Invariant measures for the geodesic flow of hyperbolic surfaces

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Abstract

These notes are available here
`https://imag.umontpellier.fr/ schapira/recherche/CIRM2025-Barbara.pdf`.

Some references?

Books

Ergodic Theory with a view towards Number Theory, by Manfred Einsiedler and Thomas Ward
Geodesic and horocyclic trajectories, by Françoise Dal'bo

Stochastic properties of dynamical systems, by Francoise Pene

Ergodic theory and dynamical systems, by Yves Coudène

Equilibrium states in negative curvature, by F. Paulin, M. Pollicott and Barbara

Schapira

Articles

Regularity of entropy, geodesic currents and entropy at infinity, avec Samuel Tapie, Annales scientifiques de l'ENS .

Pressure at infinity and strong positive recurrence in negative curvature , with Sébastien Gouezel, Samuel Tapie, and an appendix by Felipe Riquelme , Commentarii Mathematici Helvetici.

...
Many papers by Andres Sambarino

1 Goal of these lectures

Construct a family of invariant measures wrt the geodesic flow of a hyperbolic surface, whose chaotic properties reflect the chaotic behaviour of the geodesic flow, and that also allow to understand deformations of hyperbolic metrics.

2 What does chaos mean ?

2.1 Back to probability theory (5')

Consider the Lebesgue measure $\mathbb{P} = dxdy$ on the square $\Omega = [0,1]^2$. Define $A = \{(x,y) \in \Omega, 1/4 \leq x \leq 1/2\}$ and $B = \{(x,y) \in \Omega, 2/3 \leq y \leq 3/4\}$.

Observe that $P(A \cap B) = P(A)P(B)$. A and B are independent. More generally, for any intervals I, J of $[O,1]$, if $A = I \times [0,1]$ and $B = [0,1] \times J$, we still have $P(A \cap B) = P(A)P(B)$.

In other words, the coordinates x and y are independent.

Important to remember : on a product space, with respect to a product measure, coordinates x and y are independent (in the above sense).

Exercise 2.1 *Consider $\Omega = [0,1]^2$ endowed with the Dirac probability measure $\delta_{(1/4,2/3)}$ at the point $(1/4, 2/3)$. Show that the above events A and B are not independent anymore.*

Same question with the (one dimensional) Lebesgue measure on the diagonal $\Delta = \{(x,x), x \in [0,1]\}$.

2.2 Expanding dynamics (5')

Consider the angle doubling map $T : x \in [0,1] \mapsto 2x \bmod 1$. Observe that if x, y satisfy $|x - y| \sim 2^{-12}$, then after a few iterations ($n = 11$?) the distance between $T^n x, T^n y$ is macroscopic.

Exercise 2.2 *Choose distinct x and y in $[0,1]$ such that $|x-y| \leq 2^{-12}$ but the orbit $(T^n x)$ is periodic and the orbit $(T^n y)$ is dense in $[0,1]$.*

Hint: *Use binary development. Understand the effect of T on the binary development of a number $x \in [0,1[$. Find a number x whose orbit is periodic. Find another one whose orbit is dense. At the end answer the initial question.*

2.3 The geodesic flow on the hyperbolic disc (5')

2.3.1 Hyperbolic plane / disc

The hyperbolic plane is defined as $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ and endowed with the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. The geodesics are the curves which minimize the distance.

Exercise 2.3 Check these classical facts. The hyperbolic geodesics are the vertical half-lines and the half-circles orthogonal to the boundary $\mathbb{R} \times \{0\}$. The isometries preserving orientation are the homographies $z \to \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with determinant 1.

PICTURE

The model of the disk is more natural geometrically. The hyperbolic disk is the open disk $D(0,1)$ in $\mathbb{C}$, endowed with the image metric from the hyperbolic metric of $\mathbb{H}$ through the map $z \mapsto \frac{z-i}{z+i}$. In the disk model, the geodesics are the diameters and the pieces of circles orthogonal to the boundary.

PICTURE

2.3.2 Geodesic flow

Picture $\mathbb{D}$, geodesics of $\mathbb{D}$.

a vector v on $T^1\mathbb{D}$, the geodesic (at unique speed) $(c_v(t))$ determined by v (such that $c_v'(0) = v$).

The geodesic flow g_t moves v along the geodesic that it determines. If $(c_v(t))$ is the geodesic such that $c_v'(0) = v$, then

$$g_t(v) = c_v'(t)\,.$$

There is a homeomorphism, the *Hopf coordinates*,

$$v \in T^1\mathbb{D} \mapsto (v^-, v^+, s) \in S^1 \times S^1 \setminus \text{Diagonal} \times \mathbb{R}$$

This gives

- A natural geometric product structure
- Natural coordinates (v^-, v^+, s), such that the geodesic flow acts as a contraction in the direction of v^+, expansion in the v^- coordinates, translation on the real coordinate.

