

Regularity of entropy, geodesic currents and entropy at infinity

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Abstract

In this work, we introduce the notion of *entropy at infinity*, and define a wide class of noncompact manifolds with negative curvature, those which admit a critical gap between entropy at infinity and topological entropy. We call them *strongly positively recurrent manifolds* (SPR), and provide many examples. We show that dynamically, they behave as compact manifolds. In particular, they admit a finite measure of maximal entropy.

Using the point of view of currents at infinity, we show that on these SPR manifolds the topological entropy of the geodesic flow varies in a C^1 -way along (uniformly) C^1 -perturbations of the metric. This result generalizes former work of Katok (1982) and Katok-Knieper-Weiss (1991) in the compact case.

(¹)

1 Introduction

1.1 Variation of the topological entropy : An overview

The initial motivation of this work was to answer the following simple question. Consider a hyperbolic surface of finite volume and a smooth compact perturbation of the metric. Does the topological entropy of the geodesic flow vary regularly ? More generally, what happens for a smooth perturbation of the metric of a noncompact negatively curved Riemannian manifold ?

The answer has been known on compact manifolds since almost thirty years [KKW91, KKPW89, Fla95], and has been extended to the convex-cocompact case in [Tap11]. A similar argument gives the regularity of the topological entropy for a perturbation of an Anosov flow, cf [KKW91].

Compactness of the underlying space is crucial in the above results, and no result was known until now for manifolds with a non-compact non-wandering set. Even the case of a smooth compact perturbation of the metric of a finite volume hyperbolic surface was not accessible with their arguments. Let us recall the two main steps of their argument to understand why.

The key step is the following inequality, due to Katok in [Kat82] for surfaces, extended in [KKW91] to all dimensions.

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Theorem 1.1 (Katok 1982 ; Katok-Knieper-Weiss 1991). *Let g_1, g_2 be Riemannian metrics with negative sectional curvature on the same compact manifold M . Then the entropies of their geodesic flows satisfy*

$$h_{top}(g_1) \leq h_{top}(g_2) \times \int_{S^{g_1}M} \|v\|^{g_2} d\bar{m}_{BM}^{g_1}(v), \quad (1)$$

where $\|v\|^{g_2} = \sqrt{g_2(v, v)}$ and $\bar{m}_{BM}^{g_1}$ is the normalized Bowen-Margulis measure on the g_1 -unit tangent bundle $S^{g_1}M$ for the g_1 -geodesic flow.

Reversing the role of g_1 and g_2 also provides a lower bound for $h_{top}(g_1)$, and a first order power expansion gives the following smoothness result.

Theorem 1.2 (Katok-Knieper-Weiss 1991). *Let $(g_\lambda)_{\lambda \in (-1,1)}$ be a \mathcal{C}^2 -family of \mathcal{C}^2 Riemannian metrics with negative sectional curvature on the same compact manifold M . Then $\lambda \mapsto h_{top}(g_\lambda)$ is \mathcal{C}^1 , and its derivative is given by*

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} h_{top}(g_\lambda) = -h_{top}(g_0) \times \int_{S^{g_0}M} \left. \frac{d}{d\lambda} \right|_{\lambda=0} \|v\|^{g_\lambda} d\bar{m}_{BM}^{g_0}(v). \quad (2)$$

where $\bar{m}_{BM}^{g_0}$ is the normalized Bowen-Margulis measure on the g_0 -unit tangent bundle $S^{g_0}M$ for the g_0 -geodesic flow.

In the previously quoted works, the proofs of (1) strongly use the compactness of the non-wandering set. In the first part of our paper, we use a different approach to generalize it to the non-compact setting. This improves it even in the compact case, providing an explicit transformation rule for the entropies, equality which immediately implies (1), and has other interesting consequences.

The previously known proofs of (2) use the compactness of M for a crucial point: to ensure the finiteness and the continuity of the normalized Bowen-Margulis measures $\bar{m}_{BM}^{g_\lambda}$ in the weak-*topology as λ varies. Neither finiteness of the Bowen-Margulis measure nor its continuity under a variation of the metric can be ensured in general. Maybe the most striking fact of our work is that we introduce a new wide class of manifolds, which we call *SPR manifolds*, SPR meaning *strongly / stably positively recurrent*. The terminology *Stably positively recurrent* has been introduced by Gurevic-Savchenko [GS98] in the context of countable Markov shifts. Sarig [Sar01] modified it, in the same context, into *strongly positively recurrent*, terminology which has been used later by other authors as Buzzi [BBG14]. See also the very recent work of Velozo [Vel17], who follows also this terminology. Both terminologies are meaningful, and had not yet been considered in a geometric context.

The class of SPR manifolds that we define here has the remarkable property that the Bowen-Margulis measure is finite, and moreover stays finite and varies continuously along small perturbations. In particular, under uniformly \mathcal{C}^2 variation of such SPR Riemannian metrics, the topological entropy is \mathcal{C}^1 and its derivative is given by (2).

These SPR manifolds include finite volume hyperbolic manifolds, and more generally almost all known examples where the geodesic flow admits a (finite) measure of maximal entropy, as geometrically finite negatively curved manifolds with spectral gap [DOP00], Schottky product examples from [Pei03], and unpublished examples of Ancona [Anc]. The class of SPR manifolds is much larger than only the above mentioned examples. We postpone the extensive

study of SPR manifolds to a later paper [ST]. Therefore, the second half of our paper will be devoted to the presentation of a geometrical setting, as large as possible, where this finiteness and continuity of Bowen-Margulis measures can be ensured.

Let us now present our main results with more details.

1.2 Invariant measures and change of Riemannian metrics

Let (M, g_1) be a complete Riemannian manifold, and g_2 be another Riemannian metric on M such that there exists $C > 1$ with $\frac{1}{C}g_1 \leq g_2 \leq Cg_1$. We assume moreover that both g_1 and g_2 have pinched negative sectional curvatures with uniformly bounded derivatives: this implies that g_1 -geodesics are g_2 -quasi-geodesics and the visual boundary of the universal cover (\widetilde{M}, g_1) is canonically identified with the visual boundary of (\widetilde{M}, g_2) ; we will denote it by $\partial\widetilde{M}$. We will use extensively this correspondance to compare the dynamics of the geodesic flows on $S^{g_1}M$ and $S^{g_2}M$.

Let $\Gamma = \pi_1(M)$ acting on the universal cover \widetilde{M} , let m be a locally finite measure on $S^{g_1}M$, invariant by the geodesic flow $(g_1^t)_{t \in \mathbb{R}}$, and \tilde{m} its lifts to $S^{g_1}\widetilde{M}$. We write $\partial^2\widetilde{M} = (\partial\widetilde{M} \times \partial\widetilde{M}) \setminus \text{Diag}$. In g_1 -Hopf coordinates (cf Section 2), $S^{g_1}\widetilde{M} \simeq \partial^2\widetilde{M} \times \mathbb{R}$, and \tilde{m} has a local product structure of the form $d\tilde{m} = d\mu \times dt$, where μ is a Γ -invariant *geodesic current* on $\partial^2\widetilde{M}$. We write therefore $m = m_\mu^{g_1}$.

We can now define a measure $\tilde{m}_\mu^{g_2}$ on $S^{g_2}\widetilde{M}$, given in g_2 -Hopf parametrization by the same local product formula $\tilde{m}_\mu^{g_2} = d\mu \times dt$: by Γ -invariance, this induces a locally finite measure $m_\mu^{g_2}$ on $S^{g_2}M$, which is invariant for the geodesic flow $(g_2^t)_{t \in \mathbb{R}}$. The ergodic properties of $(S^{g_1}M, g_1^t, m_\mu^{g_1})$ and $(S^{g_2}M, g_2^t, m_\mu^{g_2})$ are strongly related.

Well known facts imply that if $m_\mu^{g_1}$ and $m_\mu^{g_2}$ are finite then one is ergodic or conservative if and only if the other is. The reader may believe that, since $\frac{1}{C}g_1 \leq g_2 \leq Cg_1$, then $m_\mu^{g_1}$ is finite if and only if $m_\mu^{g_2}$ is. We will indeed show that it is the case and relate the masses and entropies of these measures.

In this purpose, let us introduce the *instantaneous geodesic stretch* $\mathcal{E}^{g_1 \rightarrow g_2} : S^{g_1}\widetilde{M} \rightarrow \mathbb{R}$ defined for all $v \in S^{g_1}\widetilde{M}$ by

$$\mathcal{E}^{g_1 \rightarrow g_2}(v) = \frac{d}{dt} \Big|_{t=0^+} \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi v, \pi g_1^t \tilde{v}) = \frac{d}{dt} \Big|_{t=0^+} \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi g_1^t \tilde{v}),$$

where $\mathcal{B}_{v_+^{g_1}}^{g_2}(\cdot, \cdot)$ is the Busemann function for g_2 based at the end point of the g_1 -geodesic generated by v . By Γ invariance, it induces a map $\mathcal{E}^{g_1 \rightarrow g_2} : S^{g_1}M \rightarrow \mathbb{R}$. We will see in Section 2.3 that this is the derivative along g_1 -geodesics of a natural *Morse correspondance* $\Psi^{g_1 \rightarrow g_2} : S^{g_1}M \rightarrow S^{g_2}M$ between the g_1 and g_2 geodesic flows. This implies the following.

Proposition 1.3. *For all $m_\mu^{g_2}$ -measurable map $G : S^{g_2}M \rightarrow \mathbb{R}$, the map $G \circ \Psi^{g_1 \rightarrow g_2}$ is $m_\mu^{g_1}$ -measurable and*

$$\int_{S^{g_2}M} G dm_\mu^{g_2} = \int_{S^{g_1}M} G \circ \Psi^{g_1 \rightarrow g_2} \times \mathcal{E}^{g_1 \rightarrow g_2} dm_\mu^{g_1}.$$

In particular, the masses of $m_\mu^{g_2}$ satisfies

$$\|m_\mu^{g_2}\| = \int_{S^{g_1}M} \mathcal{E}^{g_1 \rightarrow g_2} dm_\mu^{g_1}.$$

Some other versions of the geodesic stretch has already been considered in [FF93] of [Kni95]; we explain in Section 2.3 the relationship with these references and the interest of our new definition. We then introduce in Section 3 a notion of *local entropy* for invariant measures, which is an analogous in the non-compact setting to Brin-Katok entropy, and which coincides with the classical metric entropy for Gibbs measures. This also allows us to relate the local entropies of $(S^{g_1}M, g_1^t, m_\mu^{g_1})$ and $(S^{g_2}M, g_2^t, m_\mu^{g_2})$.

Theorem 1.4. *Under the previous notations, the local entropies of $(g_1^t, m_\mu^{g_1})$ and $(g_2^t, m_\mu^{g_2})$ are related as follows.*

$$h_{loc}(m_\mu^{g_2}, g_2) = I_\mu(g_2, g_1) \cdot h_{loc}(m_\mu^{g_1}, g_1).$$

In particular, the combination of the previous theorem with the variational principle implies that

$$h_{top}(g_2) \leq \int_{S^{g_2}M} \mathcal{E}^{g_2 \rightarrow g_1}(v) \frac{dm_\mu^{g_2}(v)}{\|m_\mu^{g_2}\|} h_{top}(g_1),$$

which is an optimal improvement of 1.

Eventually, in Section 4, we show that $m_\mu^{g_1}$ is a Gibbs measure for the potential $G : S^{g_1} \rightarrow \mathbb{R}$ if and only if $m_\mu^{g_2}$ is a Gibbs measure for the potential $G \circ \Psi^{g_2 \rightarrow g_1} \times \mathcal{E}^{g_2 \rightarrow g_1}$, and give some applications of this last fact to a comparison between the length spectrum of (M, g_1) and (M, g_2) .

1.3 Entropy at infinity, SPR manifolds and Bowen-Margulis measures

Let (M, g) be a Riemannian manifold with pinched negative sectional curvatures whose derivatives are uniformly bounded. We introduce a notion of *entropy at infinity* (see section 7), which measures the highest possible complexity of the dynamics outside a compact set in the manifold.

We call the Riemannian manifold (M, g) *strongly positively recurrent*, shortly SPR, if the entropy at infinity is strictly smaller than the topological entropy of the geodesic flow. This SPR property implies that the geodesic flow admits a measure of maximal entropy, that this fact remains true under a nice small perturbation of the metric, and that these measures vary continuously in the narrow topology. Let us summarize the main results that we establish here on the SPR property. We refer to [ST] for further study of this property and its consequences.

Theorem 1.5. *Let (M, g) be a Riemannian manifold with pinched negative curvature.*

1. *The SPR property implies that the geodesic flow admits an invariant probability measure of maximal entropy m_{BM}^g , the so-called Bowen-Margulis measure. In the terminology of [PS18], the SPR property implies that the geodesic flow is positively recurrent.*

2. Geometrically finite manifolds with critical gap (see [DOP00]) have the SPR property,
3. Topologically infinite examples of [Anc] have the SPR property,
4. Schottky product examples of [Pei03] have the SPR property.

As mentioned above, this SPR property is stable in the following sense.

Theorem 1.6. *Let (M, g_0) be a SPR manifold with pinched negative curvature and bounded derivatives of the curvature. Let $(g_\varepsilon)_{\varepsilon \in (-1, 1)}$ be a C^1 -uniform variation of the metric. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the manifold (M, g_0) is SPR. Moreover, the Bowen-Margulis measures (m_{BM}^{ε}) vary continuously in the narrow topology at $\varepsilon = 0$.*

This allows us to show the following regularity property for the topological entropy, which answers our initial question. We refer to section 7 for technical details on the assumptions. Denote by $h_{top}(g)$ the topological entropy of the geodesic flow of the metric g .

Theorem 1.7. *Let (M, g_0) be a SPR manifold with pinched negative curvature and bounded derivatives of the curvature. Let (g_ε) be a C^1 -uniform variation of the metric with negative sectional curvatures. Then the map $\varepsilon \rightarrow h_{top}(g_\varepsilon)$ is C^1 near $\varepsilon = 0$, with derivative at 0*

$$\left. \frac{d h_{top}(g_\varepsilon)}{d \varepsilon} \right|_{\varepsilon=0} = -h_{top}(g_0) \int_{S^{g_0} M} \left. \frac{d \|v\|^{g_\varepsilon}}{d \varepsilon} \right|_{\varepsilon=0} d \frac{m_{BM}^{g_0}}{\|m_{BM}^{g_0}\|},$$

the measure $\frac{m_{BM}^{g_0}}{\|m_{BM}^{g_0}\|}$ being the invariant probability measure of maximal entropy for the g_0 -geodesic flow.

Let us emphasize the fact that this theorem is valid in a much greater generality than what we thought initially possible. On the one hand, SPR manifolds are a very general and interesting class of manifolds, much larger than the well known and well studied class of finite volume, or even geometrically finite hyperbolic manifolds, as illustrated by Theorem 1.5. It may be an optimal class to get such result in the sense that we guess that phase transitions for the entropy can happen when the manifold is not SPR (see [ST]).

On the other hand, we allow much more general perturbations than only compact ones since we deal with noncompact C^2 -perturbations of our metric, as soon as they are not too wild at infinity.

The paper is organized as follows. In Section 2, we develop the point of view of geodesic currents at infinity, which allows us to associate to an invariant measure $m_\mu^{g_1}$ for the geodesic flow for (M, g_1) an invariant measure $m_\mu^{g_2}$ for the geodesic flow on (M, g_2) , and compare their ergodic properties.

In Section 3, we introduce different notions of entropy and develop methods of section 2 to relate the entropies of $m_\mu^{g_1}$ and $m_\mu^{g_2}$.

In Section 4, we recall general fact about Gibbs measures on non-compact manifolds, we show that $m_\mu^{g_1}$ is a Gibbs measure if and only if $m_\mu^{g_2}$ is and give applications to the length spectrum.

In Section 5 we show some continuity results for geodesics, Busemann functions and non-normalized Bowen-Margulis measures which will be needed in the sequel.

