Exercises- Ergodic theory of the geodesic flow

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Exercise 0.1 *Consider* $\Omega = [0,1]^2$ *endowed with the Dirac probability measure* $\delta_{(1/4,2/3)}$ *at the point* $(1/4, 2/3)$. *Show that for any intervals* I, J *of* $[0,1]$, *if* $A = I \times [0,1]$ *and* $B = [0,1] \times J$, *the events* A *and* B *are not independent.*

Same question with the (one dimensional) Lebesgue measure on the diagonal $\Delta = \{(x,x), x \in [0,1]\}$.

Exercise 0.2 *Consider the angle doubling map* $T : x \in [0,1] \mapsto 2x$ *mod* 1. *Choose distinct* x *and* y *in* $[0,1]$ *such that* $|x-y| \le 2^{-12}$ *but the orbit* $(T^n x)$ *is periodic and the orbit* $(T^n y)$ *is dense in* $[0,1]$.

Hint: *Use binary development. Understand the effect of* T *on the binary development of a number* $x \in [0,1[$. *Find a number* x *whose orbit is periodic. Find another one whose orbit is dense. At the end answer the initial question.*

Exercise 0.3 *The hyperbolic plane is defined as* $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ *and endowed with the hyperbolic metric* $ds^2 = \frac{dx^2+dy^2}{y^2}$. *Show that the hyperbolic geodesics (i.e. the curves minimizing distances) are the vertical half-lines and the half-circles orthogonal to the boundary* $\mathbb{R} \times \{0\}$. *The isometries preserving orientation are the homographies* $z \to \frac{az+b}{cz+d}$ *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a matrix with determinant* 1.

Exercise 0.4 *Consider the flow* ϕ_t *on* $\mathbb{T}^2$ *defined by* $\phi_t(x) = x + v \mod \mathbb{Z}^2$. *Show that the Lebesgue measure on the torus is invariant under the flow.*
hint : come back to $\mathbb{R}^2$.

Exercise 0.5 *Show that the two versions of Birkhoff ergodic theorem in terms of integral maps or Borel set with boundary of measure zero are equivalent.*

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Exercise 0.6 *Assume that* X *is compact. Let* $\mathcal{M}^1$ *be the set of invariant probability measures on* X. *Let* (f_n) *be a countable dense family of maps in* $C(X,\mathbb{R})$. *Show that a basis of neighbourhoods of* $\mu \in \mathcal{M}^1$ *for the weak* $*$ *topology is given by*

$$\left\{ \nu \in \mathcal{M}^1, \ \forall 1 \le i \le N \ \left| \int f_i \, d\nu - \int f_i \, d\mu \right| \le \epsilon \right\}$$

Exercise 0.7 *If* c_1, c_2 *are two geodesic rays such that* $d(c_1(t), c_2(t)) \to 0$ *when* $t \to +\infty$, *then show that for every* $t \ge 0$,

$$d(c_1(t), c_2(t)) \le e^{-t} d(c_1(0), c_2(0))$$

Hint *Use the upper half plane model and come back to two vertical rays.*

[1] The notes are available here
`https://imag.umontpellier.fr/ schapira/recherche/CIRM2025-Barbara.pdf`.

Exercise 0.8 *Show that in the Hopf coordinates, the geodesic flow acts as*

$$g_t(v) \simeq (v^-, v^+, s+t)\,.$$

Consider an isometry $\gamma \in PSL(2,\mathbb{R})$*. Show that it acts as follows*

$$\gamma.v \simeq (\gamma v^-, \gamma v^+, s + \beta_{v^+}(\gamma^{-1}o, o)$$

Exercise 0.9 *Show that given a* (g^t) *invariant Radon measure* m *on* T^1S *n its lift* $\tilde{m}$ *on* $T^1\mathbb{D}$ *satisfies*

$$H_*\tilde{m} = \mathcal{C} \times dt$$

where $\mathcal{C}$ *is a* geodesic current*, i.e. a* Γ*-invariant Radon measure on* $S^1 \times S^1$*.*

Exercise 0.10 *Show that if* $\Gamma < PSL(2,\mathbb{R})$ *is a discrete group that contains at least two hyperbolic isometries with distinct axes, then there does not exist* Γ*-invariant probability measures on* S^1*.*