On the unit tangent bundle $T^1\mathbb{D}$ of the disk $\mathbb{D}$, the dynamics is not interesting. Every orbit goes straight, from the horizon at infinity to the horizon at infinity.

Compare to a linear flow on $\mathbb{R}^2$. A linear flow $\phi_t : x \in \mathbb{R}^2 \to x + \vec{v}$ has orbits that all go straight to infinity. However, the linear flow $\phi_t : x \in \mathbb{T}^2 \to x + v \mod \mathbb{Z}^2$ is interesting. If v has irrational slope, all orbits are dense.

Similarly, we will study the geodesic flow on quotients of $T^1\mathbb{D}$. Consider a surface S, and its fundamental group $\pi_1(S)$ (cf lectures F Fanoni). Consider a discrete and faithful representation $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$ of $\pi_1(S)$ as a discrete subgroup of $PSL(2,\mathbb{R})$. This allows to put a structure of Riemann surface on S, and consider S as $\mathbb{D}/\Gamma$, with $\Gamma = \pi_1(S)$. We will study the geodesic flow on the unit tangent bundle $T^1S \simeq T^1\mathbb{D}/\Gamma$, and show how the above product structure allows to prove chaotic properties as abundance of periodic orbits, positive entropy, mixing, ...

2.4 What does mean chaotic behaviour in ergodic theory (15')

Consider a flow $(\phi_t)_{t\in\mathbb{R}}$ on a space X. In dynamics, we are interested in the long time behaviour of orbits $\{\phi_t(x), t \in \mathbb{R}\}$.

This behaviour is considered as chaotic when there are many different trajectories. This can be quantified in different ways.

- (infinitely) many periodic orbits, of different lengths
- (infinitely) many different dense orbits, in different ways
- positive entropy : one fixes a precision ϵ, and count how many distinct orbits of length T one can see at the precision ϵ. Entropy is the exponential growth rate of this number.
- Topological mixing : for all open sets U and V there exists $T_0 > 0$ such that for every $t \geq T_0$, $\phi_t U \cap V \neq \emptyset$. Starting from everywhere, after a while, you can go everywhere.
- $\cdots$ (a researcher presents only the notions that are familiar to her)

In ergodic theory, one studies the dynamics from the statistical point of view, thanks to invariant measures. An invariant (probability) measure is a probability measure m on X such that for every $t \in \mathbb{R}$, $(\phi_t)_* m = m$. In other words, for every continuous map $\psi : X \to \mathbb{R}$ and every $t \in \mathbb{R}$, $\int_X \psi \circ \phi_t \, dm = \int \psi \, dm$. Or for every Borel set $A \subset X$, $m(\phi_{-t}A) = m(A)$.

Exercise 2.4 *Consider the flow ϕ_t on $\mathbb{T}^2$ defined by $\phi_t(x) = x + v \bmod \mathbb{Z}^2$. Show that the Lebesgue measure on the torus is invariant under the flow.*
hint : come back to $\mathbb{R}^2$.

A measure is said *ergodic* if it is not possible to partition X into two flow-invariant sets $X = A \sqcup A^c$ in a non trivial way. In other words, if $\phi_{-t}A = A$ for every $t \in \mathbb{R}$, then either $m(A) = 0$ or $m(A^c) = 0$. It is the basic irreducibility assumption in ergodic theory.

The first ergodic theorem is

Theorem 2.1 (Von Neumann, Birkhoff) *If (X, T, μ) is ergodic, if $\psi : X \to \mathbb{R}$ is an integrable map, then for μ-almost every $x \in X$, the ergodic average*

$$\frac{1}{T} \int_0^T \psi(\phi_t(x)) \, dt$$

converges to $\int \psi \, d\mu$.

In other words, if $A \subset X$ is a Borel set ([1]), μ almost surely, the average time spent by $(\phi_t(x))_{0 \le t \le T}$ in A converges to $\mu(A)$.

Exercise 2.5 *Show that the two versions of the above theorem are equivalent.*

In particular, the abundance of distinct ergodic invariant measures will imply the abundance of orbits typical for these distinct measures, and therefore with distinct behaviours. Let $\mathcal{M}^1$ be the set of invariant probability measures on X. It is a convex set, and *if X is compact*, it is compact wrt the weak $*$ topology ($\mu_n \to \mu$ iff for every continuous map $f : X \to \mathbb{R}$, $\int f \, d\mu_n \to \int f \, d\mu$).