In Section 6, we first show that for a fixed geodesic current μ on $\partial^2\widetilde{M}$, the metric entropy $\varepsilon \mapsto h(g_\varepsilon^t, m_\mu^{g_\varepsilon})$ is \mathcal{C}^1 under a \mathcal{C}^1 -uniform variation of the Riemannian metrics g_ε . We then show in a very similar proof that, if under a \mathcal{C}^1 -uniform variation of Riemannian metrics the normalized Bowen-Margulis measures $\overline{m}_{BM}^{g_\varepsilon}$ vary continuously in the narrow topology, then the topological entropy is also \mathcal{C}^1 .

Eventually, in Section 7, we introduce entropy at infinity and SPR manifolds, we show that they have finite Bowen-Margulis measure, and that under a small \mathcal{C}^1 -uniform variation of Riemannian metrics they remain SPR. On the way, we give some properties of the entropy at infinity of independent interest.

Theorem 1.5 follows from results of section 7.3, where we provide many examples of SPR manifolds. Theorem 1.6 is a reformulation of the second part of Theorem 7.1. At last, our main variational formula for the topological entropy, Theorem 1.7, follows from Theorems 6.4 and 1.6 (or 7.1).

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2 Hopf parametrization and geodesic currents

2.1 Hopf parametrization and geodesic flow

Let (M, g_0) be a complete manifold with pinched negative sectional curvatures satisfying $-b^2 \leq K_{g_0} \leq -b^2 < 0$, and derivatives of the curvature bounded. Let \widetilde{M} be its universal cover, equipped with the lifted metric which we will still denote by g_0 , and let $\partial_{g_0}\widetilde{M}$ be its visual boundary. Let $\Gamma = \pi_1(M)$ be the fundamental group, acting properly by diffeomorphisms on \widetilde{M} . Denote by p_Γ indistinctly the projection $\widetilde{M} \rightarrow M$ and its linear tangent map $T\widetilde{M} \rightarrow TM$. A metric g on M (or equivalently, a Γ -equivariant metric g on \widetilde{M}) will be called *admissible* if it has pinched negative sectional curvature, if the derivatives of the curvature are bounded, and if there exists a constant $C_1(g_0, g) > 1$ such that at all $x \in M$,

$$\frac{1}{C_1(g_0, g)} g_0 \leq g \leq C_1(g_0, g) g_0. \quad (3)$$

This implies that g -geodesics are g_0 -quasi-geodesics, which are contained in the $C_2(g_0, g)$ -neighbourhood of g_0 -geodesics, where $C_2(g_0, g)$ only depends on $C_1(g_0, g)$ (see [BH99, Th.1.7 p401]). In particular the visual boundary $\partial_g\widetilde{M}$ of (\widetilde{M}, g) is canonically identified to the visual boundary of (\widetilde{M}, g_0) , and they will therefore both be denoted by $\partial\widetilde{M}$. Moreover, this identification is H lder continuous w.r.t the visual distances induced by both g_0 and g , so that $\partial\widetilde{M}$ has a natural H lder structure.

The *limit set* $\Lambda_\Gamma \subset \partial\widetilde{M}$ is the set of accumulation points of any orbit $\Gamma.x$ on the boundary. It does not depend either of the chosen admissible metric. The *radial limit set* $\Lambda_\Gamma^r \subset \Lambda_\Gamma$ is the set of endpoints of geodesics which, on the quotient manifold M , return infinitely often to some compact set. It does not depend either on the chosen admissible metric.

Let us fix once for all a point $o \in \widetilde{M}$. Let g be any admissible metric on M , and d^g the distance induced by g on M and \widetilde{M} . Denote by $S^g M$ (resp. $S^g \widetilde{M}$) the unit tangent bundle of (M, g) (resp. (\widetilde{M}, g)), and $\partial^2 \widetilde{M} = (\partial \widetilde{M} \times \partial \widetilde{M}) \setminus \text{Diag}$. We write $\pi : TM \rightarrow M$ and $\pi : T\widetilde{M} \rightarrow \widetilde{M}$ the projections from the tangent bundle to its base, and by $(g^t)_{t \in \mathbb{R}}$ the geodesic flow on $S^g M$ or $S^g \widetilde{M}$. For any $v \in S^g \widetilde{M}$, write v_-^g and v_+^g for the endpoints in $\partial \widetilde{M}$ of the geodesic $\{\pi g^t v; t \in \mathbb{R}\}$.

Remark 2.1. We keep track in our notations of the metric g since we will soon compare these quantities for two different admissible metrics g_1 and g_2 .

For all $\xi \in \partial \widetilde{M}$, let \mathcal{B}_ξ^g be the Busemann function at ξ defined, for any $x, y \in \widetilde{M}$, by

$$\mathcal{B}_\xi^g(x, y) = \lim_{z \rightarrow \xi} d^g(x, z) - d^g(y, z).$$

The map

$$H^g : v \mapsto \left(v_-^g, v_+^g, \mathcal{B}_{v_+^g}^g(o, \pi v) \right)$$

is a Hölder homeomorphism from $S^g \widetilde{M}$ to $\partial^2 \widetilde{M} \times \mathbb{R}$, called the *Hopf parametrization* of the unit tangent bundle.

The action of Γ by (differentials of) isometries on $S^g \widetilde{M}$ can be written in these coordinates as

$$\gamma \cdot (v_-^g, v_+^g, t) = \left(\gamma \cdot v_-^g, \gamma \cdot v_+^g, t + \mathcal{B}_{v_+^g}^g(o, \gamma^{-1} \cdot o) \right).$$

Let us emphasize the fact that this action of Γ on $\partial^2 \widetilde{M} \times \mathbb{R}$, and more specifically on the third factor, depends strongly on the cocycle \mathcal{B}^g , and therefore on the metric g .

2.2 Geodesic currents and invariant measures

In the coordinates given by the Hopf parametrization of $S^g \widetilde{M}$, the geodesic flow (g^t) acts by translation on the last factor: for all $v \in S^g \widetilde{M}$, and $s \in \mathbb{R}$,

$$\text{if } H^g(v) = (v_-, v_+, t) \text{ then } H^g(g^s v) = (v_-, v_+, t + s).$$

Therefore, any positive Radon measure m on $S^g M$ invariant by the flow lifts to a measure \widetilde{m} on $S^g \widetilde{M}$ of the form $\widetilde{m} = (H^g)^*(\mu \times dt)$, where dt is the Lebesgue measure on \mathbb{R} , and μ is a Γ -invariant locally finite positive measure on $\partial^2 \widetilde{M}$.

Definition 2.2 (Geodesic current). *A Γ -invariant geodesic current, or simply geodesic current, is a Γ -invariant positive Radon measure on $\partial^2 \widetilde{M}$.*

Given any geodesic current μ and any admissible metric g on M , we will denote by m_μ^g the unique measure on $S^g M$ invariant by the geodesic flow (g^t) whose lifts on $S^g \widetilde{M}$ is $\widetilde{m}_\mu^g = (H^g)^*(d\mu \times dt)$. The *non-wandering set* $\Omega^g \subset S^g M$ of the geodesic flow (g^t) is

$$\Omega^g = (H^g)^{-1}((\Lambda_\Gamma \times \Lambda_\Gamma) \setminus \text{Diag} \times \mathbb{R}).$$

It was shown in Eberlein [Ebe72] that for the geodesic flow of a negatively curved manifold, this definition coincides with the usual definition of the nonwandering set of a flow.

It follows from (3) and [BH99, Thm 1.7 p401] that Ω^g is compact (i.e. (M, g) is *convex-cocompact*) if and only if Ω^{g_0} is. We will mainly be interested in the case where Ω^g is *not compact*.

The measure m_μ^g is locally finite, but may have infinite mass as soon as (M, g) is not convex-cocompact. We will use all over this paper the fact that many properties of the measure m_μ^g only depend on the geodesic current μ and not on the chosen admissible metric g .

Recall first that an invariant measure is *ergodic* if every invariant set either has measure zero or its complementary set has measure zero. An invariant measure is *periodic* if it is (proportional to) the Lebesgue measure on a periodic orbit. The measure m is *conservative* if it satisfies the conclusion of Poincaré recurrence Theorem: for all sets A of positive measure $m(A) > 0$, and m -almost all vectors v , the orbit $(g^t v)$ returns infinitely often in A . The measure m has a *product structure* if the associated geodesic current is equivalent to a product of measures on $\partial\tilde{M}$. The measure m is *strongly mixing* if it is finite and satisfies $m(A \cap g^t B) \rightarrow m(A)m(B)$ when $t \rightarrow \pm\infty$ for all Borel sets A, B . It is *weakly mixing* if it is finite and $\frac{1}{T} \int_0^T |m(A \cap g^t B) - m(A)m(B)|$ goes to 0 when $T \rightarrow \pm\infty$ for all Borel sets A, B .

First well known properties are given in the following proposition.

Proposition 2.3. *Let μ be a geodesic current, let g_1 and g_2 be two admissible metrics on M . Then*

1. *the measure $m_\mu^{g_1}$ is supported by a (finite number of) closed geodesic(s) if and only if $m_\mu^{g_2}$ is ;*
2. *the measure $m_\mu^{g_1}$ is ergodic for the geodesic flow (g_1^t) if and only if $m_\mu^{g_2}$ is ergodic for the geodesic flow (g_2^t) ;*
3. *the measure $m_\mu^{g_1}$ is conservative for the geodesic flow (g_1^t) if and only if $m_\mu^{g_2}$ is conservative for the geodesic flow (g_2^t) ;*
4. *the measure $m_\mu^{g_1}$ has a local product structure iff the measure $m_\mu^{g_2}$ has a local product structure.*

Proof. The measure $m_\mu^{g_1}$ is supported by a closed geodesic if and only if μ is carried by the Γ -orbit of a couple $(\xi_-, \xi_+) \in \partial^2\tilde{M}$ where ξ_- and ξ_+ are the fixed point of a hyperbolic element $\gamma \in \Gamma$. Since this property does not depend on g_1 , it shows 1.

The measure $m_\mu^{g_1}$ is ergodic for the geodesic flow (g_1^t) if and only if μ is ergodic under the action of Γ on $\partial^2\tilde{M}$ (cf for instance [Rob03, p. 19]). This property only depends on μ , which shows 2.

The measure $m_\mu^{g_1}$ is conservative for the geodesic flow (g_1^t) if and only if μ gives full measure to $\Lambda_\Gamma^r \times \Lambda_\Gamma^r$ [Rob03, proof of (b) page 19] where Λ_Γ^r is the radial limit set, which does not depend on the (admissible) metric g_i . This shows 3. \square

One should note that in general an invariant measure m_μ^g , even with finite total mass, has no reason to be a probability measure.

We will see further nontrivial relationships between $m_\mu^{g_1}$ and $m_\mu^{g_2}$ later. It would be interesting to know if this kind of result can be extended to mixing property. Explicit examples of mixing measures have all a local product structure. But there exist mixing measures without such a product structure.

2.3 Geodesic stretches

Let g_1 and g_2 be two admissible metrics. For all $v \in S^{g_1}M$, define the quantity

$$e^{g_1 \rightarrow g_2}(v) = \inf_{t > 0} \frac{d^{g_2}(\pi \tilde{v}, \pi g_1^t \tilde{v})}{t}, \quad (4)$$

where \tilde{v} is a lift of v to $S^g \tilde{M}$. This does not depend on the choice of \tilde{v} . Knieper showed in [Kni95] that if m is any invariant measure for (g_1^t) , then for m -almost every $v \in S^{g_1}M$,

$$e^{g_1 \rightarrow g_2}(v) = \lim_{t \rightarrow +\infty} \frac{d^{g_2}(\pi \tilde{v}, \pi g_1^t \tilde{v})}{t}. \quad (5)$$

This asymptotic geodesic stretch has been studied by many authors, among which [FF93], [Kni95], [Glo15]. Sambarino uses a different point of view of reparametrization of the geodesic flow (see for example [Sam15]) which is very close from our point of view below.

Recall that, $\xi \in \partial \tilde{M}$ being fixed, the Busemann function $\mathcal{B}_\xi^g(\cdot)$ is C^2 on \tilde{M}^2 [HIH77, prop. 3.1]. Therefore, for all $v \in S^{g_1}M$, we can define

$$\mathcal{E}^{g_1 \rightarrow g_2}(v) = \left. \frac{d}{dt} \right|_{t=0^+} \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi v, \pi g_1^t \tilde{v}) = \left. \frac{d}{dt} \right|_{t=0^+} \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi g_1^t \tilde{v}), \quad (6)$$

where $\tilde{v} \in S^{g_1} \tilde{M}$ is any lift of v , and $\mathcal{B}_{v_+^{g_1}}^{g_2}(\cdot, \cdot)$ is the Busemann function for g_2 based at the end point of the g_1 -geodesic generated by v . This definition was inspired by Ledrappier's paper [Led94]. In his notations, our geodesic stretch satisfies $\mathcal{E}^{g_1 \rightarrow g_2}(v) = \alpha^{g_2}(v)$, where α^{g_2} is the harmonic 1-form on the g_1 -stable foliation associated to the Busemann cocycle of the metric g_2 .

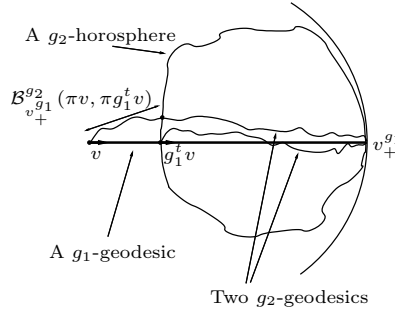


Figure 1: Geodesic stretch

Definition 2.4 (Geodesic stretch). *The maps $e^{g_1 \rightarrow g_2} : S^{g_1}M \rightarrow \mathbb{R}$ and $\mathcal{E}^{g_1 \rightarrow g_2} : S^{g_1}M \rightarrow \mathbb{R}$ will be called respectively the asymptotic and instantaneous geodesic stretch of g_2 with respect to g_1 .*

Anyway, we will most of the time call them both without distinction *geodesic stretch*.

By construction, for all $v \in S^{g_1}M$, $\mathcal{E}^{g_1 \rightarrow g_1}(v) = e^{g_1 \rightarrow g_1}(v) = 1$. Observe that there is no obvious relation from the definition between $e^{g_1 \rightarrow g_2}$ (resp. $\mathcal{E}^{g_1 \rightarrow g_2}$) and $e^{g_2 \rightarrow g_1}$ (resp. $\mathcal{E}^{g_2 \rightarrow g_1}$).

If m is ergodic, then $e^{g_1 \rightarrow g_2}$ is m -almost everywhere constant. Of course its value strongly depends on the measure m . On the opposite, the map $\mathcal{E}^{g_1 \rightarrow g_2}$ is defined everywhere and does not depend on the chosen measure. It is in general non-constant, globally Hölder on $S^{g_1}M$ [Bal95, Appendix of Brin], [PPS15, thm 7.3], and C^1 along g_1 -geodesics (as Busemann functions are C^2 , see [HH77]). We will need the following basic estimate.