Exercise 0.11 *Show that the Poincaré series*

$$P_{(\Gamma,f)}(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^\gamma \tilde{f}}$$

associated with (Γ, f) *converges for* $s > \delta^f$ *and diverges for* $s < \delta^f$*, where*

$$\delta^f = \lim_{t\to\infty} \frac{1}{t} \log \sum_{\gamma\in\Gamma, d(o,\gamma o)\in[t,t+1]} e^{\int_o^{\gamma o} \tilde{f}}$$

Exercise 0.12 *Show that the measure* ν^f *obtained through the Patterson Sullivan Gibbs construction is supported on* S^1

Exercise 0.13 *Show that* ν^f *gives full measure to* Λ_Γ*.*

Exercise 0.14 *Use the geodesic rays* c_x *and* c_y *from* x *(resp* y*) to* ξ *to give a rigorous meaning to the quantity*

$$\rho^f_\xi(x,y) = \lim_{t\to\infty} \int_x^\xi \tilde{f} - \int_y^\xi \tilde{f} \quad \text{and} \quad \beta^f = \delta^f\beta - \rho^f$$

Exercise 0.15 *Show that the measure* ν^f *is* Γ *quasi invariant and that for a.e.* ξ *and all* $\gamma \in \Gamma$*,*

$$\frac{d\gamma_*\nu^f}{d\nu^f}(\xi) = \exp(-\beta^f_\xi(\gamma o, o)) = \exp(-\delta^f\beta_\xi(\gamma o, o) + \rho^f_\xi(\gamma o, o))\,.$$

Exercise 0.16 *Show that the measure* $\mathcal{C}^f$ *on* $S^1 \times S^1$ *defined by*

$$d\mathcal{C}^f(\xi,\eta) = \exp\left(\beta^f_\eta(o,x) + \beta^f_\xi(o,x)\right) d\nu^f(\xi)\, d\nu^f(\eta), \qquad \text{for any point } x \in (\xi,\eta)\,,$$

is a geodesic current. (Admit that it gives zero measure to the diagonal of $S^1 \times S^1$*.)*

Exercise 0.17 *Show that the measure* m^f *is supported on*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma \subset T^1S$$

Exercise 0.18 *Show that the surface* $S = \mathbb{D}/\Gamma$ *is convex-cocompact if and only if* $H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})$ *is cocompact, i.e.*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma$$

is compact.

Hint*: First observe that* $\Omega \subset T^1\mathcal{C}^{core}/\Gamma$ *and deduce that one direction of the equivalence is easy. For the other direction, use the fact that triangles are thin to show that any point of* C^{core} *is at uniformly bounded distance of a geodesic joining two points of* Λ_Γ*.*

Exercise 0.19 *The measure* m^f *is ergodic iff it satisfies the conclusion of Birkhoff ergodic theorem.*

Exercise 0.20 *Show that* $E(\psi|\mathcal{I}) \equiv \int \psi, dm^f$ *for every* $\psi \in L^1(m^f)$ *iff* $E(\psi|\mathcal{I}) = \int \psi\, dm^f$ *for every* $\psi \in C_c(T^1S)$*.*

Exercise 0.21 *For* $\psi \in C_c(T^1S)$*, define*

$$\psi^{\pm}(v) = \limsup_{T\to\pm\infty} \frac{1}{T}\int_0^T \psi \circ g^t v dt$$

Prove that ψ^+ *and* ψ^- *are* (g^t) *invariant.*
Prove that if v *and* w *are on the same stable horocycle, i.e.* $d(g^tv, g^tw) \to 0$ *when* $t \to +\infty$*, then* $\psi^+(v) = \psi^+(w)$*. Prove the analogous property for* ψ^- *when* $t \to -\infty$*.*

Exercise 0.22 *Try to show a property like*

$$B(v,T,\epsilon) \asymp \mathcal{O}_{\pi(v)}(B(\pi(g^Tv),\epsilon)) \times \mathcal{O}_{\pi(g^Tv)}(B(\pi(v),\epsilon)) \times [-\epsilon,\epsilon]$$

Exercise 0.23 *Prove*

- $m_{\mathcal{C}}^{g_1}$ *and* $m_{\mathcal{C}}^{g_2}$ *are simultaneously periodic*
- $m_{\mathcal{C}}^{g_1}$ *and* $m_{\mathcal{C}}^{g_2}$ *are simultaneously ergodic*
- $m_{\mathcal{C}}^{g_1}$ *and* $m_{\mathcal{C}}^{g_2}$ *are simultaneously of full support*
- $m_{\mathcal{C}}^{g_1}$ *and* $m_{\mathcal{C}}^{g_2}$ *are simultaneously product measures or not.*