Exercise 2.6 *Assume that X is compact. Let (f_n) be a countable dense family of maps in $C(X, \mathbb{R})$. Show that a basis of neighbourhoods of $\mu \in \mathcal{M}^1$ for the weak $*$ topology is given by*

$$\left\{ \nu \in \mathcal{M}^1, \ \forall 1 \le i \le N \ \left| \int f_i \, d\nu - \int f_i \, d\mu \right| \le \epsilon \right\}$$

From the ergodic point of view, the dynamical system has chaotic features if

- $\mathcal{M}^1$ is large
- there are measures of positive entropy, of maximal entropy (see later)
- there are measures with positive Lyapounov exponents (outside of these lectures)
- there are mixing measures, i.e. measures such that past and future are asymtotically independent: for all Borel sets A, B,

$$\mu(A \cap \phi_{-t}(B)) \to \mu(A)\mu(B) \, .$$

[1] with $\mu(\partial A) = 0$

3 Invariant measures for the geodesic flow

3.1 Hopf coordinates 5'

Picture : disk, boundary at infinity, several geodesics with same endpoint $\xi \in S^1$.

Exercise 3.1 *If c_1, c_2 are two geodesic rays such that $d(c_1(t), c_2(t)) \to 0$ when $t \to +\infty$, then show that for every $t \geq 0$,*

$$d(c_1(t), c_2(t)) \leq e^{-t} d(c_1(0), c_2(0))$$

Hint *Use the upper half plane model and come back to two vertical rays.*

The *Busemann cocycle* is the following map, defined on $S^1 \times \mathbb{D} \times \mathbb{D}$:

$$\beta_\xi(x, y) = \lim_{t \to +\infty} d(x, c_x(t)) - d(y, c_x(t)),$$

where (c_x) is a geodesic ray at unit speed from x to ξ.

Picture

A level set of a function $x \to \beta_\xi(x, y)$ is a horocycle.

Denote by o the center of the disk $\mathbb{D}$. If $v \in T^1\mathbb{D}$, $\pi(v)$ is the basepoint of v. The Hopf coordinates are given by

$$v \in T^1\mathbb{D} \mapsto \left(v^-, v^+, \beta_{v^+}(o, \pi(v))\right) .$$

In these coordinates, the geodesic flow acts as follows. If $v \simeq (v^-, v^+, s)$, then

$$g_t(v) \simeq (v^-, v^+, s + t) .$$

Consider an isometry $\gamma \in PSL(2, \mathbb{R})$. In these coordinates it acts as follows

$$\gamma.v \simeq (\gamma v^-, \gamma v^+, s + \beta_{v^+}(\gamma^{-1} o, o)$$

Exercise 3.2 *Check and prove the above formulas.*

3.2 Invariant measures and geodesic currents 5'

Consider now a hyperbolic surface $S = \mathbb{D}/\Gamma$, with Γ a discrete subgroup of $PSL(2, \mathbb{R})$ without torsion. The unit tangent bundle T^1S identifies with $T^1\mathbb{D}/\Gamma$.

A (g^t) invariant Radon measure m on T^1S can be lifted into a (g^t)-invariant and Γ-invariant Radon measure $\tilde{m}$ on $T^1\mathbb{D}$. In the Hopf coordinates, we deduce that $H_*\tilde{m}$ can be written as

$$H_*\tilde{m} = \mathcal{C} \times dt$$

where $\mathcal{C}$ is a *geodesic current*, i.e. a Γ-invariant Radon measure on $S^1 \times S^1$.

Exercise 3.3 *Provide details to the above assertion.*

NB : for geometers, a geodesic current is usually a measure on $S^1 \times S^1 / \sim$ where $(x, y) \sim (y, x)$.

Said differently, understanding invariant measures on T^1S and understanding the geodesic currents on $S^1 \times S^1$ (that give zero measure to the diagonal) is essentially the same.

NB : When S is not closed, one does not see on $\mathcal{C}$ if m will be finite or not.

3.3 Patterson Sullivan Gibbs construction

Exercise 3.4 *Show that if Γ contains at least two hyperbolic isometries with distinct axes, then there does not exist Γ-invariant probability measures on S^1.*

We shall construct a family of measures on T^1S by constructing first quasi-invariant measures on S^1, second geodesic currents on $S^1 \times S^1$.