Lemma 2.5. *Let g_1 and g_2 be two admissible metrics, and m any g_1 -invariant measure. For m -almost all $v \in S^{g_1}\widetilde{M}$,*

$$e^{g_1 \rightarrow g_2}(v) \leq \int_{S^{g_1}M} \|v\|^{g_2} dm,$$

whereas for all $v \in S^{g_1}\widetilde{M}$,

$$\mathcal{E}^{g_1 \rightarrow g_2}(v) \leq \|v\|^{g_2}.$$

Proof. The first estimate was shown in [Kni95, p.44]. The second follows from triangular inequality. Indeed, for all $t \geq 0$, $\mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^t v)) \leq d^{g_2}(\pi(v), \pi(g_1^t v))$, and these two quantities vanish at $t = 0$ so that their derivatives at $t = 0$ satisfy the same inequality. Moreover, $d^{g_2}(\pi(v), \pi(g_1^t v))$ is smaller than the g_2 -length of the curve $(\pi(g_1^s v))_{0 \leq s \leq t}$, whose derivative at zero is exactly $\|v\|_{g_2}$. \square

Lemma 2.6 and Corollary 2.8 below justify the common name of *geodesic stretch* given to the two maps $e^{g_1 \rightarrow g_2}$ and $\mathcal{E}^{g_1 \rightarrow g_2}$. Before stating them, recall a well known feature of negative curvature. On a geodesic space X , each triangle (x, y, z) admits an interior triangle (p, q, r) such that $d(r, x) = d(q, x)$, $d(q, z) = d(p, z)$ and $d(p, y) = d(r, y)$. If g is a metric with negative curvature, there exists a universal constant $\Delta(g)$ such that for any geodesic triangle (x, y, z) in \widetilde{M} , the associated interior triangle has sides smaller than $\Delta(g)$.

Lemma 2.6. *There exists $C_3 = C_3(g_1, g_2) > 0$, depending only on the constant $C_2(g_1, g_2)$ and the hyperbolicity constant $\Delta(g_2)$, such that for all $\tilde{v} \in S^{g_1}\widetilde{M}$ and for all $T > 0$,*

$$\left| d^{g_2}(\pi\tilde{v}, \pi g_1^T \tilde{v}) - \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi\tilde{v}, \pi g_1^T \tilde{v}) \right| = \left| d^{g_2}(\pi\tilde{v}, \pi g_1^T \tilde{v}) - \int_0^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt \right| \leq C_3(g_1, g_2).$$

Proof. Let $\tilde{v} \in S^{g_1}\widetilde{M}$ and $T > 0$ be fixed. We write $x = \pi\tilde{v}$, $x_T = \pi g_1^T \tilde{v}$, and z_T be the intersection between the g_2 -geodesic $(x, v_+^{g_1})^{g_2}$ and the g_2 -horosphere centered at $v_+^{g_1}$ passing through x_T .

We will need at several occasions the following estimate.

Fact 2.7. *With above notations, $d^{g_2}(x_T, z_T) \leq 2C_2(g_1, g_2) + \Delta(g_2)$.*

Let us first prove this fact. Consider the g_2 -geodesic triangle $x, x_T, v_+^{g_1}$ and its interior triangle, say $p \in (x_T, v_+^{g_1})$, $q \in (x, x_T)$, $r \in (x, v_+^{g_1})$.

Then by definition of z_T , $d^{g_2}(z_T, r) = d^{g_2}(x_T, q)$, so that $d^{g_2}(x_T, z_T) \leq 2d^{g_2}(x_T, q) + d^{g_2}(q, r)$. Now, the definition of (p, q, r) implies $d^{g_2}(x_T, q) \leq d^{g_2}(x_T, (x, v_+^{g_1})) \leq C_2(g_1, g_2)$. The fact follows.

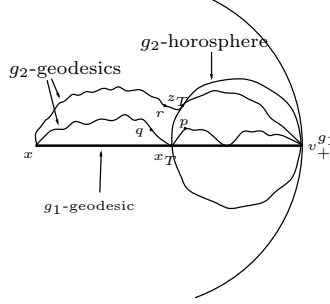


Figure 2: Proof of lemma 2.6

By definition (6),

$$\int_0^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi \tilde{v}, \pi g_1^T \tilde{v}) = d^{g_2}(x, z_T).$$

Thanks to the above fact, we get

$$|d^{g_2}(x, x_T) - d^{g_2}(x, z_T)| \leq d^{g_2}(x_T, z_T) \leq 2C_2(g_1, g_2) + \Delta(g_2).$$

The result of the lemma follows, with $C_3(g_1, g_2) = 2C_2(g_1, g_2) + \Delta(g_2)$. \square

Corollary 2.8. *Let m be an ergodic probability measure on $S^{g_1}M$, invariant by the geodesic flow (g_1^t) . Then*

$$\int_{S^{g_1}M} e^{g_1 \rightarrow g_2}(v) dm(v) = \int_{S^{g_1}M} \mathcal{E}^{g_1 \rightarrow g_2}(v) dm(v).$$

Moreover, for m -almost every $v \in S^{g_1}M$ and all lifts $\tilde{v} \in S^{g_1}\tilde{M}$ of v ,

$$\lim_{T \rightarrow +\infty} \frac{d^{g_2}(\pi \tilde{v}, \pi g_1^T \tilde{v})}{T} = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt = \int_{S^{g_1}M} \mathcal{E}^{g_1 \rightarrow g_2}(w) dm(w).$$

Proof. It follows from the previous lemma that for all $\varepsilon > 0$, there exists $T_0 > 0$ such that for all $T \geq T_0$ and all $\tilde{v} \in S^{g_1}\tilde{M}$,

$$\frac{1}{T} \left| d^{g_2}(\pi \tilde{v}, \pi g_1^T \tilde{v}) - \int_0^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt \right| \leq \varepsilon.$$

Since $m_\mu^{g_1}$ is ergodic, for $m_\mu^{g_1}$ -almost all vector $v \in S^{g_1}$,

$$\int_{S^{g_1}M} \mathcal{E}^{g_1 \rightarrow g_2}(v) dm(v) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt$$

and

$$\int_{S^{g_1}M} e^{g_1 \rightarrow g_2}(v) dm(v) = \lim_{T \rightarrow +\infty} \frac{d^{g_2}(\pi \tilde{v}, \pi g_1^T \tilde{v})}{T},$$

which concludes the proof of the corollary. \square

Let us emphasize the fact that the measures which we will consider will usually have finite mass, but may not be probability measures. We will denote by $\|m\|$ the mass of a finite measure m on TM .

Definition 2.9 (Geodesic stretch with respect to a geodesic current). *Let μ be a geodesic current on $\partial^2\widetilde{M}$ such that m_μ^g is finite. We will call (average) geodesic stretch of g_2 relative to g_1 with respect to μ the quantity*

$$I_\mu(g_1, g_2) = \frac{1}{\|m_\mu^{g_1}\|} \int_{S^{g_1}M} \mathcal{E}^{g_1 \rightarrow g_2}(v) dm_\mu^{g_1}(v) = \frac{1}{\|m_\mu^{g_1}\|} \int_{S^{g_1}M} e^{g_1 \rightarrow g_2}(v) dm_\mu^{g_1}(v).$$

By Corollary 2.8, $I_\mu(g_1, g_2)$ coincides with the definition of the geodesic stretch studied in [Kni95] (note that Knieper only considers invariant *probability* measures).

When (M, g) has finite volume and μ is the Liouville geodesic current of g_1 , then

$$I_\mu(g_1, g_2) \cdot \text{Vol}(S^{g_1}M) = i(g_1, g_2),$$

where $i(g_1, g_2)$ is the *intersection* between the metrics g_1 and g_2 studied in [FF93].

It follows from the definition that for all geodesic current μ such that m_μ^g is finite, $I_\mu(g_1, g_1) = 1$.

Remark 2.10 (Geodesic stretches and Thurston metric). Given two negatively curved metrics g_1 and g_2 on a compact surface S , the Thurston distance $d_{Th}(g_1, g_2)$ is defined as the supremum over all periodic orbits of the ratios of their lengths:

$$d_{Th}(g_1, g_2) = \sup_\gamma \left(\frac{\ell^{g_2}(\gamma)}{\ell^{g_1}(\gamma)}, \frac{\ell^{g_1}(\gamma)}{\ell^{g_2}(\gamma)} \right).$$

With our notations, this distance could also be defined as the following supremum

$$d_{Th}(g_1, g_2) = \sup_\mu (I_\mu(g_1, g_2), I_\mu(g_2, g_1))$$

over all currents μ associated to ergodic measures. Indeed, considering periodic measures immediately shows that Thurston distance is smaller than the above supremum. In the other direction, the density of periodic measures in the set of ergodic measures gives the above equality.

2.4 Morse correspondances and geodesic stretches

To compare dynamics of the geodesic flows on $S^{g_1}M$ and $S^{g_2}M$, it is natural to consider their dynamics modulo the Γ -action on $S^{g_1}\widetilde{M}$ and $S^{g_2}\widetilde{M}$. Hopf coordinates are a good motivation to consider the map

$$\widetilde{\Phi}^{g_1 \rightarrow g_2} := (H^{g_2})^{-1} \circ H^{g_1} : S^{g_1}\widetilde{M} \rightarrow S^{g_2}\widetilde{M}.$$

It is a Hölder homeomorphism, but it is unfortunately not Γ -equivariant, as both Γ -actions on each unit tangent bundle $S^{g_i}\widetilde{M}$ are different. In other words, as said earlier, on $\partial^2\widetilde{M} \times \mathbb{R}$, these Γ -actions involve different cocycles on the \mathbb{R} component.

Despite its non-invariance, this map is sometimes useful, because it has the nice property to commute with both geodesic flows. But we need to find another

map from $S^{g_1}\widetilde{M}$ to $S^{g_2}\widetilde{M}$ which will be Γ -equivariant. We proceed as follows. For all $v \in S^{g_1}\widetilde{M}$, let $w = \widetilde{\Psi}^{g_1 \rightarrow g_2}(v)$ be the unique vector in $S^{g_2}\widetilde{M}$ on the g_2 -geodesic joining $v_-^{g_1}$ to $v_+^{g_1}$ satisfying $\mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(w)) = 0$.

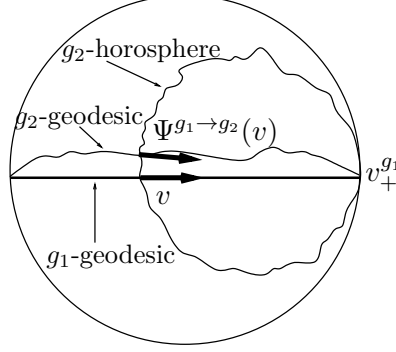


Figure 3: Morse correspondance

Lemma 2.11. *The map $\Psi^{g_1 \rightarrow g_2}$ is Hölder continuous. Moreover, for all $v \in S^{g_1}\widetilde{M}$, we have*

$$d^{g_2}(v, \Psi^{g_1 \rightarrow g_2}(v)) \leq C_3(g_1, g_2),$$

where $C_3(g_1, g_2)$ is the constant given by Lemma 2.6.

Proof. It is Hölder continuous as composition of the maps $\Phi^{g_1 \rightarrow g_2}$ and some time g_2^t of the geodesic flow, with $t = t(v)$ depending Hölder-continuously of v .

The bound on $d^{g_2}(v, \Psi^{g_1 \rightarrow g_2}(v))$ has already been proved in Fact 2.7. \square

By construction, the correspondance $\widetilde{\Psi}^{g_1 \rightarrow g_2}$ is Γ -invariant. We denote by $\Psi^{g_1 \rightarrow g_2}$ the induced map from $S^{g_1}M$ to $S^{g_2}M$. It is a homeomorphism homotopic to identity sending (g_1^t) -orbits to (g_2^t) -orbits, i.e. a (g_1, g_2) -Morse correspondance in the sense of [FF93].

By definition of both correspondances, the following lemma holds. It says that the geodesic flows (g_1^t) and (g_2^s) on the unit tangent bundles $S^{g_i}\widetilde{M}$ are conjugated by $\Phi^{g_1 \rightarrow g_2}$, and conjugated up to reparametrization by the Morse correspondance $\Psi^{g_1 \rightarrow g_2}$.

Lemma 2.12. *With the above notations, we have for all $v \in S^{g_1}\widetilde{M}$*

1. $\Phi^{g_1 \rightarrow g_2} \circ g_1^t(v) = g_2^t \circ \Phi^{g_1 \rightarrow g_2}(v)$.
2. $\Phi^{g_2 \rightarrow g_1} = (\Phi^{g_1 \rightarrow g_2})^{-1}$.
3. $\Psi^{g_1 \rightarrow g_2} \circ g_1^t(v) = g_2^{s^{g_1 \rightarrow g_2}(t, v)} \circ \Psi^{g_1 \rightarrow g_2}(v)$, with $s^{g_1 \rightarrow g_2}(t, v) = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^t v))$.
4. $\Psi^{g_1 \rightarrow g_2}(v) = g_2^{\tau^{g_1 \rightarrow g_2}(v)} \circ \Phi^{g_1 \rightarrow g_2}(v)$, with

$$\tau^{g_1 \rightarrow g_2}(v) = \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi(v)) - \mathcal{B}_{v_+^{g_1}}^{g_1}(o, \pi(v)).$$
5. $\Psi^{g_2 \rightarrow g_1} \circ \Psi^{g_1 \rightarrow g_2}(v) = g_1^{\sigma^{g_1 \rightarrow g_2}(v)}(v)$, with $\sigma^{g_1 \rightarrow g_2}(v) = \mathcal{B}_{v_+^{g_1}}^{g_1}(\pi(v), \pi(\Psi^{g_1 \rightarrow g_2} v))$.

Let us emphasize that $\Phi^{g_1 \rightarrow g_2}$ and its inverse are not Γ -invariant, $\Psi^{g_1 \rightarrow g_2}$ and its inverse are Γ -invariant, the map $\tau^{g_1 \rightarrow g_2}$ is not Γ -invariant, whereas $\sigma^{g_1 \rightarrow g_2}$ and the cocycle $s^{g_1 \rightarrow g_2}(t, v)$ are Γ -invariant.

Proof. The fact that $\Phi^{g_1 \rightarrow g_2}$ commutes with the geodesic flows of g_1 and g_2 is immediate by definition of Hopf coordinates. The property about its inverse is also obvious.

By definition of $\Psi^{g_1 \rightarrow g_2}$, the vectors $\Psi^{g_1 \rightarrow g_2}(g_1^t v)$, for $t \in \mathbb{R}$, all lie on the g_2 -geodesic joining $v_-^{g_1}$ to $v_+^{g_2}$. The only question is to compute

$$s^{g_1 \rightarrow g_2}(t, v) = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(\Psi^{g_1 \rightarrow g_2}(v)), \pi(\Psi^{g_1 \rightarrow g_2}(g_1^t v))).$$

By definition of $\Psi^{g_1 \rightarrow g_2}$,

$$\mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(\Psi^{g_1 \rightarrow g_2}(g_1^t v)), \pi(g_1^t v)) = 0 = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(\Psi^{g_1 \rightarrow g_2}(v)), \pi(v)).$$

Using the cocycle properties of $\mathcal{B}_{v_+^{g_1}}^{g_2}$, we deduce immediately that $s^{g_1 \rightarrow g_2}(t, v)$ is the algebraic g_2 -distance $\mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^t v))$.

The next affirmation follows from the computation

$$\begin{aligned} \tau^{g_1 \rightarrow g_2}(v) &= \mathcal{B}_{v_+^{g_1}}^{g_2}(\Phi^{g_1 \rightarrow g_2}(v), \Psi^{g_1 \rightarrow g_2}(v)) = \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi(v)) - \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \Phi^{g_1 \rightarrow g_2}(v)) \\ &= \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi(v)) - \mathcal{B}_{v_+^{g_1}}^{g_1}(o, v). \end{aligned}$$

The last statement follows easily from the previous one. \square

2.5 Change of mass

We will need the following variant of Lemma 2.6, which shows once more that $\mathcal{E}^{g_1 \rightarrow g_2}$ behaves asymptotically as the infinitesimal reparametrization of the flow given by Morse correspondance $\Psi^{g_1 \rightarrow g_2} : S^{g_1} \widetilde{M} \rightarrow S^{g_2} \widetilde{M}$.

Proposition 2.13. *Let $G : S^{g_2} M \rightarrow \mathbb{R}$ be a continuous map and $\widetilde{G} : S^{g_2} \widetilde{M} \rightarrow \mathbb{R}$ be its (Γ -invariant) lift to $S^{g_2} \widetilde{M}$. Then for all $v \in S^{g_1} \widetilde{M}$, $T \geq 0$, and $w = \Psi^{g_1 \rightarrow g_2}(v)$, we have*

$$\int_0^{s^{g_1 \rightarrow g_2}(T, v)} \widetilde{G}(g_2^s w) ds = \int_0^T G \circ \Psi^{g_1 \rightarrow g_2}(g_1^t v) \times \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t v) dt,$$

with $s^{g_1 \rightarrow g_2}(T, v) = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^T v))$ as in Lemma 2.12.

