)\, dt = +\infty$.*
4. *the action of Γ on $\Lambda_\Gamma \times \Lambda_\Gamma \setminus \{Diagonal\}$ is ergodic and conservative.*

Theorem 4.2 (The Hopf Dichotomy - convergent case) *The following assertions are equivalent.*

1. *The group Γ is convergent*
2. *The Patterson-Sullivan measure ν_o gives zero measure to the radial limit set: $\nu_o(\Lambda_{rad}) = 0$.*
3. *The geodesic flow is totally dissipative w.r.t the Bowen-Margulis measure m_{BM}, i.e. for m_{BM}-almopst all $v \in T^1M$, and all neighbourhoods K of v, $\int_0^\infty \mathbf{1}_K(g^t v)\, dt < +\infty$.*
4. *the action of Γ on $\Lambda_\Gamma \times \Lambda_\Gamma \setminus \{Diagonal\}$ is totally dissipative*

We could add more precisions, but we prefer to concentrate the presentation on the key points.

Remark 4.3 By Poincaré recurrence theorem, a finite invariant measure is conservative. By the above theorems, we deduce that when the Bowen-Margulis measure is finite, it is ergodic and conservative.

A variant of some arguments in the proof of the Hopf dichotomy allows to show the following

Theorem 4.4 *When the Bowen-Margulis measure is ergodic and conservative, it is the unique measure given by the Patterson-Sullivan construction. In other words, when Γ is divergent, there is a unique δ_Γ-conformal family of measures $(\nu_x)_{x\in\mathbb{H}^n}$ on the boundary.*

This theorem gives a justification to the terminology "the Bowen-Margulis measure" used in this text.

We will sketch the proof of the Hopf dichotomy later. Let us add some complements about finiteness, and major statements true under this assumption.

4.1.1 Finiteness criteria

Theorem 4.5 (Sullivan) *When $M = \Gamma\backslash\mathbb{H}^n$ is geometrically finite, the Bowen-Margulis measure is finite.*

Note that the above result is false in variable negative curvature.

Theorem 4.6 (Peigné) *There exist geometrically infinite hyperbolic manifolds whose Bowen-Margulis measure is finite.*

4.2 Extraordinary properties of the Bowen-Margulis measure

We present here two major statements that we will try to prove later. We refer the reader to the proofs to get rigorous definitions of the notions involved here.

4.2.1 About entropy

Theorem 4.7 (Otal-Peigné) *Let Γ be a nonelementary Kleinian group. If the Bowen-Margulis measure is finite, then it is the unique measure of maximal entropy of the geodesic flow, also called the* Bowen-Margulis measure.

Otherwise, the geodesic flow admits no finite invariant measure maximizing entropy.

In any case, the critical exponent δ_Γ is the topological entropy of the geodesic flow.

4.2.2 Counting

Theorem 4.8 (Roblin) *Let Γ be a nonelementary Kleinian group. If the Bowen-Margulis measure is finite, then the critical exponent δ_Γ is the exponential growth rate of the orbits of Γ in $\mathbb{H}^n$, and also the exponential growth rate of the periodic orbits of T^1M.*

In fact, Roblin gave also a (less precise) counting result in the case where the measure is infinite.

4.3 Proof of the Hopf dichotomy

As said before, we will just give the main ideas, and skip some subtle technical difficulties.

Before the proof, let us recall the following probabilistic statement.

Lemma 4.9 (Borel-Cantelli) *Let (A_n) be a sequence of events on the probability space $(\Omega, \mathcal{A}, P)$.*

1. *If $\sum_n P(A_n) < \infty$, then $P(\limsup A_n) = 0$*
2. *If the (A_n) are independent and $\sum_n P(A_n) = +\infty$, then $P(\limsup A_n) = 1$.*

Recall that by definition, a vector $v \in T^1M$ has a lift (and therefore all lifts) pointing towards the radial limit set, i.e. $v^+ \in \Lambda_{rad}(\Gamma)$ if and only if the geodesic ray $(g^t v)_{t\geq 0}$ returns infinitely often in some compact set. Therefore, using this simple idea and the construction of the Bowen-Margulis measure through the Patterson-Sullivan measure ν_o on the boundary, the following equivalences are easy to show (and we will not do it).

• $\nu_o(\Lambda_{rad}(\Gamma)) = 0$ iff the action of Γ on $\Lambda_\Gamma \times \Lambda_\Gamma \setminus \{diagonal\}$ is totally dissipative wrt the measure $\nu_o \otimes \nu_o$, iff the action of the geodesic flow (g^t) on Ω is totally dissipative wrt m_{BM}.