3.3.1 Hölder maps, Poincare series

Consider a Hölder continuous map $f : T^1S \to \mathbb{R}$, and its Γ-invariant lift $\tilde{f}$ on $T^1\mathbb{D}$. If $a, b \in \mathbb{D}$, denote by $\int_a^b f$ the integral of f along the unique geodesic from a to b. More precisely, if $c : [0, d(a,b)] \to \mathbb{D}$ is this geodesic,

$$\int_a^b f := \int_0^{d(a,b)} f(c'(t))dt$$

If you are very new in the subject, feel free to consider $f \equiv 0$.

Define the Poincaré series associated with (Γ, f) as

$$P_{(\Gamma,f)}(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^\gamma \tilde{f}}$$

Let

$$\delta^f = \lim_{t\to\infty} \frac{1}{t} \log \sum_{\gamma\in\Gamma, d(o,\gamma o)\in[t,t+1]} e^{\int_o^{\gamma o} \tilde{f}}$$

Exercise 3.5 *Show that this series converges for $s > \delta^f$ and diverges for $s < \delta^f$.*

Define a probability measure ν_f^s on $\mathbb{D} \subset \mathbb{D} \cup \partial\mathbb{D}$ by

$$\nu_s^f = \frac{1}{P_{(\Gamma,f)}(s)} \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^\gamma \tilde{f}} \mathcal{D}_{\gamma o}$$

where $\mathcal{D}_x$ is the Dirac measure at the point x.

By compactness of $\mathbb{D} \cup S^1$, one can find decreasing sequences $s_n \to \delta^f$ such that $\nu_{s_n}^f \to \nu^f$.

Theorem 3.1 (Patterson) *One can modify P slightly without changing δ^f and get that $P_{(\Gamma,f)}$ diverges at $s = \delta^f$.*

Exercise 3.6 *Deduce that ν^f is supported on S^1*

Define the *limit set* $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o$. It is the smallest Γ-invariant set on S^1.
PICTURE.

Exercise 3.7 *Show that ν^f gives full measure to Λ_Γ.*

Denote by ρ^f the cocycle on $S^1 \times \mathbb{D} \times D$ defined by

$$\rho_\xi^f(x,y) = \lim_{t\to\infty} \int_x^\xi \tilde{f} - \int_y^\xi \tilde{f} \quad \text{and} \quad \beta^f = \delta^f\beta - \rho^f$$

Exercise 3.8 *Use the geodesic rays c_x and c_y from x (resp y) to ξ to give a rigorous meaning to the above expression.*

Exercise 3.9 *Show that the measure ν^f is Γ quasi invariant and that for a.e. ξ and all $\gamma \in \Gamma$,*

$$\frac{d\gamma_* \nu^f}{d\nu^f}(\xi) = \exp(-\beta^f_\xi(\gamma o, o)) = \exp(-\delta^f \beta_\xi(\gamma o, o) + \rho^f_\xi(\gamma o, o)).$$

A key property of the measure is the so-called Shadow lemma. A shadow $\mathcal{O}_x(B(y,R))$ is the set of points $\xi \in S^1$ such that

Theorem 3.2 (Sullivan, Hamenstadt, Ledrappier (?)) *There exists R_0 such that for $R \geq R_0$, there exists $C > 0$ such that for every $\gamma \in \Gamma$,*

$$\frac{1}{C}\exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right) \leq \nu^f(\mathcal{O}_o(B(\gamma o, R)) \leq C\exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right)$$

Proof :

$$\nu^f(\mathcal{O}_o(B(\gamma o, R)) = \gamma_*\nu^f(\gamma^{-1}(\mathcal{O}_o(B(\gamma o, R)) = \gamma_*\nu^f(\mathcal{O}_{\gamma^{-1}o}(B(o,R))).$$

Now, on $\mathcal{O}_{\gamma^{-1}o}(B(o,R))$ the Radon Nikodym derivative $d\gamma_*\nu^f/d\nu^f$ is uniformly close to $\exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right)$. We deduce that

$$\nu^f(\mathcal{O}_o(B(\gamma o, R)) \asymp exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right) \times \nu^f(\mathcal{O}_{\gamma^{-1}o}(B(o,R)).$$

It remains to show that this measure is uniformly bounded from above and below. The upper bound, 1, is obvious. The lower bound is more subtle. It uses the fact (not proven here) that ν^f has no atoms, to deduce that for R large enough, there exists some $\alpha > 0$, such that for every $y \in \mathbb{D} \cup S^1$, $\nu^f(\mathcal{O}_y(B(o,R)) \geq \alpha > 0$

□

3.3.2 Product measure

Exercise 3.10 *Show that the measure $\mathcal{C}^f$ on $S^1 \times S^1$ defined by*

$$d\mathcal{C}^f(\xi,\eta) = \exp\left(\beta^f_\eta(o,x) + \beta^f_\xi(o,x)\right) d\nu^f(\xi)\, d\nu^f(\eta), \quad \textit{for any point } x \in (\xi,\eta),$$

is a geodesic current. (Admit that it gives zero measure to the diagonal of $S^1 \times S^1$.)