If moreover G is bounded, then there exists $C = C(G, g_1, g_2)$ such that for all $v \in S^{g_1} \widetilde{M}$, $T \geq 0$, and $w = \Psi^{g_1 \rightarrow g_2}(v)$, we have

$$\left| \int_0^{d^{g_2}(v, g_1^T v)} \widetilde{G}(g_2^s w) ds - \int_0^T \widetilde{G} \circ \Psi^{g_1 \rightarrow g_2}(g_1^t v) \times \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t v) dt \right| \leq C.$$

If G is not bounded, then for all compact sets $K \subset S^{g_1} M$ there exists another constant $C' = C'(G, K, g_1, g_2)$ such that for all $v \in S^{g_1} \widetilde{M}$ and $T \in \mathbb{R}$ such that both v and $g_1^T v$ project inside K , $\widetilde{K} \subset S^{g_1} \widetilde{M}$, we have

$$\left| \int_0^{d^{g_2}(v, g_1^T v)} \widetilde{G}(g_2^s w) ds - \int_0^T \widetilde{G} \circ \Psi^{g_1 \rightarrow g_2}(g_1^t v) \times \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t v) dt \right| \leq C'.$$

The geodesic stretch $\mathcal{E}^{g_1 \rightarrow g_2}$ can therefore be understood as the instantaneous reparametrization of the flow (g_1^t) in the correspondance $\Psi^{g_1 \rightarrow g_2}$.

Proof. The first equality is a simple change of variable using Lemma 2.12. The second follows using Lemma 2.6 and the fact that F is bounded. Indeed,

$$\begin{aligned} & \left| \int_0^{d^{g_2}(v, g_1^T v)} G(g_2^s w), ds - \int_0^T F \circ \Psi^{g_1 \rightarrow g_2}(g_1^t v) \times \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t v) dt \right| \\ &= \left| \int_{s(T, v)}^{d^{g_2}(v, g_1^T v)} G(g_2^s w) ds \right| \\ &\leq \|G\|_\infty \times \left| d^{g_2}(v, g_1^T v) - \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^T v)) \right| \\ &= \|G\|_\infty \times \left| d^{g_2}(v, g_1^T v) - \int_0^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t v) dt \right| \\ &\leq C_1 \|G\|_\infty. \end{aligned}$$

The last assertion is a variation on the second one. If v and $g_1^T v$ are in a compact set K , for any parameter s such that $|s| \leq C_3(g_1, g_2)$, $g_1^{T \pm s} v$ belongs to the $C_3(g_1, g_2)$ -neighbourhood of K , on which G is bounded. The above computation therefore applies verbatim. \square

Remark 2.14. Proposition 1.3 follows immediately: given any $m_\mu^{g_2}$ -measurable map $G : S^{g_2} M \rightarrow \mathbb{R}$, the map $G \circ \Psi^{g_1 \rightarrow g_2}$ is $m_\mu^{g_1}$ -measurable and G on $S^{g_2} M$, and we have

$$\int_{S^{g_2} M} G dm_\mu^{g_2} = \int_{S^{g_1} M} G \circ \Psi^{g_1 \rightarrow g_2} \times \mathcal{E}^{g_1 \rightarrow g_2} dm_\mu^{g_1}.$$

The corollary below follows immediately from the above Remark. It gives a nice interpretation of the geodesic stretch $I_\mu(g_1, g_2)$.

Corollary 2.15 (Mass transformation law). *Let μ be a geodesic current such that $m_\mu^{g_1}$ is ergodic and has finite total mass, denoted by $\|m_\mu^{g_1}\|$. Then*

$$\|m_\mu^{g_2}\| = I_\mu(g_1, g_2) \times \|m_\mu^{g_1}\|.$$

In particular $m_\mu^{g_1}$ has finite mass if and only if $m_\mu^{g_2}$ has finite mass. Moreover, when it is the case,

$$I_\mu(g_1, g_2) = \frac{1}{I_\mu(g_2, g_1)} = \frac{\|m_\mu^{g_2}\|}{\|m_\mu^{g_1}\|}.$$

Remark 2.16. The previous formula is very natural if $I_\mu(g_2, g_1)$ is interpreted as the average dilation of the reparametrization of the flow via the Morse correspondance $\Psi^{g_1 \rightarrow g_2}$. Indeed, in the case where (g_1^t) and (g_2^t) are suspension flows over a (fixed) compact basis for distinct ceiling functions, the above formula is well known [Abr59].

2.6 Periodic orbits and geodesic stretch

In this section we relate geodesic stretch and lengths of periodic orbits. The results will not be useful in the sequel of the paper, but are enlightening about the geodesic stretch.

For $i = 1, 2$, for any hyperbolic element $\gamma \in \Gamma$, let γ^{g_i} be the closed g_i -geodesic associated to the conjugacy class of γ . Let $\ell^{g_i}(\gamma)$ be its g_i -length, and $d\ell_\gamma^{g_i}$ be the Lebesgue measure along the geodesic γ^{g_i} . Observe that, up to normalizing constants, the periodic measure $d\ell_\gamma^{g_i}$, $i = 1, 2$, induce the same current at infinity.

Since $m_\mu^{g_1}$ is finite and ergodic, there exists a sequence $(\gamma_k)_{k \in \mathbb{N}}$ of hyperbolic elements such that in the weak topology,

$$\lim_{k \rightarrow \infty} \frac{d\ell_{\gamma_k}^{g_1}}{\ell^{g_1}(\gamma_k)} = \frac{m_\mu^{g_1}}{\|m_\mu^{g_1}\|},$$

see for instance [CS10, Lemma 2.2]. This convergence holds a priori in the dual of continuous functions with compact support. But as all measures involved above are probability measures, this convergence also holds in the dual of bounded continuous functions of $S^{g_1}M$.

We can moreover suppose that $\lim_{k \rightarrow \infty} \ell^{g_1}(\gamma_k) = +\infty$.

The following proposition shows that the same happens on $S^{g_2}M$, and that the ratio of lengths of periodic orbits in both metrics allows to recover the geodesic stretch.

Proposition 2.17. *Let (M, g_i) , $i = 1, 2$, be two admissible Riemannian structures with pinched negative curvature. Let μ be a geodesic current such that both measures $m_\mu^{g_i}$ are finite. Let (γ_k) be a sequence of hyperbolic elements such that $\frac{d\ell_{\gamma_k}^{g_1}}{\ell^{g_1}(\gamma_k)}$ converges weakly to $\frac{m_\mu^{g_1}}{\|m_\mu^{g_1}\|}$ in the dual of bounded continuous functions. Then $\frac{d\ell_{\gamma_k}^{g_2}}{\ell^{g_2}(\gamma_k)}$ converges weakly to $\frac{m_\mu^{g_2}}{\|m_\mu^{g_2}\|}$ in the dual of bounded continuous functions.*

Moreover, the ratios of lengths satisfy

$$\lim_{k \rightarrow +\infty} \frac{\ell^{g_2}(\gamma_k)}{\ell^{g_1}(\gamma_k)} = I_\mu(g_1, g_2).$$

The proof is separated in two lemmas. The first one asserts that viewed on $S^{g_2}M$, the sequence of periodic probability measures associated to (γ_k) also converges to $\frac{m_\mu^{g_2}}{\|m_\mu^{g_2}\|}$ in the dual of bounded continuous functions. The second says that the ratio of lengths $\ell^{g_2}(\gamma_k)/\ell^{g_1}(\gamma_k)$ converges to the average geodesic stretch $I_\mu(g_1, g_2)$.

Lemma 2.18. *With the previous notations, for the same sequence (γ_k) , in the dual of continuous bounded functions of $S^{g_2}M$,*

$$\lim_{k \rightarrow \infty} \frac{d\ell_{\gamma_k}^{g_2}}{\ell^{g_2}(\gamma_k)} = \frac{m_\mu^{g_2}}{\|m_\mu^{g_2}\|}.$$

Proof. First, as the sequence of probability measures $\frac{d\ell_{\gamma_k}^{g_1}}{\ell^{g_1}(\gamma_k)}$ converges to the probability measure $\frac{m_\mu^{g_1}}{\|m_\mu^{g_1}\|}$, the Γ -invariant lift of $\frac{d\ell_{\gamma_k}^{g_1}}{\ell^{g_1}(\gamma_k)}$ to $S^{g_1}\widetilde{M}$ converges

in the dual of continuous functions with compact support towards $\frac{\tilde{m}_\mu^{g_1}}{\|m_\mu^{g_1}\|}$. Using Hopf coordinates, we deduce that the geodesic current on $\partial^2 \widetilde{M}$ associated through H^{g_1} to $\frac{d\ell_{\gamma_k}^{g_1}}{\ell^{g_1}(\gamma_k)}$ converges weakly (in the dual of continuous functions with compact support) to μ . Using the same reasoning in the other direction, we obtain that the sequence of probability measures $\frac{d\ell_{\gamma_k}^{g_2}}{\ell^{g_2}(\gamma_k)}$ converges weakly (in the dual of continuous functions with compact support) to some multiple of $m_\mu^{g_2} / \|m_\mu^{g_2}\|$.

It is not exactly the desired result. To get the convergence towards the probability measure $m_\mu^{g_2} / \|m_\mu^{g_2}\|$, and in the dual of bounded continuous functions, we need to avoid a possible loss of mass at infinity. To establish this convergence, it is necessary and sufficient to prove that $\frac{d\ell_{\gamma_k}^{g_2}}{\ell^{g_2}(\gamma_k)}$ does not diverge. In other words, we want to check that for all $\varepsilon > 0$, there exists a compact set $\mathcal{K}_\varepsilon \subset S^{g_2} M$, such that for all $k \geq 0$ large enough,

$$\frac{d\ell_{\gamma_k}^{g_2}}{\ell^{g_2}(\gamma_k)}(\mathcal{K}_\varepsilon) \geq 1 - \varepsilon.$$

It follows easily from the fact that there exists a constant $C = C(g_1, g_2)$ such that any g_2 -geodesic of \widetilde{M} stays in a $C(g_1, g_2)$ -neighbourhood of the g_1 -geodesic with same endpoints at infinity. Let us write the detail of the argument.

Choose first some $\varepsilon > 0$, and some compact set $K_1 \subset M$ such that $\frac{m_\mu^{g_1}(S^{g_1} K_1)}{\|m_\mu^{g_1}\|} \geq 1 - \varepsilon/2$. By convergence of $\frac{d\ell_{\gamma_k}^{g_1}}{\ell^{g_1}(\gamma_k)}$, for all $k \geq k_0$ large enough, we also have $\frac{\ell_{\gamma_k}^{g_1}(S^{g_1} K_1)}{\ell^{g_1}(\gamma_k)} \geq 1 - \varepsilon$.

Now, choose a relatively compact preimage $\widetilde{K}_1 \in \widetilde{M}$, its g_2 -convex closure \widetilde{K}_2 and $\widetilde{K}_3 \supset \widetilde{K}_2$ a larger compact convex set of \widetilde{M} containing a $2C(g_1, g_2)$ -neighbourhood of \widetilde{K}_2 for both metrics g_1 and g_2 .

Consider a lift $\widetilde{\gamma}_k^{g_1}$ of the g_1 -geodesic $\gamma_k^{g_1}$ which intersects \widetilde{K}_1 , and the associated lift $\widetilde{\gamma}_k^{g_2}$ of the g_2 -geodesic $\gamma_k^{g_2}$, at distance at most $C(g_1, g_2)$ from $\widetilde{\gamma}_k^{g_1}$. Let a, b be two points on $\widetilde{\gamma}_k^{g_2}$ such that the length $\ell_{\gamma_k}^{g_2}((a, b)) = \ell^{g_2}(\gamma_k)$. We want to estimate the proportion of g_2 -length of $[a, b]$ outside $\Gamma \cdot \mathcal{K} = \Gamma \cdot S^{g_2} K_3$.

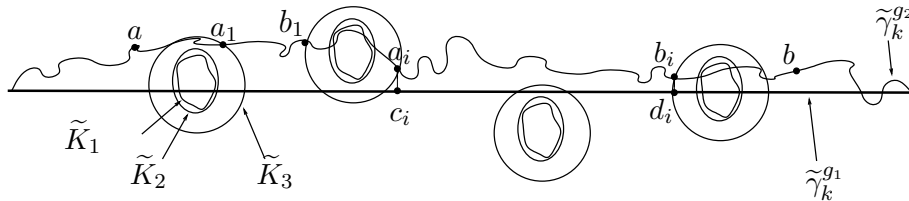


Figure 4: Proof of Lemma 2.18

By convexity of \widetilde{K}_3 , we can write $(a, b) \cap (\Gamma \cdot \widetilde{K}_3)^c$ as the disjoint union $\sqcup (a_i, b_i)$ of finitely many intervals. Thus, we have to show that

$$\frac{\ell_{\gamma_k}^{g_2}(\mathcal{K}^c)}{\ell^{g_2}(\gamma_k)} = \frac{\sum_i \ell_{\gamma_k}^{g_2}(a_i, b_i)}{\ell^{g_2}(\gamma_k)} \leq \varepsilon.$$

Choose two points c_i and d_i on $\tilde{\gamma}_k^{g_1}$ whose projections (for the metric g_2) on the g_2 -geodesic $\tilde{\gamma}_k^{g_2}$ are exactly a_i and b_i . Such points are not necessarily unique but always exist: take c_i in the intersection of $\tilde{\gamma}_k^{g_1}$ with the hyperplane orthogonal to $\tilde{\gamma}_k^{g_2}$ at a_i . Denote by (c_i, d_i) the g_1 -geodesic segment on $\tilde{\gamma}_k^{g_1}$, and let $\lambda > 0$ be such that $\frac{1}{\lambda}g_2 \leq g_1 \leq \lambda g_2$. We have

$$\ell_{\gamma_k}^{g_2}(a_i, b_i) = d^{g_2}(a_i, b_i) \leq d^{g_2}(c_i, d_i) \leq \ell^{g_2}(c_i, d_i) \leq \sqrt{\lambda} \ell^{g_1}(c_i, d_i).$$

We deduce that

$$\frac{\ell_{\gamma_k}^{g_2}(\mathcal{K}^c)}{\ell^{g_2}(\gamma_k)} \leq \lambda \sum_i \frac{\ell^{g_1}(c_i, d_i)}{\ell^{g_1}(\gamma_k)} \leq \lambda \frac{\ell^{g_1}((S^{g_1}\Gamma.\tilde{K}_1)^c) \cap \tilde{\gamma}_k^{g_1}}{\ell^{g_1}(\gamma_k)},$$

the last inequality coming from the fact that, as a_i and b_i are in the boundary of $\Gamma.\tilde{K}_3$, and c_i and d_i are at distance at most $C(g_1, g_2)$ resp. from a_i and d_i , they cannot belong to $\Gamma.\tilde{K}_2$, so that the segment (c_i, d_i) does not intersect $\Gamma.\tilde{K}_1$. This proves that

$$\frac{\ell_{\gamma_k}^{g_2}(\mathcal{K}^c)}{\ell^{g_2}(\gamma_k)} \leq \lambda \varepsilon,$$

which concludes the proof (up to changing ε in ε/λ). \square

Moreover, the lengths $\ell^{g_1}(\gamma_k)$ and $\ell^{g_2}(\gamma_k)$ are related as follows.