• $\nu_o(\Lambda_{rad}) = 1$ iff (g^t) is conservative w.r.t. the measure m_{BM}.

• The measure m_{BM} is ergodic and conservative iff the action of Γ on $\partial^2\mathbb{H}^n$ is ergodic and conservative w.r.t $\nu_o \otimes \nu_o$.

Therefore, the key steps of the proof are stated in the following lemms.

Convergent-case **Lemma 4.10** *If $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(x,\gamma x)} < \infty$, then $\nu_o(\Lambda_{rad}) = 0$.*

Divergent-case **Lemma 4.11** *If $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(x,\gamma x)} = +\infty$, then $\nu_o(\Lambda_{rad}(\Gamma)) > 0$.*

Lemma 4.12 *If $\nu_o(\Lambda_{rad}(\Gamma) > 0$, then $\nu_o(\Lambda_{rad}(\Gamma)) = 1$.*

This last statement will not be proven, because the arguments of the proof are not used elsewhere.

Let us prove Lemma 4.10. The analogy with the Borel Cantelli lemma let think that it is the easy part.

Proof : Observe that $\Lambda_{rad}(\Gamma) = \cup_{N\in\mathbb{N}} \Lambda^N_{rad}(\Gamma)$, where $\Lambda^N_{rad}(\Gamma)$ is the set of points $\xi \in \Lambda_{rad}(\Gamma)$ such that there exists a sequence $\gamma_n.o$ converging to ξ while staying at distance at most N from the geodesic $[o,\xi)$. It is therefore enough to show that $\Lambda^N_{rad}(\Gamma)$ has measure zero. But $\Lambda^N_{rad}(\Gamma) \subset \cup_{\gamma\in\Gamma, d(o,\gamma.o)\geq T} \mathcal{O}_o(B(\gamma.o,N))$ so that, by the Shadow Lemma,

$$\nu_o(\Lambda^N_{rad}(\Gamma)) \leq C_{o,N} \sum_{\gamma\in\Gamma, d(o,\gamma.o)\geq T} e^{-\delta_\Gamma d(o,\gamma.o)} .$$

The right sum is the rest of a convergent series, so that it goes to zero when $T \to +\infty$. It concludes the proof of the lemma. □

Let us give some hints about the proof of Lemma 4.11.

Proof : We want to prove that $\nu_o(\Lambda_{rad}) > 0$, or equivalently, that there is a set of positive m_{BM}-measure of vectors v that return infinitely often in a compact set. It is enough to prove that for $K = T^1B(o,R) \subset T^1M$, the set $\{v \in K, \int_0^+ \infty \mathbf{1}_K(g^t v)\, dt = +\infty\}$ has positive measure. In particular, if this is true, then it implies

$$\int_K \int_0^{+\infty} \mathbf{1}_K(v)\mathbf{1}_K(g^t v)\, dt\, dm_{BM}(v) = +\infty .$$

We will not prove the lemma, but only (the heuristic of) this weaker statement, because the proof is shorter but contains the key ideas.

Lift K to $\widetilde{K}$ which is the unit tangent bundle of a ball, still denoted by $T^1B(o,R)$. We have

$$\int_0^\infty \mathbf{1}_K(v)\mathbf{1}_K(g^t v)\, dt = \sum_{\gamma\in\Gamma} \int_0^{+\infty} \mathbf{1}_{\widetilde{K}}(v)\mathbf{1}_{\gamma\widetilde{K}}(g^t v)\, dt .$$

Moreover, by construction of the Bowen-Margulis measure, observe that, up to some constants, when $d(o,\gamma o) \simeq t$, $K \cap g^{-t}\gamma K$ almost coincides with the set of vectors $\{v \in T^1M, \pi(v) \in K, v+ \in \mathcal{O}_o(B(\gamma o, R))\}$ so that, using in a crucial way the product structure of the Bowen-Margulis measure $m_{BM} \sim \nu_o \times \nu_o \times dt$, we have

$$\tilde{m}_{BM}(\widetilde{K} \cap g^{-t}\gamma\widetilde{K}) \asymp \nu_o(\mathcal{O}_o(B(\gamma o, R))) \asymp e^{-\delta_\Gamma d(o,\gamma o)}$$

Therefore, we can estimate our integral, as follows

$$\begin{aligned}\int_0^\infty \mathbf{1}_K(v)\mathbf{1}_K(g^t v)\, dt &= \sum_{\gamma\in\Gamma}\int_0^{+\infty} \mathbf{1}_{\widetilde{K}}(v)\mathbf{1}_{\gamma\widetilde{K}}(g^t v)\, dt \\ &\asymp \sum_{n\in\mathbb{N}} \sum_{\gamma\in\Gamma,\, n\le d(o,\gamma o)<n+1} e^{-\delta_\Gamma d(o,\gamma o)}\end{aligned}$$

This last term is infinite because the group Γ is assumed to be divergent. This proves the above assertion. The proof of the lemma uses refinements of this idea.