It allows to define a measure $\tilde{m}^f$ on $T^1\mathbb{D}$, as

$$\tilde{m}^f = (H^{-1})_*(\tilde{C}^f \otimes dt).$$

This measure $\tilde{m}^f$ is Γ-invariant and (g^t) invariant.Therefore it induces a measure m^f on $T^1S = T^1\mathbb{D}/\Gamma$, that is a Radon measure, i.e. gives finite mass to compact sets.

Theorem 3.3 *$m^f = m^g$ iff $f = g + cste + coboundary$, where "coboundary" means a function which is the derivative of another map in the direction of the flow.*

NB : If S is compact, T^1S is also compact, and therefore m^f is finite.

Exercise 3.11 *Show that the measure m^f is supported on*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma \subset T^1S$$

Exercise 3.12 *Show that the surface $S = \mathbb{D}/\Gamma$ is convex-cocompact if and only if $H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})$ is cocompact, i.e.*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma$$

is compact.

Hint*: First observe that $\Omega \subset T^1\mathcal{C}^{core}/\Gamma$ and deduce that one direction of the equivalence is easy. For the other direction, use the fact that triangles are thin to show that any point of C^{core} is at uniformly bounded distance of a geodesic joining two points of Λ_Γ.*

NB : If S is convex cocompact, then m^f is finite.

If S is not compact, there are criteria, and sufficient conditions, to ensure that m^f is finite, and examples where it is (or not) the case.

When it is finite, we assume that this measure is **normalized** into a probability measure.

These family of measures have many interesting features.

Theorem 3.4 (Hopf, Sullivan, Hamenstadt, Ledrappier, Babillot, Otal-Peigné, PPS...) *If m^f is finite, then it satisfies the following properties :*

- *it is ergodic (Hopf argument) ,*
- *it is mixing, (Babillot)*
- *when S is compact, it is exponentially mixing (Dolgopyat)*
- *It is the unique measure maximizing the pressure, i.e. realizing the following supremum :*

$$\delta^f = \sup_{m \in \mathcal{M}^1} \left(h(m) + \int f \, dm \right)$$

- *It satisfies the Gibbs property : for every $v \in T^1S$, denote by $B(v, T, \epsilon) = \{w \in T^1S, \forall 0 \le t \le T, d(g^t v, g^t w) \le \epsilon)\}$. Then for* **every** *$v \in T^1S$,*

$$m^f(B(v, T, \epsilon)) \asymp \exp\left(-\delta^f T + \int_0^T f(g^t v)\right) dt$$

- *When $f = 0$ we recover the so-called Bowen-Margulis-Sullivan measure.*
- *Weighted equidistribution of periodic orbits, counting (PPS, Schapira-Tapie)*
- *More interesting examples later.*
- *These measures are involved in many deformation problems.*

3.4 The Hopf argument for ergodicity

Exercise 3.13 *The measure m^f is ergodic iff it satisfies the conclusion of Birkhoff ergodic theorem.*

Let us prove that when m^f is finite, then m^f is ergodic.

First, the measure m^f (when finite) always satisfies the following property (admitted).

Theorem 3.5 (Non ergodic Birkhoff Theorem) *every $\psi \in L^1(m^f)$,and a.e. $v \in T^1 S$,*

$$\frac{1}{T}\int_0^T \psi \circ g^t v\, dt \to \mathbb{E}(\psi|\mathcal{I})(v) \quad \textit{when} \quad T \to \pm\infty$$

Showing ergodicity is therefore equivalent to show that $E(\psi|\mathcal{I}) \equiv \int \psi, dm^f$ for every $\psi \in L^1(m^f)$.

Exercise 3.14 *By density, it is enough to prove that $E(\psi|\mathcal{I}) = \int \psi\, dm^f$ for every $\psi \in C_c(T^1 S)$.*

For $\psi \in C_c(T^1 S)$, define

$$\psi^{\pm}(v) = \limsup_{T\to\pm\infty} \frac{1}{T}\int_0^T \psi \circ g^t v dt$$

Exercise 3.15 *Prove that ψ^+ and ψ^- are (g^t) invariant.*
Prove that if v and w are on the same stable horocycle, i.e. $d(g^t v, g^t w) \to 0$ when $t \to +\infty$, then $\psi^+(v) = \psi^+(w)$. Prove the analogous property for ψ^- when $t \to -\infty$.