Lemma 2.19. *With the previous notations, for the same sequence (γ_k) ,*

$$\lim_{k \rightarrow +\infty} \frac{\ell^{g_2}(\gamma_k)}{\ell^{g_1}(\gamma_k)} = I_\mu(g_1, g_2).$$

Proof. For all $k \in \mathbb{N}$, let $v_k^{g_1}$ (resp. $v_k^{g_2}$) be a tangent vector to $\gamma_k^{g_1}$ (resp. $\gamma_k^{g_2}$) such that $d^{g_2}(\pi v_k^{g_1}, \pi v_k^{g_2}) \leq C(g_1, g_2)$, where $C(g_1, g_2) > 0$ is the constant coming from (3). Let $\tilde{v}_k^{g_1} \in S^{g_1}\tilde{M}$ and $\tilde{v}_k^{g_2} \in S^{g_2}\tilde{M}$ be lifts of $v_k^{g_1}$ and $v_k^{g_2}$ such that again, $d^{g_2}(\pi \tilde{v}_k^{g_1}, \pi \tilde{v}_k^{g_2}) \leq C(g_1, g_2)$. It follows from Proposition 2.13 applied to $F \equiv 1$ that there exists $C_1 > 0$, only depending on $C(g_1, g_2)$ and the bounds on the curvature, such that

$$\left| \ell^{g_2}(\gamma_k) - \int_0^{\ell^{g_1}(\gamma_k)} \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt \right| \leq C_1.$$

Therefore,

$$\left| \frac{\ell^{g_2}(\gamma_k)}{\ell^{g_1}(\gamma_k)} - \frac{1}{\ell^{g_1}(\gamma_k)} \int_0^{\ell^{g_1}(\gamma_k)} \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt \right| \leq \frac{C_1}{\ell^{g_1}(\gamma_k)}.$$

By Lemma 2.18, as $\mathcal{E}^{g_1 \rightarrow g_2}$ is bounded and continuous, we know that

$$\frac{1}{\ell^{g_1}(\gamma_k)} \int_0^{\ell^{g_1}(\gamma_k)} \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t \tilde{v}) dt \rightarrow I_\mu(g_1, g_2),$$

so that the conclusion follows. \square

3 Entropy of finite measures

In this section, given two admissible metrics g_1 and g_2 as before, and a geodesic current μ on $\partial^2 \widetilde{M}$, we wish to compare the entropies of the measures $m_\mu^{g_1}$ and $m_\mu^{g_2}$. Theorem 3.11 establishes that their ratio is the average geodesic stretch between g_1 and g_2 w.r.t μ , but in the reverse direction compared to the relation between their masses, which leads to Corollary 3.12, which states that the product of the entropy of $m_\mu^{g_i}$ by its mass $\|m_\mu^{g_i}\|$ remains constant under an admissible change of metric.

First, we will recall some definitions and relations between dynamical balls (subsection 3.1). In subsection 3.2, we compare two notions of entropy of a measure, the Kolmogorov-Sinai entropy and the local Brin-Katok entropy, recalling well and less known results of Brin-Katok and Riquelme. It allows us to prove Theorem 3.11 and Corollary 3.12 in subsection 3.3.

3.1 Dynamical balls and shadows

If (φ^t) is a continuous dynamical system on a metric space (X, d) , a dynamical ball is a ball for the dynamical distance $d_T(x, y) = \sup_{0 \leq t \leq T} d(\varphi^t x, \varphi^t y)$.

We will restrict ourselves to geodesic flows associated to a Riemannian metric g on $S^g M$. For such geometric dynamical systems, it is more convenient to work with the Riemannian distance induced by the metric g on M or \widetilde{M} instead of the distance coming from the Sasaki metric on TM or $T\widetilde{M}$. We refer to [Bal95, p.70] and [PPS15, p.19-20] for a discussion about the fact that it is the good thing to do in this case.

For all $\varepsilon, T > 0$ and $v \in S^g \widetilde{M}$, we will call *dynamical ball* of center v , diameter ε and length T the set

$$B^g(v, T, \varepsilon) = \{w \in S^g \widetilde{M}, d^g(\pi(g^t v), \pi(g^t w)) \leq \varepsilon, \text{ for all } 0 \leq t \leq T\}.$$

Note that $B^g(v, 0, \varepsilon)$ is the ε -ball with center v for the distance d^g defined above.

Remark 3.1. On the quotient, for $v \in S^g M$, one can either consider the quotient dynamical ball $B^g(v, T, \varepsilon) = p_\Gamma(B^g(\tilde{v}, T, \varepsilon))$, \tilde{v} being any lift of v to $S^g \widetilde{M}$. There is also a more dynamical definition, as

$$B_{dyn}^g(v, T, \varepsilon) = \{w \in S^g M, d^g(\pi(g^t v), \pi(g^t w)) \leq \varepsilon, \text{ for all } 0 \leq t \leq T\}.$$

Of course, if $\tilde{v} \in S^g \widetilde{M}$ and $v = p_\Gamma(\tilde{v}) \in S^g M$, one has the obvious inclusion

$$B^g(v, T, \varepsilon) = p_\Gamma(B^g(\tilde{v}, T, \varepsilon)) \subset B_{dyn}^g(v, T, \varepsilon). \quad (7)$$

One can easily see that this inclusion is an equality when the injectivity radius of M is uniformly bounded from below, as soon as ε is small enough. However, when the injectivity radius of M is not bounded from below, one can build examples where this inclusion is not an equality [Bel17, Vel17].

It turns out that in many cases, the most natural dynamical ball to consider is the small ball $p_\Gamma(B^g(\tilde{v}, T, \varepsilon))$. Therefore, we will call it the *small dynamical ball* and denote it by $B^g(v, T, \varepsilon)$.

This problem has not been emphasized in [PPS15], where only these small dynamical balls are considered (see [PPS15, 3.15]). However, in various definitions of local entropies, the large dynamical balls have to be considered.

We will also need the following variant, for $v \in S^g \widetilde{M}$ and $T, T' > 0$:

$$B^g(v; T, T', \varepsilon) = \{w \in S^g \widetilde{M}, d^g(\pi(v), \pi(w)) \leq \varepsilon, \text{ for all } -T' \leq t \leq T\}.$$

Observe that $B^g(v; T, T', \varepsilon) = g^{T'}(B^g(g^{-T'}v, T + T', \varepsilon))$. As mentioned in the above remark 3.1, we consider on $S^g \widetilde{M}$ the small dynamical balls $B^g(v; T, T', \varepsilon) = p_\Gamma(B^g(\widetilde{v}; T, T', \varepsilon))$.

Recall the following well known fact in negative curvature.

Lemma 3.2. *Let (M, g) be a manifold with pinched negative curvature. For all $0 < a < b$, there exists a constant $c = c(a, b) > 0$ such that for all vectors $v, w \in S^g \widetilde{M}$, and all $T > 2c$, if $d^g(\pi(g_tv), \pi(g_tw)) \leq b$ for all $0 < t < T$, then $d^g(g_tv, g_tw) \leq a$ for all $c < t < T - c$.*

Proof. This is an exercise using standard comparison results. Note that the constant $c(a, b)$ also depends on the upper bound of the curvature. \square

Lemma 3.3. *Let (M, g) be a manifold with pinched negative curvature. For all $0 < \varepsilon_1 < \varepsilon_2$, there exists $C(g, \varepsilon_1, \varepsilon_2) > 0$ such that for all $v \in S^g \widetilde{M}$ and $T, T' > 0$, we have*

$$B^g(v; T + C(g, \varepsilon_1, \varepsilon_2), T' + C(g, \varepsilon_1, \varepsilon_2), \varepsilon_2) \subset B^g(v; T, T', \varepsilon_1) \subset B^g(v; -T, T', \varepsilon_2).$$

Proof. The right inclusion is obvious. The left one comes from Lemma 3.2 above. \square

The shadow $\mathcal{O}_x^g(B^g(y, R))$ of the ball $B^g(y, R)$ viewed from x w.r.t. the metric g is the set of positive endpoints in $\partial \widetilde{M}$ of g -geodesic rays starting from x and intersecting $B^g(y, R)$.

Recall Lemma 3.17 from [PPS15].

Lemma 3.4 ([PPS15]). *For all $r, \alpha > 0$ and $T, T' > 0$, and $v \in S^g \widetilde{M}$ such that $\mathcal{B}_{v_+^g}^g(\pi(v), o) = 0$, if x_t denotes the footpoint of $g_t(v)$, we have*

$$B^g(v; T, T', r) \subset (H^g)^{-1} \left(\mathcal{O}_{x_{-T'}}^g(B^g(x_T, 2r)) \times \mathcal{O}_{x_T}^g(B^g(x_{-T'}, 2r)) \times] - r, r[\right), \quad \text{and}$$

$$(H^g)^{-1} \left(\mathcal{O}_{x_{-T'}}^g(B^g(x_T, r)) \times \mathcal{O}_{x_T}^g(B^g(x_{-T'}, r)) \times] - \alpha, \alpha[\right) \subset B^g(v; T, T', 2r + 2\alpha).$$

When g_1 and g_2 are two admissible negatively curved metrics on M , recall that any g_1 -geodesic between any two points is at distance at most $C_2(g_1, g_2)$ of the g_2 -geodesic joining the same endpoints, and vice versa, for some constant $C_2(g_1, g_2)$ depending only on g_1 and g_2 . This leads immediately to the following lemma.

Lemma 3.5. *Let g_1 and g_2 be two admissible negatively curved metrics on M , and x, y two points on \widetilde{M} . Then*

$$\mathcal{O}_x^{g_1}(B^{g_1}(y, R)) \subset \mathcal{O}_x^{g_2}(B^{g_2}(y, R + C_2(g_1, g_2))) \subset \mathcal{O}_x^{g_1}(B^{g_1}(y, R + 2C_2(g_1, g_2))).$$

These lemmas will have the following very convenient corollary.

Corollary 3.6. *Let g_1 and g_2 be two admissible negatively curved metrics on M . For all $\varepsilon > 0$, there exists $C > 0$ and $\varepsilon' = \varepsilon'(\varepsilon) > 0$ such that for all $v \in S^{g_1}M$, we have*

$$B^{g_2}(\Psi^{g_1 \rightarrow g_2}(v), S+C, S'+C, \varepsilon) \subset \Psi^{g_1 \rightarrow g_2}(B^{g_1}(v, T, T', \varepsilon)) \subset B^{g_2}(\Psi^{g_1 \rightarrow g_2}(v), S, S', \varepsilon'),$$

where $S = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^T v))$, $S' = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^{-T'} v))$ and $\varepsilon' = \varepsilon(5 + C_1(g_1, g_2)) + C_2(g_1, g_2) + 2C_3(g_1, g_2)$.

Proof. As the sets considered in the above statement are typically small, we can prove them on \widetilde{TM} instead of TM . Without loss of generality, we can assume that $\pi(v) = o$. Indeed, all lemmas stated above are valid with o an arbitrary point, for example the basepoint of v . In particular, we have $\Psi^{g_1 \rightarrow g_2}(v) = \Phi^{g_1 \rightarrow g_2}(v)$.

We start with the right inclusion. Given $u \in B^{g_1}(v, T, T', \varepsilon)$, we want to control the distance $d^{g_2}(g_2^s \Psi^{g_1 \rightarrow g_2} u, g_2^s \Psi^{g_1 \rightarrow g_2} v)$. As $u \in B^{g_1}(v, T, T', \varepsilon)$, $\tau^{g_1, g_2}(u) \leq \varepsilon(1+C_1(g_1, g_2))$ so that $d^{g_2}(\Phi^{g_1 \rightarrow g_2}(u), \Psi^{g_1 \rightarrow g_2}(u)) \leq \varepsilon(1+C_1(g_1, g_2))$. Therefore, $\Psi^{g_1 \rightarrow g_2} B^{g_1}(v; T, T', \varepsilon)$ is included in the $\varepsilon(1+C_1(g_1, g_2))$ -neighbourhood of $\Phi^{g_1 \rightarrow g_2}(B^{g_1}(v; T, T', \varepsilon))$.

Let $w = \Psi^{g_1 \rightarrow g_2}(v) = \Phi^{g_1 \rightarrow g_2}(v)$. Denote by w_s (resp. z_s) the basepoint $\pi(g_2^s w)$, for $s \in \mathbb{R}$, of $g_2^s w$. (resp. of $g_2^s z$). Let $S = \mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi(g_1^T v))$ and $S' = -\mathcal{B}_{v_+^{g_1}}^{g_2}(o, \pi(g_1^{-T'} v))$. By lemma 2.6, we know that

$$|S - d^{g_2}(\pi(v), \pi(g_1^T v))| \leq C_3(g_1, g_2) \quad \text{and} \quad |S' - d^{g_2}(\pi(v), \pi(g_1^{-T'} v))| \leq C_3(g_1, g_2).$$

Moreover, the distances $d^{g_2}(w_S, v_T)$ and $d^{g_2}(w_{-S'}, v_{-T'})$ are uniformly bounded. Indeed, by Lemma 2.12, $\Psi^{g_1 \rightarrow g_2}(g_1^T v) = g_2^S \Psi^{g_1 \rightarrow g_2}(v)$ so that $d^{g_2}(w_S, v_T) = d^{g_2}(\pi(\Psi^{g_1 \rightarrow g_2}(g_1^T v)), \pi(g_1^T v)) \leq C_3(g_1, g_2)$.

Lemma 3.5 and elementary geometric considerations in negative curvature give the inclusion

$$\begin{aligned} & \mathcal{O}_{v_{-T'}}^{g_1}(B^{g_1}(v_T, 2\varepsilon) \times \mathcal{O}_{v_T}^{g_1}(B^{g_1}(x_{-T'}, 2\varepsilon)) \times] - \varepsilon, \varepsilon[\subset \\ & \mathcal{O}_{w_{-S'}}^{g_2}(B^{g_2}(w_S, 2\varepsilon + C_2(g_1, g_2) + 2C_3(g_1, g_2)) \times \\ & \mathcal{O}_{y_S}^{g_2}(B^{g_2}(y_{-S'}, 2\varepsilon + C_2(g_1, g_2) + 2C_3(g_1, g_2))) \times] - \varepsilon, \varepsilon[. \end{aligned}$$

Lemma 3.4 implies the right inclusion $\Phi^{g_1 \rightarrow g_2} B^{g_1}(v; T, T', \varepsilon) \subset B^{g_2}(w; S, S', 4\varepsilon + C_2(g_1, g_2) + 2C_3(g_1, g_2))$. The relation between $\Phi^{g_1 \rightarrow g_2}$ and $\Psi^{g_1 \rightarrow g_2}$ gives

$$\Psi^{g_1 \rightarrow g_2} B^{g_1}(v; T, T', \varepsilon) \subset B^{g_2}(\Psi^{g_1 \rightarrow g_2}(v), S, S', \varepsilon(5+C_1(g_1, g_2)) + C_2(g_1, g_2) + 2C_3(g_1, g_2)).$$

We proceed in the same way for the left inclusion, but we need in addition the help of Lemma 3.3.

Reasoning similarly as above gives the inclusion

$$B^{g_2}(w; S, S', \varepsilon) \subset \Psi^{g_1 \rightarrow g_2}(B^{g_1}(v; T, T', (4\varepsilon + C_2(g_1, g_2) + 2C_3(g_1, g_2))(1 + C_1(g_1, g_2))).$$

As T, T', ε are arbitrary, using lemma 3.3, we obtain easily the existence of a constant $C > 0$ such that

$$B^{g_2}(w; S + C, S' + C, \varepsilon) \subset \Psi^{g_1 \rightarrow g_2}(B^{g_1}(v; T, T', \varepsilon)).$$

□

3.2 Kolmogorov-Sinai, Brin-Katok and topological entropies

The Kolmogorov-Sinai entropy of a dynamical system T w.r.t an invariant probability measure μ is the supremum over all measurable partitions of the exponential growth rate of the complexity of a partition, when iterated by T , and measured by μ . By Shannon-McMillan-Breiman Theorem, it also equals (the supremum over all partitions of) the exponential decay rate of a typical atom of the iterated partition.

Instead of iterating a measurable partition, when X is a metric space, endowed with the Borel σ -algebra, one can consider exponential decay rate of the measure of typical dynamical balls, which will give us a notion of local entropy, introduced by [BK83].