□

4.4 The Hopf argument

In this section, we want to present this famous argument, which allows to prove that when m_{BM} is finite or infinite and conservative, it is ergodic. A variant allows to show that when the measure m_{BM} is ergodic, it is uniquely defined.

The proof seems simple but hides some very subtle difficulties.

Proof : Assume that m_{BM} is finite. We will explain rapidly at the end what are the differences in the case where it is infinite conservative. And recall that it is -up to a density that we will neglect in the proof - equal to the product measure $\nu_o \times \nu_o \times dt$.

Ergodicity of the Bowen-Margulis measure means that an integrable invariant map should be constant m_{BM}-almost surely. Let us prove it.

Let $f : T^1M \to \mathbb{R}$ be an integrable map. It is enough to prove that its conditional expectation $E(f|\mathcal{I})$ given the σ-algebra of invariant sets is constant. As uniformly continuous maps are dense in $L^1(T^1M, m_{BM})$, it is enough to restrict to the case where f is uniformly continuous.

Consider the following maps

$$f^+(v) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(g^s v)\, ds \quad \text{and} \quad f^-(v) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(g^s v)\, ds\,.$$

By Birkhoff theorem these maps f^+ and f^- are well defined m_{BM}-almost surely and coincide m_{BM}-almost surely.

As f is uniformly continuous, it is easy to check that f^+ is invariant along geodesic orbits and stable manifolds, and therefore depends only on v^+ in the Hopf coordinates $v = (v^-, v^+, t)$. Similarly, f^- depends only on v^-. Now, it seems easy to conclude that f is constant m_{BM}-almost surely, as it coincides m_{BM}-almost surely with a function depending only on v^+, and also m_{BM}-almost surely with another function depending only on v^-. The rigorous proof of this intuitive fact crucially uses the fact that m_{BM} is a product measure through the use of Fubini theorem.

To convince yourself that it is crucial, consider a measure μ which is the sum of the measures on two disjoint periodic orbits. It is nonergodic. Consider a map f which takes two different values on the two disjoint periodic orbits. It is invariant, but nonconstant. Moreover, w.r.t the measure μ, it depends almost surely only on v^+, and almost surely only on v^-.

We do not add details, but we hope that the reader is now convinced that it is less obvious than it seems. □

5 About entropy

5.1 Several definitions of entropy

Entropy is an invariant measuring the exponential growth rate of the complexity of the dynamics. But there are several ways to understand this sentence, and therefore several definitions, which coincide in good cases.

All good definitions satisfy the relation $h(g^t) = |t|h(g^1)$, so that we consider the time-one map of the geodesic flow.

Historically, the first notion of entropy is the Kolmogorov-Sinai entropy, but it is not the simplest to define, so that we begin with the topological entropy.

The following definition is due to Bowen. Let d be a distance on $X = T^1M$. Let $K \subset X$ be a compact set. A set $E \subset K$ is said (ε, N, K) separated if $E \subset K$, and for all $x \neq y$ in E, and all $0 \leq k \leq n$, one has $d(g^k x, g^k y) \geq \varepsilon$.

The *topological entropy* of $g : (X, d) \to (X, d)$ is defined as

$$h^d(g) = \sup_K \sup_{\varepsilon > 0} \limsup_{N \to +\infty} \frac{1}{N} \log \max \#E\,,$$

the maximum being over all (ε, N, K) separated sets of K. Let us emphasize the fact that when X is non compact, this definition strongly defines on the distance d. Given a topology on X, one has $\sup_d h^d(g) = +\infty$, the supremum being considered over all distances defining the topology.

Therefore, the good notion of topological entropy is

$$h_{top}(g) = \inf_d h^d(g)\,.$$

Definition Kolmogorov Sinai entropy
Handel-Kitchens
Critical exponent

Theorem 5.1 (Otal-Peigné [?])

Gurevic entropy : growth rate of periodic orbits

Theorem 5.2 (Roblin)

6 Proof of the variational principle

In this section, we give details on theorem ??.

7 Proofs of the Counting results

References

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