By definition of $\psi^{\pm}$ and $E(\psi|\mathcal{I})$, we have

$$\psi^+ = E(\psi|\mathcal{I}) \quad m^f - a.e. \quad \text{and} \quad \psi^- = E(\psi|\mathcal{I}) \quad m^f - a.e.$$

Lift these functions on $T^1\mathbb{D}$ (as Γ-invariant functions), and use Hopf coordinates. We deduce that $E(\psi|\mathcal{I})$ is (g^t)-invariant, and therefore does not depend on the real coordinate. Moreover, m^f almost surely it depends only on v^-, and m^f-a.e. it depends only on v^+.

As m^f has a product structure, by a Fubini like argument, we deduce that it depends neither on s nor on v^- or v^+, i.e. it is constant. See my notes of my last lectures at CIRM 10 years ago here
`https://perso.univ-rennes1.fr/barbara.schapira/recherche/texteCIRM-Hasselblatt.pdf`

3.5 (Local) entropy of a measure

We introduced earlier dynamical balls $B(v, T, \epsilon)$.

Exercise 3.16 *Try to show a property like*

$$B(v, T, \epsilon) \asymp \mathcal{O}_{\pi(v)}(B(\pi(g^T v), \epsilon)) \times \mathcal{O}_{\pi(g^T v)}(B(\pi(v), \epsilon)) \times [-\epsilon, \epsilon]$$

The topological entropy of the flow can be defined as

$$h_{top}(g^t) = \sup_K \limsup \frac{1}{T} \log N(K, T, \epsilon)$$

where $N(K, T, \epsilon)$ is the maximal number of disjoint dynamical balls $B(v, T, \epsilon)$ included in the compact set K / the min number of dyn balls needed to cover K.

The Katok entropy of an ergodic invariant probability measure is defined as

$$h_{Kat}(m) = \limsup_T \frac{1}{T} \log M(T, \epsilon, \alpha)$$

where $M(T, \epsilon, \alpha)$ is the min number of dynamical balls needed to cover a set of mass at least α.

The local (upper) Brin Katok entropy of an ergodic invariant probability measure is defined as

$$h_{loc}(m) = \text{supess}_{v\in T^1S} \limsup_{T\to\infty} -\frac{1}{T} \log m(B(v, T, \epsilon))$$

Theorems of Katok, Brin-Katok and Riquelme in the noncompact case ensure that these entropies coincide with the usual (Kolmogorov Sinai entropy) and we have

Theorem 3.6 (OP, PPS) *When m^f is finite, it is the unique measure realizing the supremum below.*

$$\delta^f = \sup_{m \in \mathcal{M}^1_{erg}} \left(h(m) + \int f \, dm \right) = h(m^f) + \int f \, dm^f \, .$$

Theorem 3.7 (GST, unpublished) *Gibbs measures have positive entropy*

4 Change of metric and Gibbs measures

4.1 Identifying boundaries

Consider a closed surface with genus ≥ 2, and two hyperbolic metrics, denoted by g_1 and g_2, on S. Consider the universal cover $\tilde{S}$ as a differentiable surface, endowed with two distinct Riemannian metrics, instead of $\mathbb{D}$. Observe that the tangent bundle is a differentiable notion, but the unit tangent bundle depends on the metric. Thus we will denote them by $T^1_{g_i}\tilde{S}$ and $T^1_{g_i}S$.

A classical result ensures that g_1-geodesics and g_2-geodesics stay at uniformly bounded distance. Therefore, $\partial_\infty(\tilde{S}, g_1) \simeq \partial_\infty(\tilde{S}, g_2)$. We denote them by $\partial\tilde{S}$.

A contrario, observe that the Busemann cocycle $\beta^{g_i}_\xi(x, y)$ is a metric notion.

Moreover, as S is compact, the limit set satisfies $\Lambda_\Gamma = \partial\tilde{S}$.

The Hopf coordinates give two distinct homeomorphisms

$$H_1 : \; T^1_{g_1}\tilde{S} \;\rightarrow\; \partial\tilde{S} \times \partial\tilde{S} \times \mathbb{R} \;\leftarrow\; T^1_{g_2}\tilde{S} \;: H_2$$

Therefore, there is a $1-1$-correspondance between $\mathcal{M}^1(T^1_{g_1}S)$ and $\mathcal{C}urrents(\partial\tilde{S} \times \partial\tilde{S})$ on the one hand, and between $\mathcal{M}^1(T^1_{g_2}S)$ and $\mathcal{C}urrents(\partial\tilde{S} \times \partial\tilde{S})$ on the other hand.