When T is a continuous map on a *compact* space X , Brin-Katok [BK83] showed that this Kolmogorov-Sinai entropy coincides with the exponential decay of dynamical balls, also called the local entropy. This equality also holds when T is a Lipschitz map of a noncompact manifold, as has been verified in [Riq16, Thm 1.32].

We shall not define the classical Kolmogorov-Sinai entropy, denoted by $h_{KS}(T, m)$, because we do not really use it in this work. But we recall below some definitions of local entropy and the statements of Brin-Katok and Riquelme.

For $(\varphi^t) : X \rightarrow X$ a dynamical system and m a finite invariant measure, define the lower local entropy

$$\underline{h}_{loc}(T, m) = \operatorname{ess\,inf}_{x \in X} \lim_{\varepsilon \rightarrow 0} \liminf_{T \rightarrow \infty} -\frac{1}{T} \log m(B_{dyn}(x, T, \varepsilon)), \quad (8)$$

and the upper local entropy relative to compact sets

$$\overline{h}_{loc}^{comp}(T, m) = \sup_K \operatorname{ess\,sup}_{x \in K} \lim_{\varepsilon \rightarrow 0} \limsup_{T \rightarrow \infty, \varphi^T x \in K} -\frac{1}{T} \log m(B_{dyn}(x, T, \varepsilon)) \quad (9)$$

For the geodesic flow in negative curvature, dynamical balls should be defined relatively to a distance on $S^g M$, but, as mentioned in the above subsection, the "natural" Sasaki distance on $S^g M$ is equivalent to the distance $d(v, w) = \sup_{-1 \leq t \leq 0} d^g(\pi(g_t v), \pi(g_t w))$, so that, when studying asymptotic quantities as entropy, we can use the distance d^g on M instead of the Sasaki distance on $S^g M$.

The following result is essentially due to Brin-Katok and Riquelme.

Theorem 3.7 (Brin-Katok [BK83], Riquelme [Riq16] [Riq18]). *Let M be a Riemannian manifold with pinched negative curvature.*

$$h_{KS}(m, g) = \underline{h}_{loc}(m, g) = \overline{h}_{loc}^{comp}(m, g). \quad (10)$$

Proof. This result is due to Brin-Katok in the compact case. Their proof of the inequality $h_{KS}(m, g) \leq \underline{h}_{loc}(m, g)$ extends verbatim to the noncompact case. In [Riq16, Th.1.32], Riquelme proved the equality $h_{KS}(m, g) = \underline{h}_{loc}(m, g)$ for any Lipschitz dynamical system. In [Riq16, Th 1.41], he established the inequality $\underline{h}_{loc}(m, g) \leq \overline{h}_{comp}^{loc}(m, g)$, and the inequality $\overline{h}_{comp}^{loc}(m, g) \leq h_{KS}(m, g)$ is established in the proof of [Riq16, Th 1.42]. \square

As observed in Remark 3.1 there are two notions of dynamical balls and the small ones are more relevant for us. Therefore, we define what we will call the local entropy, denoted by $h_{\Gamma}^{loc}(m, g)$ in the sequel, as follows.

$$h_{\Gamma}^{loc}(m, g) = \sup_{K \subset S^g M} \operatorname{ess\,sup}_{v \in K} \lim_{\varepsilon \rightarrow 0} \limsup_{T \rightarrow \infty, g^T v \in K} -\frac{1}{T} \log m(B^g(v, T, \varepsilon)) \quad (11)$$

It follows from Theorem 3.7 and inclusion 7 that

$$h_{KS}(m, g) = h_{comp}^{loc}(m, g) \leq h_{\Gamma}^{loc}(m, g), \quad (12)$$

with equality as soon as M has an injectivity radius bounded from below or m has compact support.

Remark 3.8. Let us emphasize that all definitions of entropies above are sensitive to the scaling of the metric but not sensitive to the scaling of the measure. In particular, if m is finite but not a probability measure, then

$$h_{\Gamma}^{loc}(m, g) = h_{\Gamma}^{loc}(\lambda m, g) \quad \text{and} \quad h_{\Gamma}^{loc}(m, \lambda g) = \frac{h_{\Gamma}^{loc}(m, g)}{\sqrt{\lambda}}.$$

Remark 3.9. Observe that contrarily to Kolmogorov-Sinai entropy, the above definitions of local entropy make perfectly sense for an infinite invariant ergodic and conservative Radon measure. In particular, the Bowen-Margulis measure (see section 3.4) which, when finite, is the measure of maximal entropy of the geodesic flow, always has a local entropy with respect to small dynamical balls and return times into compact sets which coincides with the topological entropy of the geodesic flow, see Proposition 3.16.

Lemma 3.3 allows us to choose some $\varepsilon > 0$ without need to take the limit when $\varepsilon \rightarrow 0$. Moreover, the invariance of the measure allows to consider shifted dynamical balls. It is the result below.

Lemma 3.10. *Let (M, g) be a manifold with pinched negative curvature, and μ a geodesic current. Let m_{μ}^g be the g -invariant measure associated to μ on $S^g M$. One can compute its local entropy as*

$$h_{\Gamma}^{loc}(m_{\mu}^g, g) = \sup_K \sup_{v \in K} \operatorname{ess\,sup}_{T+T' \rightarrow \infty, g^T v \in K, g^{-T'} v \in K} \limsup -\frac{1}{T+T'} \log m_{\mu}^g(B^g(v; T, T', \varepsilon)).$$

Geometers usually are more interested in topological entropy than measure-theoretic entropy. We shall not define topological entropy topologically, but through the variational principle. Denote by $\mathcal{M}^1(g)$ the set of invariant probability measures for the metric g .

The topological entropy of the geodesic flow (g^t) , denoted by $h_{top}(g)$, satisfies

$$h_{top}(g) = \sup_{m \in \mathcal{M}^1(g)} h_{KS}(m, g). \quad (13)$$

This variational principle is due first to [Din70, Goo71], [Mis76] and later Handel-Kitchen [HK95] on noncompact spaces. It follows from [OP04] that this supremum is achieved iff the so-called Bowen-Margulis measure is finite (see later subsection 3.4 for details). In this case, it is the unique measure maximizing entropy.

3.3 Entropy transformation law

Our goal is to prove the following result.

Theorem 3.11. *Let (M, g_i) , $i = 1, 2$ be two admissible Riemannian metrics with pinched negative curvature on M . Let μ be a geodesic current and $m_\mu^{g_i}$ the associated invariant measure on $S^{g_i}M$ under the geodesic flow (g_i^t) . Assume that these measures are finite and ergodic. Then their local entropies are related as follows.*

$$h_\Gamma^{loc}(m_\mu^{g_2}, g_2) = I_\mu(g_2, g_1) \cdot h_\Gamma^{loc}(m_\mu^{g_1}, g_1).$$

Thanks to Proposition 2.15, the corollary below follows.

Corollary 3.12. *Under the same assumptions, we have*

$$h_\Gamma^{loc}(m_\mu^{g_2}, g_2) \times \|m_\mu^{g_2}\| = h_\Gamma^{loc}(m_\mu^{g_1}, g_1) \times \|m_\mu^{g_1}\|.$$

Let us prove Theorem 3.11.

Proof. Without loss of generality, we will assume μ to be ergodic. It follows from Lemma 3.10 that the entropy may be computed as

$$h_\Gamma^{loc}(m_\mu^{g_2}, g_2) = \sup_K \sup_{v \in K} \operatorname{ess} \limsup_{T+T' \rightarrow \infty, g_2^T v \in K, g_2^{-T'} v \in K} -\frac{1}{T+T'} \log m_\mu^{g_2}(B^{g_2}(v; -T, T', \varepsilon))$$

for some fixed $\varepsilon > 0$, the essential supremum being relative to $m_\mu^{g_2}$. The above limsup is constant along (g_2) -orbits, so that by ergodicity, it is $m_\mu^{g_2}$ -almost surely constant. Observe also that when K grows, the quantity on the right also grows.

Choose some large compact set $K \subset TM$ large enough to contain an open subset of $\Omega^{g_i} \cap S^{g_i}M$ for $i = 1, 2$, and to have positive $m_\mu^{g_i}$ measure. Choose it large enough so that it allows to estimate entropies $h_\Gamma^{loc}(m_\mu^{g_i}, g_i)$, up to some small arbitrary α . In other words,

$$\left| h_\Gamma^{loc}(m_\mu^{g_i}, g_i) - \sup_{v \in K \cap S^{g_i}M} \operatorname{ess} \limsup_{T+T' \rightarrow \infty, g_2^T v \in K, g_2^{-T'} v \in K} -\frac{1}{T+T'} \log m_\mu^{g_i}(B^{g_2}(v; -T, T', \varepsilon)) \right| \leq \alpha.$$

Choose a typical $v \in S^{g_1}M \cap K$, which realizes the above essential supremum on K , and the almost sure conclusion of Corollary 2.8 when $T \rightarrow \pm\infty$. With the notations of Corollary 3.6, let $w = \Psi^{g_1 \rightarrow g_2}(v)$. As observed in the preceding section, we have $m_\mu^{g_2} = \mathcal{E}^{g_1 \rightarrow g_2} \times \Psi_*^{g_1 \rightarrow g_2} m_\mu^{g_1}$. But $\mathcal{E}^{g_1 \rightarrow g_2}$ is uniformly close to 1 on $B^{g_1}(v, \varepsilon)$.

Thus, up to some constants $e^{\pm c(v, \varepsilon)}$, by corollary 3.6, we have

$$e^{-c(v, \varepsilon)} m_\mu^{g_2}(B^{g_2}(w; S+C, S'+C, \varepsilon)) \leq m_\mu^{g_1}(B^{g_1}(v; T, T', \varepsilon)) \leq e^{c(v, \varepsilon)} m_\mu^{g_2}(B^{g_2}(w; S, S', \varepsilon))$$

with $w = \Phi^{g_1 \rightarrow g_2}(v)$, $S = d^{g_2}(\pi(w), \pi(g_2^T w)) \pm C_3(g_1, g_2)$ and $S' = d^{g_2}(\pi(v), \pi(g_2^{-T'} w)) \pm C_3(g_1, g_2)$.

Observe also that the condition $g_1^T v \in K$ (resp. $g_1^{-T'} v \in K$) implies that $g_2^S w$ (resp. $g_2^{-S'} w$) belongs to the $C_3(g_1, g_2)$ -neighbourhood of K for any of the two metrics g_1 or g_2 . It remains true for $g_2^{S+C} w$ and $g_2^{-S'-C} w$ inside the $C_3(g_1, g_2) + C$ -neighbourhood of K for the metric g_2 . Set $K' = \mathcal{V}_{C_3(g_1, g_2)+C}(K) \supset K$.

By definition of T, T', S, S' , we also have $T + T' \rightarrow +\infty$ iff $S + S' \rightarrow \infty$. Therefore, taking the limsup of $\frac{1}{S+S'} \log$ of the above quantity, we get

$$\begin{aligned} & \limsup_{S+S' \rightarrow \infty, g_2^S w \in K', g_2^{-S'} w \in K'} -\frac{1}{S+S'} \log m_\mu^{g_2}(B^{g_2}(w, S, S', \varepsilon')) \\ = & \limsup_{T+T' \rightarrow \infty, g_1^T v \in K, g_1^{-T'} v \in K} \frac{T+T'}{S+S'} \times \frac{-1}{T+T'} \log m_\mu^{g_1}(B^{g_1}(v, T, T', \varepsilon)) \end{aligned}$$

By Corollary 2.8 we know that

$$\frac{T+T'}{S+S'} = \frac{T+T'}{\mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(g_1^{-T'} v), \pi(g_1^T v))} = \frac{T+T'}{\int_{-T'}^T \mathcal{E}^{g_1 \rightarrow g_2}(g_1^t v) dt}$$

converges when $T + T' \rightarrow +\infty$ to

$$\frac{1}{\int_{S^{g_1} M} \mathcal{E}^{g_1 \rightarrow g_2} \frac{dm_\mu^{g_1}}{\|m_\mu^{g_1}\|}} = \int_{S^{g_2} M} \mathcal{E}^{g_2 \rightarrow g_1} \frac{dm_\mu^{g_2}}{\|m_\mu^{g_2}\|}.$$

We deduce easily, by taking the supremum in K , that

$$h_\Gamma^{loc}(m_\mu^{g_2}, g_2) = \int_{S^{g_2} M} \mathcal{E}^{g_2 \rightarrow g_1} \frac{dm_\mu^{g_2}}{\|m_\mu^{g_2}\|} h_\Gamma^{loc}(m_\mu^{g_1}, g_1) = I_\mu(g_2, g_1) \times h_\Gamma^{loc}(m_\mu^{g_1}, g_1).$$

□

3.4 Bowen-Margulis measures and comparison of topological entropies

We define now the so-called *Bowen-Margulis measure*, and use it to deduce from Theorem 3.11 a corollary about the comparison of topological entropies of two metrics g_1 and g_2 . The construction below is due to Patterson [Pat76] for compact surfaces, to Sullivan [Sul79, Sul84] for geometrically finite hyperbolic manifolds, and Yue [Yue96] extended Sullivan's work in variable negative curvature.

Let (M, g) be a negatively curved manifold, with pinched negative curvature. Choose some point $o \in \widetilde{M}$. Consider the Poincaré series

$$P_\Gamma^g(s) = \sum_{\gamma \in \Gamma} e^{-sd^g(o, \gamma o)}.$$

Let $\delta(g)$ be its critical exponent. This exponent is finite, and when Γ is nonelementary, it is positive. The following lemma is immediate from the definition of δ .

Lemma 3.13. *Let $(g_\varepsilon)_{-1 \leq \varepsilon \leq 1}$ be a family of negatively curved metrics on $M = \widetilde{M}/\Gamma$, such that $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$. Then $e^{-\varepsilon/2} \delta(g_0) \leq \delta(g_\varepsilon) \leq e^{\varepsilon/2} \delta(g_0)$.*

We need to ensure that the above series diverges at $s = \delta(g)$, which could be false. We will modify $P_\Gamma^g(s)$ into $\widetilde{P}_\Gamma^g(s)$ as follows. The Patterson trick [Pat76] is the following. Define a continuous map $h : (0, +\infty) \rightarrow (0, +\infty)$ as

the exponential of continuous piecewise affine maps: $h(x) = \exp(\varepsilon_k x)$ on the interval I_k , with $\varepsilon_k \rightarrow 0$ and I_k a sequence of adjacent intervals of increasing length. It is possible to do it in such a way that h is positive, increasing, continuous, with slow growth, $\frac{h(t_0+t)}{h(t_0)}$ is bounded by $\exp(\varepsilon_k t)$, and therefore converges to 1 when $t \rightarrow +\infty$ uniformly in $t_0 > 0$. Moreover, ε_k and I_k can be chosen in order to ensure that

$$\tilde{P}_\Gamma^g(s) = \sum_{\gamma \in \Gamma} h(d^g(o, \gamma \cdot o)) e^{-s d^g(o, \gamma o)}$$

has exponent $\delta(g)$ but now diverges at $s = \delta(g)$.

Define for all $x \in \widetilde{M}$ and $s > \delta(g)$ a probability measure

$$\nu_x^s = \frac{1}{\tilde{P}_\Gamma^g(s)} \sum_{\gamma \in \Gamma} h(d^g(o, \gamma \cdot o)) e^{-s d^g(x, \gamma o)} \Delta_{\gamma o}$$

on $\overline{M} = \widetilde{M} \cup \partial \widetilde{M}$, where Δ_x denotes the Dirac mass at the point x . Choose a decreasing sequence $s_k \rightarrow \delta(g)$ such that $\nu_o^{s_k}$ converges to a probability measure ν_o^g on \overline{M} . Choose for all $x \in \widetilde{M}$ a subsequence s_{k_j} of s_k such that $\nu_x^{s_{k_j}}$ converges to a measure ν_x^g on \overline{M} . By construction, as $\tilde{P}(\delta(g))$ diverges, all these measures are equivalent finite measures supported on $\Lambda_\Gamma \subset \partial \widetilde{M}$, the measure ν_o^g is a probability measure, and this family $(\nu_x^g)_{x \in \widetilde{M}}$ satisfies two crucial properties for all $x, y \in \widetilde{M}$, almost all $\xi \in \partial \widetilde{M}$ and all $\gamma \in \Gamma$:

$$\frac{d\nu_x^g}{d\nu_y^g}(\xi) = \exp(-\delta(g) \mathcal{B}_\xi(x, y)) \quad \text{and} \quad \gamma_* \nu_x^g = \nu_{\gamma x}^g.$$

From these properties follows the Sullivan's Shadow Lemma.

Proposition 3.14 (Sullivan [Sul79]). *Let (ν_x^g) be a family of measures on Λ_Γ obtained as above. Then for all $R > 0$ large enough, there exists a constant $c = c(R) > 0$ such that*

$$\frac{1}{c} \exp(-\delta(g) d^g(o, \gamma o)) \leq \nu_o^g(\mathcal{O}_o(B(\gamma o, R))) \leq c \exp(-\delta(g) d^g(o, \gamma o)).$$

A *Bowen-Margulis measure* on $S^g \overline{M}$ is a measure obtained from such a family (ν_x^g) by the following formula on $S^g \overline{M}$, with $v = (H^g)^{-1}(v_-, v_+, t)$

$$d\tilde{m}_{BM}^g(v) = \exp\left(\delta(g) \mathcal{B}_{v_+^g}(o, \pi(v)) + \delta(g) \mathcal{B}_{v_-^g}(o, \pi(v))\right) d\nu_o^g(v_+^g) d\nu_o^g(v_-^g) dt. \quad (14)$$

This formula being Γ -invariant, it induces on the quotient a Bowen-Margulis measure m_{BM}^g on $S^g M$.

It is well known (see the above references, or Roblin [Rob03] for the most general version) that P_Γ^g diverges at $s = \delta(g)$ iff the Bowen-Margulis measure is ergodic and conservative, and in this case, the family of measures (ν_x^g) is in fact unique. In particular, when this measure m_{BM}^g is finite, it is ergodic and conservative and P_Γ^g diverges at $\delta(g)$.

Otal-Peigné proved the following result, due to Sullivan in the case of geometrically finite hyperbolic manifolds.

Theorem 3.15 (Sullivan [Sul84], Otal-Peigné [OP04]). *Let (M, g) be a manifold with pinched negative curvature and bounded derivatives of the curvature. Then*

$$\delta(g) = h_{top}(g)$$

is the topological entropy of g . Moreover, when m_{BM}^g is finite and normalized into a probability measure, it is the unique measure maximizing entropy in the sense that $h_{KS}(\frac{m_{BM}^g}{\|m_{BM}^g\|}, g) = h_{top}(g)$. When m_{BM}^g is infinite, there is no probability measure maximizing entropy.

It follows from [PPS15, Prop. 3.16] and [OP04] that, finite or not, the Bowen-Margulis measure satisfies the following equality.

Proposition 3.16. *Let (M, g) be a negatively curved manifold with pinched negative curvature and bounded derivatives of the curvature. Let m_{BM}^g be a Bowen-Margulis measure. Then*

$$h_{\Gamma}^{loc}(m_{BM}^g, g) = \delta(g) = h_{KS}(m_{BM}^g, g). \quad (15)$$

Proof. The first equality is a computation done in [PPS15, Prop. 3.16], the second is one of the main results of [OP04]. \square

This equality suggests that we could be able to prove a variational principle for infinite measures, using local entropies instead of Kolmogorov-Sinai entropies. We will not do it here.

Corollary 3.17. *Let (M, g_i) , $i = 1, 2$ be two admissible Riemannian metrics with pinched negative curvature on M . Let μ be a geodesic current and $m_{\mu}^{g_i}$ the associated invariant measure on $S^{g_i}M$ under the geodesic flow (g_i^t) . Assume that these measures are finite and ergodic. Then*

$$\begin{aligned} h_{top}(g_2) = \delta(g_2) = h_{\Gamma}^{loc}(m_{BM}^{g_2}, g_2) &= I_{\mu_{BM}^{g_2}}(g_2, g_1) \times h_{\Gamma}^{loc}(m_{\mu_{BM}^{g_2}}^{g_1}, g_1) \\ &\leq I_{\mu_{BM}^{g_2}}(g_2, g_1) \times h_{top}(g_1). \end{aligned}$$

Proof. Let us first note that by Theorem 4.2, the measure $m_{\mu_{BM}^{g_2}}^{g_1}$ is a Gibbs measure. Moreover, [PPS15, Thm 1.3] ensures that the Gibbs measure associated to a given potential, when finite, is the unique equilibrium measure of this potential. Therefore $h_{\Gamma}^{loc}(m_{\mu_{BM}^{g_2}}^{g_1}) = h_{KS}(m_{\mu_{BM}^{g_2}}^{g_1})$, and the variational principle ensures that $h_{KS}(m_{\mu_{BM}^{g_2}}^{g_1}) \leq h_{top}(g_1)$, which gives the last inequality. \square

In the compact case, the inequality $h_{top}(g_2) \leq I_{\mu_{BM}^{g_2}}(g_2, g_1) \times h_{top}(g_1)$ is due to Knieper [Kni95]. Katok had a similar weaker inequality [Kat82], proving that

$$h_{top}(g_2) \leq \int_{S^{g_2}M} \|v\|^{g_1} dm_{BM}^{g_2} \times h_{top}(g_1).$$

Our inequality above is valid on any manifold, compact or not, with finite Bowen-Margulis measure. It follows from lemma 2.5 that it implies Katok's inequality. Let us mention however that it is this weaker version which is really used in the proof of our main theorem of differentiability of entropy.

4 Gibbs measures

This section, particularly Theorem 4.2, is crucial in the proof of Corollary 3.17, and therefore in our approach of Theorem 1.7.

Theorem 4.2 is new on noncompact manifolds, the explicit change of potential being new even on compact manifolds. Corollary 4.4 is new even on compact manifolds.

Gibbs measures are, for a hyperbolic dynamical system, a family of measures with strong stochastic properties, each one associated to a weight, i.e. some Hölder continuous potential, describing somehow that all possible dynamical behaviours can happen. For the geodesic flow on the unit tangent bundle of a compact manifold, their geometric construction, adapted from the Patterson-Sullivan construction described in the above section, has been done by Ledrappier in [Led95]. He proved there, on compact manifolds, that being a Gibbs measure does not depend on the metric. In other words, if g_1 and g_2 are negatively curved metrics on M , an invariant measure $m_\mu^{g_1}$ on $S^{g_1}M$ is a Gibbs measure iff the measure $m_\mu^{g_2}$ on $S^{g_2}M$ is also a Gibbs measure. However, his proof strongly relies on the compactness of M . Our goal in this section is to prove this result differently on noncompact manifolds.

4.1 Definitions

We refer to [PPS15] for details on all notions presented here. Let (M, g) be a negatively curved manifold, with pinched negative curvatures and bounded derivatives of the curvature. Let $F : S^gM \rightarrow \mathbb{R}$ be a Hölder continuous map. The *pressure of F* is the quantity

$$P^g(F) = \sup_{m \in \mathcal{M}^1(g)} \left(h_{KS}(m, g) + \int_{S^gM} F dm \right), \quad (16)$$

the supremum being considered over all invariant probability measures $m \in \mathcal{M}^1(g)$. An invariant probability measure m is an *equilibrium state for F* if it realizes the above supremum.

Assume that $P^g(F)$ is finite. An invariant measure m under the geodesic flow (g^t) satisfies the *Gibbs property* for the potential F if for all compact sets $K \subset S^gM$ and $\varepsilon > 0$ there exists a constant $C(K, \varepsilon) > 0$ such that for all $v \in K$ and $T > 0$ with $g^T v \in K$, we have

$$\begin{aligned} \frac{1}{C(K, \varepsilon)} \exp \left(\int_0^T F(g^t v) dt - TP^g(F) \right) &\leq m(B^g(v, T, \varepsilon)) \\ &\leq C(K, \varepsilon) \exp \left(\int_0^T F(g^t v) dt - TP^g(F) \right) \end{aligned} \quad (17)$$

A variant of the Patterson-Sullivan construction presented in subsection 3.4 provides a measure m_F which satisfies (17) see [PPS15, Prop. 3.16]. Moreover, when finite and normalized into a probability measure, it is the unique equilibrium state, i.e. the unique measure realizing the supremum in (16) (see [PPS15, Th. 6.1]). When this measure m_F is infinite, there is no equilibrium state for F . Let us summarize what is useful in the present work in the following proposition.

Proposition 4.1. *Let (M, g) be a negatively curved manifold with pinched negative curvature and bounded derivatives of the curvature. Let $F : S^g M \rightarrow \mathbb{R}$ be a Hölder potential. If the measure m_F is finite and normalized, then*

$$P^g(m_F) = h_{KS}(m_F, g) + \int_{S^g M} F dm_F = h_{\Gamma}^{loc}(m_F, g) + \int_{S^g M} F dm_F.$$

4.2 Being a Gibbs measure does not depend on the metric

Theorem 4.2. *Let (M, g_i) be two admissible metrics with pinched negative curvature and bounded derivatives of the curvature on M . Let $F : S^{g_1} M \rightarrow \mathbb{R}$ be a Hölder map, and $m_F^{g_1}$ the associated Gibbs measure. We assume $m_F^{g_1}$ ergodic and conservative. Let $\mu_F^{g_1}$ be the associated current on $\partial^2 \widehat{M}$. Let $m_{\mu_F^{g_1}}^{g_2}$ be the g_2 -invariant measure associated to the same current.*

Then $m_{\mu_F^{g_1}}^{g_2}$ is also ergodic and conservative, and satisfies the Gibbs property (17) for the Hölder potential

$$G = (F - P^{g_1}(F)) \circ \Psi^{g_2 \rightarrow g_1} \times \mathcal{E}^{g_2 \rightarrow g_1}.$$

Moreover, $P^{g_2}(G) = 0$. In other words, for all compact subsets $K \subset S^{g_2} M$ and $\varepsilon > 0$ there exists $C > 0$ such that for all $w \in K$ and $S > 0$ with $g^S w \in K$, we have

$$\frac{1}{C} e^{\int_0^S G(g_2^s w) ds} \leq m_{\mu_F^{g_1}}^{g_2}(B_{\Gamma}^{g_2}(w, S, \varepsilon)) \leq C e^{\int_0^S G(g_2^s w) ds}.$$

If we assume moreover that the measure $m_F^{g_1}$ is finite, and is therefore the equilibrium measure associated to F , then $m_{\mu_F^{g_1}}^{g_2} / \|m_{\mu_F^{g_1}}^{g_2}\|$ is the equilibrium measure associated to G .

Remark 4.3. Reversing the role of g_1 and g_2 , we observe that the same result holds with the potential $H = ((F - P^{g_1}(F)) \times (\mathcal{E}^{g_1 \rightarrow g_2})^{-1}) \circ (\Psi^{g_1 \rightarrow g_2})^{-1}$. Therefore, they must be cohomologous.

Proof. Conservativity and ergodicity depend only on the current at infinity and not on the (admissible) metric, as said in Proposition 2.3.

Gibbs property for the potential G follows from Corollary 3.6. Let us explain it more in details. We stated Theorem 4.2 in the most natural way, starting from g_1 and going to g_2 , but in view of all the statements proved above that we shall use, we will reverse the role of g_1 and g_2 , F and G , in the proof below. Assume that $m_G^{g_2}$ is a Gibbs measure w.r.t. the potential G on $S^{g_2} M$, let $\mu = \mu_G^{g_2}$ be its current at infinity, and let us prove that $m_{\mu}^{g_1}$ is a Gibbs measure w.r.t. the potential $F = (G - P^{g_2}(G)) \circ \Psi^{g_1 \rightarrow g_2} \times \mathcal{E}^{g_1 \rightarrow g_2}$.

First choose some compact set $K^{g_1} \subset S^{g_1} M$ and some $\varepsilon > 0$. Let $v \in K^{g_1}$ and $T > 0$ such that $g^T v \in K^{g_1}$. Define a compact set K^{g_2} as the C -neighbourhood of $\Psi^{g_1 \rightarrow g_2}(K^{g_1}) \cup (\Psi^{g_2 \rightarrow g_1})^{-1} K^{g_1}$, where C is given by Corollary 3.6.

We will use Corollary 3.6 and first part of Proposition 2.13, and the fact that $m_{\mu}^{g_2} = \Psi_*^{g_1 \rightarrow g_2}(\mathcal{E}^{g_1 \rightarrow g_2} \times m_{\mu}^{g_1})$.

As $\mathcal{E}^{g_1 \rightarrow g_2}$ is continuous, it is uniformly continuous on K^{g_1} so that for all

$v \in K^{g_1}$ and $u \in B^{g_1}(v, \varepsilon)$, $\mathcal{E}^{g_1 \rightarrow g_2}(u) = e^{\pm c(K^{g_1}, \varepsilon)} \mathcal{E}^{g_1 \rightarrow g_2}(v)$. We deduce that

$$\begin{aligned} \frac{e^{-c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} m_\mu^{g_2}(\Psi^{g_1 \rightarrow g_2}(B^{g_1}(v, T, \varepsilon))) &\leq m_\mu^{g_1}(B^{g_1}(v, T, \varepsilon)) \\ &\leq \frac{e^{c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} m_\mu^{g_2}(\Psi^{g_1 \rightarrow g_2}(B_\Gamma^{g_1}(v, T, \varepsilon))). \end{aligned}$$

Now, using Corollary 3.6, with $w = \Psi^{g_1 \rightarrow g_2}v$, and $S = \mathcal{B}_{v_+^{g_1}}^{g_2}(\pi(v), \pi(g_1^T v))$, we get

$$\begin{aligned} \frac{e^{-c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} m_\mu^{g_2}(B^{g_2}(w : S + C, C, \varepsilon)) &\leq m_\mu^{g_1}(B^{g_1}(v, T, \varepsilon)) \\ &\leq \frac{e^{c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} m_\mu^{g_2}(B^{g_2}(w, S, \varepsilon')). \end{aligned}$$

As $m_\mu^{g_2}$ is a Gibbs measure, and $w, g_2^S w$, but also $g_2^{-C} w$ and $g_2^{S+C} w$ belong to K^{g_2} , there exists a constant $C(G, K^{g_2}, \varepsilon')$ coming from the Gibbs property, such that

$$\begin{aligned} \frac{e^{-c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} \frac{e^{\int_{-C}^{S+C} (G - P^{g_2}(G))(g_2^s w) ds}}{C(G, K^{g_2}, \varepsilon')} &\leq m_\mu^{g_1}(B^{g_1}(v, T, \varepsilon)) \\ &\leq \frac{e^{c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} C(G, K^{g_2}, \varepsilon') e^{\int_0^S (G - P^{g_2}(G))(g_2^s w) ds}. \end{aligned}$$

As G is (Hölder) continuous, it is bounded on K^{g_2} , so that the integral $\int_{-C}^{S+C} (G - P^{g_2}(G))(g_2^s w) ds$ is, up to a constant c , uniformly close to $\int_0^S (G - P^{g_2}(G))(g_2^s w) ds$. The next ingredient is Proposition 2.13, which gives

$$\begin{aligned} \frac{e^{-c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} \frac{e^{-c}}{C(G, K^{g_2}, \varepsilon')} e^{\int_0^T F(g_1^t w) dt} &\leq m_\mu^{g_1}(B^{g_1}(v, T, \varepsilon)) \\ &\leq \frac{e^{c(K^{g_1}, \varepsilon)}}{\mathcal{E}^{g_1 \rightarrow g_2}(v)} C(G, K^{g_2}, \varepsilon') e^{\int_0^T F(g_1^t w) dt}, \end{aligned}$$

with $F = (G - P^{g_2}(G)) \circ \Psi^{g_1 \rightarrow g_2} \times \mathcal{E}^{g_1 \rightarrow g_2}$. It is exactly the Gibbs property for $m_\mu^{g_1}$ w.r.t. F .