Remark 4.1 Be very careful about the fact that the action of Γ on $\partial\tilde{S} \times \partial\tilde{S}$ does not depend on the metric g_i but the action of Γ on $\partial\tilde{S} \times \partial\tilde{S}$ involves the Busemann cocyle, and therefore does depend on the metric.

4.2 The set of invariant measures does not depend (so much) on the metric

As said above, there is a $1-1$-correspondance between $\mathcal{M}^1_{g_1}(T^1_{g_1}S)$, $\mathcal{C}urrents(\partial\tilde{S} \times \partial\tilde{S})$, and $\mathcal{M}^1_{g_1}(T^1_{g_1}S)$.

For a given current, say $\mathcal{C}$ in $\mathcal{C}urrents(\partial\tilde{S} \times \partial\tilde{S})$, denote by $m^{g_i}_{\mathcal{C}}$ the associated (g_i^t)-invariant measures.

It is natural to ask which properties depend really on the metric g_i and which are independent of the metric.

We have the following

Theorem 4.2

- *$m^{g_1}_{\mathcal{C}}$ and $m^{g_2}_{\mathcal{C}}$ are simultaneously periodic*
- *$m^{g_1}_{\mathcal{C}}$ and $m^{g_2}_{\mathcal{C}}$ are simultaneously ergodic*
- *$m^{g_1}_{\mathcal{C}}$ and $m^{g_2}_{\mathcal{C}}$ are simultaneously of full support*

- *$m_\mathcal{C}^{g_1}$ and $m_\mathcal{C}^{g_2}$ are simultaneously product measures or not.*
- *$m_\mathcal{C}^{g_1}$ and $m_\mathcal{C}^{g_2}$ have simultaneously positive (or zero) entropy)*
- *$m_\mathcal{C}^{g_1}$ and $m_\mathcal{C}^{g_2}$ are simultaneously Gibbs measures (or not), for different potentials.*

Exercise 4.1 *Prove the first four properties. (The last two are proven in [Schapira-Tapie]*

4.3 Geodesic stretch

Proposition 4.3 (Ledrappier, ?) *The following quantity is well defined, for every $v \in T^1_{g_1}\tilde{M}$,*

$$\mathcal{E}^{g_1\to g_2}(v) = \frac{d}{dt}_{|t=0} \beta^{g_2}_{v_+^{g_1}}(\pi(v), \pi(g_1^t v))$$

We call it the infinitesimal geodesic stretch

Given a geodesic current $\mathcal{C}$ and the associated invariant measures $m_\mathcal{C}^{g_i}$, we define the average

$$I_\mathcal{C}(g_1, g_2) = \frac{1}{\|m_\mathcal{C}^{g_1}\|} \int_{T^1_{g_1}S} \mathcal{E}^{g_1\to g_2}\, dm_\mathcal{C}^{g_1}$$

PICTURE

As noticed above, the actions of Γ on $\partial\tilde{S} \times \partial\tilde{S} \times \mathbb{R}$ associated with g_1 and g_2 are not the same. Therefore, the map $\Phi^{g_1\to g_2} := H^{g_2})^{-1} \circ H^{g_1} : T^1_{g_1}\tilde{S} \to T^1_{g_2}\tilde{S}$ is not Γ-equivariant (but satisfies $g_2^t \circ \Phi^{g_1\to g_2} = \Phi^{g_1\to g_2} \circ g_1^t$).

One can define a Γ-equivariant map, called Morse correspondance, as follows. $\Psi^{g_1\to g_2}(v)$ is the unique vector w on the g_2 geodesic joining $v_{g_1}^-$ to $v_{g_1}^+$ such that $\beta^{g_2}_{v_{g_1}^+}(\pi(v), \pi(w)) = 0$.

Then one can show the following properties :

Theorem 4.4

- $\int_{T^1_{g_2}S} G\, dm_\mathcal{C}^{g_2} = \int_{T^1_{g_1}S} G \circ \Psi^{g_1\to g_2} \times \mathcal{E}^{g_1\to g_2}\, dm_\mathcal{C}^{g_1}$
- $\|m_\mathcal{C}^{g_2}\| = \frac{1}{\|m_\mathcal{C}^{g_1}\|} \int_{T^1_{g_1}S} \mathcal{E}^{g_1\to g_2}\, dm_\mathcal{C}^{g_1} \times \|m_\mathcal{C}^{g_1}\|$
- *If (γ_k) is a sequence of periodic orbits such that $\frac{d\ell^{g_1}_{\gamma_k}}{\ell^{g_1}(\gamma_k)}$ converges to $\frac{1}{\|m_\mathcal{C}^{g_1}\|} m_\mathcal{C}^{g_1}$, then $\frac{d\ell^{g_2}_{\gamma_k}}{\ell^{g_2}(\gamma_k)}$ converges to $\frac{1}{\|m_\mathcal{C}^{g_1}\|} m_\mathcal{C}^{g_2}$ AND $\ell^{g_2}(\gamma_k)/\ell^{g_1}(\gamma_k)$ converges to $\frac{1}{\|m_\mathcal{C}^{g_1}\|} \int_{T^1_{g_1}S} \mathcal{E}^{g_1\to g_2}\, dm_\mathcal{C}^{g_1}$.*
- $h(m_\mathcal{C}^{g_1}) = \frac{1}{\|m_\mathcal{C}^{g_1}\|} \int_{T^1_{g_1}S} \mathcal{E}^{g_1\to g_2}\, dm_\mathcal{C}^{g_1} \times h(m_\mathcal{C}^{g_2})$.
- *If $m_\mathcal{C}^{g_1} = m^f$ is the Gibbs measure associated with $f : T^1_{g_1}S \to \mathbb{R}$, then $m_\mathcal{C}^{g_2}$ is also a Gibbs measure, associated with the Hölder potential*

$$g = (f - \delta^f_{g_1}) \circ \Psi^{g_2\to g_1} \times \mathcal{E}^{g_2\to g_1}$$

Corollary 4.5 *Consider the map $f \equiv 0$ on $T^1_{g_1}S$. The measure m^0 is called the Bowen-Margulis-Sullivan measure. As S is a hyperbolic surface of finite measure, it coincides with the Lebesgue / Liouville / Haar measure. Denote by $\mathcal{C}^{g_1}_{BMS}$ the associated current. Then for any periodic orbit p and any vector $w \in T^1_{g_2}$ on it, we have*

$$m^{g_2}_{\mathcal{C}^{g_1}_{BMS}}(B^{g_2}(w_\gamma, T, \epsilon) \asymp e^{-\delta^{g_1}_\Gamma T \frac{\ell^{g_1}(p)}{\ell^{g_2}(p)}}$$

The key argument in all proofs is very elementary : A shadow for g_1 is a shadow for g_2, but at different distances. It allows to show that a dynamical ball $B^{g_1}(v, T, \epsilon)$ is sent by $\Psi^{g_1 \to g_2}$ to a dynamical ball $B^{g_2}(\Psi^{g_1 \to g_2}(v), S, \epsilon')$, and S/T is essentially the ergodic average $\frac{1}{T} \int_0^T \mathcal{E}^{g_1 \to g_2}(g_1^t v)\, dt$.

Our initial goal was to prove

Theorem 4.6 (Katok Knieper Weiss, Schapira Tapie) (g_ϵ) C^1 *variation of the metric. Then* $\epsilon \to \delta_\Gamma^{g_\epsilon}$ *is* C^1 *and*

$$\frac{d}{dt}_{|t=0} \delta_\Gamma(g^\epsilon) = -\delta_\Gamma^{g_0} \times \int_{T^1_{g_0} S} \frac{d\|v\|^{g_\epsilon}}{d\epsilon} \times dm^{g_0}_{BMS}$$

5 Higher Teichmuller spaces and Gibbs measures

We saw rank one situations where Gibbs measures are useful. They are also useful in higher rank, for proving several statements. To give a flavour, let me state a few results.

5.1 Examples of statements

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Let $\rho : \Gamma = \pi_1(S) \to PGL(d, \mathbb{R})$ be a deformation of $\rho_0 \pi_1(S) \to PGL(2, \mathbb{R}) \to PGL(d, \mathbb{R})$ where $PGL(2, \mathbb{R}) \to PGL(d, \mathbb{R})$ is the irreducible embedding. Assume that the representation is P_1-Anosov. There exists $h > 0$ and $c > 0$ such that

$$\#\{\gamma \in \Gamma, \|\rho(\gamma)\| \leq R\} \sim \frac{R^h}{c} .$$

And h is independent of the norm chosen on $\mathbb{R}^d$.

$$\#\{[\gamma] \in [\Gamma], \lambda_1(\rho(\gamma)) \leq t\} \sim \frac{e^{ht}}{ht}$$

Sambarino

5.2 A few words on the strategy

Representation $\rho \to$ cocycle $\to$ potential $\to$ measure $\to$ many properties.