It remains to show that $P^{g_1}(F) = 0$. To simplify notations, let us assume that $P^{g_2}(G) = 0$. Let ρ be any geodesic current on $\partial^2 \widetilde{M}$. By definition,

$$P^{g_1}(F) = \sup_\rho \left(h_{KS}(m_\rho^{g_1}, g_1) + \int_{S^{g_1} M} F dm_\rho^{g_1} \right),$$

the supremum being taken over all currents ρ such that $m_\rho^{g_1}$ is an invariant probability measure. The change of mass and change of entropy (Corollary 2.15 and Theorem 3.11) give

$$P^{g_1}(F) = \sup_\rho I_\rho(g_2, g_1) \left(h(m_\rho^{g_2} / \|m_\rho^{g_2}\|, g_2) + \int G dm_\rho^{g_2} / \|m_\rho^{g_2}\| \right) \leq 0.$$

The same computations with $\rho = \mu = \mu_G^{g_2}$ give $P^{g_1}(F) = 0$. \square

4.3 Length spectrum and change of metrics

Let g_1 and g_2 be two quasi-isometric negatively curved metrics. There is a particular case where the above results have an easy but striking illustration

Corollary 4.4. *Let (M, g_i) be two quasi-isometric complete negatively curved metrics on the same connected manifold M . Assume that the Bowen-Margulis measure of g_1 is ergodic and conservative, and let $\mu_{BM}^{g_1}$ be the associated geodesic current. Then the measure $m_{\mu_{BM}^{g_1}}^{g_2}$ is also ergodic and conservative. It is a Gibbs measure associated with the potential $G = -h_{top}(g_1)\mathcal{E}^{g_2 \rightarrow g_1}$.*

Moreover, for all primitive hyperbolic elements $\gamma \in \Gamma$, if w_γ is a periodic vector of $S^{g_2}M$ associated to γ , for all $\varepsilon > 0$ there exists $C \geq 1$ such that for all $T > 0$, we have

$$\frac{1}{C} e^{-h_{top}(g_1)T \frac{\ell^{g_1}(\gamma)}{\ell^{g_2}(\gamma)}} \leq m_{\mu_{BM}^{g_1}}^{g_2}(B^{g_2}(w_\gamma, T, \varepsilon)) \leq C e^{-h_{top}(g_1)T \frac{\ell^{g_1}(\gamma)}{\ell^{g_2}(\gamma)}}.$$

Proof. It is an immediate application of Theorem 4.2 with $F = 0$. First write T as $T = n\ell^{g_2}(\gamma) + r$, with $0 \leq r < \ell^{g_2}(\gamma)$. The only thing to notice is that $\int_0^{\ell^{g_2}(\gamma)} \mathcal{E}^{g_2 \rightarrow g_1}(g_2^s w_\gamma) ds = \ell^{g_2}(\gamma) \times e^{g_2 \rightarrow g_1}(\gamma)$ so that

$$- \int_0^T h_{top}(g_1) \mathcal{E}^{g_2 \rightarrow g_1}(g_2^s v_\gamma) ds = -h_{top}(g_1) \times T \times \frac{\ell^{g_1}(\gamma)}{\ell^{g_2}(\gamma)} \pm \text{constant},$$

the error term in the above inequality being smaller than $h_{top}(g_1)\ell^{g_2}(\gamma)\|\mathcal{E}^{g_2 \rightarrow g_1}\|_\infty$. \square

5 Convergence of geodesics, Busemann functions and invariant measures

In this section, we study the continuity of geodesics, Busemann functions, and Bowen-Margulis measures under a Lipschitz perturbation of the metric with uniform negative curvatures.

Let $(g_\varepsilon)_{-1 \leq \varepsilon \leq 1}$ be a family of metrics on \widetilde{M} with sectional curvatures satisfying $K_{g_\varepsilon} \leq -a^2$, such that $\forall \varepsilon > 0$, at all $x \in \widetilde{M}$, $e^{-\varepsilon}g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$.

We first show that the g_ε -geodesic between two points at infinity converge uniformly in the Hausdorff topology of \widetilde{M} to the g_0 -geodesic with same extremities, and that the Busemann functions of g_ε converge uniformly on compact sets to the Busemann functions of g_0 .

When the variation of metrics is continuous in \mathcal{C}^1 -topology, this also implies that the Morse-correspondances $\Phi^{g_0 \rightarrow g_\varepsilon}$ and $\Psi^{g_0 \rightarrow g_\varepsilon}$ converge to the identity uniformly on compact sets in the \mathcal{C}^0 -topology of $S^g \widetilde{M}$, and that the geodesic stretch $\mathcal{E}^{g_0 \rightarrow g_\varepsilon}$ converges to 1.

Eventually, we show that under suitable assumptions, the Bowen-Margulis measures vary continuously in the weak-* topology.

5.1 Convergence of geodesics and Busemann functions

The following lemma is a classical and very useful consequence of the uniform upper bound on the curvature.

Lemma 5.1. *Let $a > 0$ and (\widetilde{M}, g) be a complete simply connected manifold with sectional curvatures satisfying $K_g \leq -a^2$.*

1. *For all $C > 0$, all $\xi \in \partial\widetilde{M}$, $x, y \in \widetilde{M}$ with $d^g(x, y) \leq C$, and $t \geq C$, if $x_t = \gamma_{x, \xi}(t)$, we have*

$$\left| \mathcal{B}_\xi^g(x, y) - (d^g(x, x_t) - d^g(y, x_t)) \right| \leq 2Ce^{-at}.$$

2. *For all $T, K, \alpha > 0$, for all $R \geq R_0 = T - \frac{1}{a} \ln \frac{\alpha}{4Ke^K}$, if $(\gamma_1(t))_{t \in \mathbb{R}}$ and $(\gamma_2(t))_{t \in \mathbb{R}}$ are g -geodesics with*

$$d^g(\gamma_1(-R), \gamma_2(-R)) \leq K \quad \text{and} \quad d^g(\gamma_1(R), \gamma_2(R)) \leq K,$$

then for all $t \in [-T, T]$,

$$d^g(\gamma_1(t), \gamma_2) \leq \alpha.$$

Proof. We will omit the subscript g in the proof. Let us first prove 1.

Assume $d(x, y) \leq C$. We can also assume that $\mathcal{B}_\xi(x, y) \geq 0$. Denote by x' the unique point on $[x, \xi]$ such that $\mathcal{B}_\xi(x', y) = 0$. By convexity of the horoball, $d(x, x') \leq C$ and $d(x', y) \leq C$. Let x_s (resp. y_s) be the points on $[x', \xi]$ (resp. $[y, \xi]$) at distance s of x' (resp. y). It follows from [HIH77] that for all $s \geq C$,

$$d(x_s, y_s) \leq d(x', y)e^{-as} \leq Ce^{-as}.$$

Observe also that $\left| \mathcal{B}_\xi^g(x, y) - (d^g(x, x_s) - d^g(y, y_s)) \right| = |\mathcal{B}_\xi(x_s, y_s)| \leq d(x_s, y_s)$, so that $\left| \mathcal{B}_\xi^g(x, y) - (d^g(x, x_s) - d^g(y, y_s)) \right| \leq 2d(x_s, y_s) \leq 2Ce^{-as}$.

To prove 2, denote by x_s the point of $[\gamma_1(-R), \gamma_1(R)]$ at distance s from $\gamma_1(-R)$, y_s the point of $[\gamma_1(-R), \gamma_2(R)]$ at distance t from $\gamma_1(-R)$ and distance say d_s from $\gamma_2(R)$ and z_s the point of $[\gamma_2(-R), \gamma_2(R)]$ at distance d_s from $\gamma_2(R)$. Observe immediately that $|d_s - 2R + s| \leq K$.

By the above, we have $d(x_s, y_s) \leq d(x_{2R}, y_{2R})e^{-as}$. But elementary considerations in the triangle $(x_{2R}, y_{2R}, \gamma_2(R))$ lead to

$$d(x_{2R}, y_{2R}) = d(\gamma_1(R), y_{2R}) \leq d(\gamma_1(R), \gamma_2(R)) + d(\gamma_2(R), y_{2R}) \leq 2K.$$

Thus $d(x_s, y_s) \leq 2Ke^{-as}$.

Similarly we get $d(y_s, z_s) \leq 2Ke^{-ad_s} \leq 2Ke^K e^{-a(2R-s)}$. We deduce that

$$d(x_s, \gamma_2) \leq d(x_s, z_s) \leq 2Ke^K (e^{-as} + e^{-a(2R-s)}).$$

Now, choose $R_0 = T - \frac{1}{a} \ln \frac{\alpha}{4Ke^K}$. For $t \in [-T, T]$, we have $\gamma_1(t) = x_{R+t}$ and $R+t \geq R_0 - T$ and $2R - (R+t) \geq R_0 - T$, so that

$$d(\gamma_1(t), \gamma_2) \leq d(\gamma_1(t), z_{R+t}) \leq 4Ke^K e^{-a(R_0-T)} \leq \alpha.$$

□

Let us now show that the g_ε -geodesic segments converge to the g_0 -geodesic segments in the Hausdorff topology of \widetilde{M} .

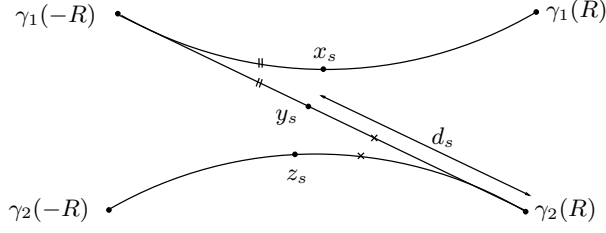


Figure 5: Proof of Lemma 5.1

Proposition 5.2. *Let g_0 be a complete metric on \widetilde{M} with $K_{g_0} \leq 0$. For all $0 < \varepsilon \leq 1$ small enough let g_ε be a complete metric on \widetilde{M} such that at all $x \in \widetilde{M}$, $e^{-\varepsilon}g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$.*

Then for all $x, y \in \widetilde{M}$, the minimizing g_ε -geodesic γ_ε joining x to y is contained in the D_ε -neighbourhood of the g_0 -geodesic $[x, y]_0$ from x to y , with $D_\varepsilon \leq \min(\sqrt{\varepsilon}d^{g_0}(x, y), C(g_1, g_0))$.

Proof. Let g_0 and g_ε as above, and $x, y \in \widetilde{M}$. Set $L_0 = d^{g_0}(x, y)$ and $L_\varepsilon = d^{g_\varepsilon}(x, y)$. Let $\gamma_0 : [0, L_0] \rightarrow \widetilde{M}$ and $\gamma_\varepsilon : [0, L_\varepsilon] \rightarrow \widetilde{M}$ be minimizing geodesics from x to y respectively for g_0 and g_ε , parametrized with unit speed. Note that γ_0 is unique. Let $l \in [0, L_\varepsilon]$ be such that

$$d^{g_0}(\gamma_\varepsilon(l), [x, y]_0) = \max_{t \in [0, L_\varepsilon]} d^{g_0}(\gamma_\varepsilon(t), [x, y]_0) = D_\varepsilon.$$

We call $z = \gamma_\varepsilon(l) \in \widetilde{M}$. Consider the g_0 -geodesic triangle with vertices x, y, z . Set $l_1 = d^{g_0}(x, z)$ and $l_2 = d^{g_0}(z, y)$.

We have

$$l_1 \leq \int_0^l \|\dot{\gamma}_\varepsilon(t)\|_{g_0} dt \quad \text{and} \quad l_2 \leq \int_l^{L_\varepsilon} \|\dot{\gamma}_\varepsilon(t)\|_{g_0} dt. \quad (18)$$

Since $e^{-\varepsilon}g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$, we have $\|\dot{\gamma}_\varepsilon(t)\|_{g_0} \leq e^{\varepsilon/2}$ for all $t \in [0, L_\varepsilon]$ and $\|\dot{\gamma}_0(t)\|_{g_\varepsilon} \leq e^{\varepsilon/2}$ for all $t \in [0, L_0]$. Therefore, by Equation (18),

$$L_\varepsilon \leq \int_0^{L_\varepsilon} \|\dot{\gamma}_\varepsilon(t)\|_{g_\varepsilon} dt \leq e^{\varepsilon/2} L_\varepsilon \quad \text{and} \quad l_1 + l_2 \leq \int_0^{L_\varepsilon} \|\dot{\gamma}_\varepsilon(t)\|_{g_0} dt \leq e^\varepsilon L_\varepsilon.$$

Since $K_{g_0} \leq 0$, the distance d^{g_0} satisfies $CAT(0)$ -triangle comparison property (cf [BH99] p161): D_ε is less than the height \bar{D} from \bar{z} of the comparison triangle $(\bar{x}, \bar{y}, \bar{z})$ in the Euclidean plane with side lengths $d^{eucl}(\bar{x}, \bar{y}) = L_0$, $d^{eucl}(\bar{x}, \bar{z}) = l_1$ and $d^{eucl}(\bar{y}, \bar{z}) = l_2$. Moreover, for all such Euclidean triangles with $l_1 + l_2 \leq e^\varepsilon L_0$, the height \bar{D} is maximal if and only if $l_1 = l_2 = \frac{e^\varepsilon L_0}{2}$. Therefore,

$$D_\varepsilon^2 \leq \bar{D}^2 \leq \frac{e^{2\varepsilon} L_0^2}{4} - \frac{L_0^2}{4} \leq \varepsilon L_0^2$$

as soon as $e^{2\varepsilon} - 1 \leq 4\varepsilon$, which ends the proof of Proposition 5.2. \square

Proposition 5.2 together with Lemma 5.1 imply that when the curvatures have a uniform negative upper bound, the complete geodesics on \widetilde{M} converge uniformly for the g_0 -Hausdorff topology under a variation of the metric. Let $a > 0$ be fixed.

Proposition 5.3. *Let $(g_\varepsilon)_{-1 < \varepsilon < 1}$ be a family of metrics on \widetilde{M} with sectional curvatures satisfying $K_{g_\varepsilon} \leq -a^2$, such that $\forall \varepsilon \in (-1, 1)$, at all $x \in \widetilde{M}$, $e^{-\varepsilon}g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$. Then there exists $\alpha : (-1, 1) \rightarrow [0, +\infty)$, with $\lim_{\varepsilon \rightarrow 0} \alpha(\varepsilon) = 0$, such that for all $\varepsilon \in (-1, 1)$ and all $(\eta, \xi) \in \partial^2 \widetilde{M}$, the g_ε -geodesic with extremities η and ξ is contained in the $\alpha(\varepsilon)$ -neighbourhood of the g_0 -geodesic with extremities η and ξ .*

Proof. First, recall (see section 2.1) that the geodesics for g_0 and g_ε are at uniform bounded distance $C_1(g_0, g_\varepsilon) \leq C_1(g_0, g_1)$. Let γ_0 be the g_0 -geodesic from ξ to η . Choose its origin $\gamma_0(0)$ arbitrarily. For any large $\rho > 0$, we have $d(\gamma_0(\pm\rho), \gamma_\varepsilon) \leq C_1(g_0, g_1)$. Consider the g_0 -geodesic segment γ_1 joining the nearest point to $\gamma_0(\rho)$ on γ_ε with the nearest point to $\gamma_0(-\rho)$ on γ_ε . This geodesic segment has g_0 -length equal to $2R = 2\rho \pm 2C_1(g_0, g_1)$. Choose its origin in such a way that $d^{g_0}(\gamma_0(\pm R), \gamma_1(\pm R)) \leq 2C_1(g_0, g_1)$.

For all $\alpha > 0$, Lemma 5.1 applied with $K = 2C_1(g_0, g_1)$, $\alpha/2$ and $T = 1$ gives some $R_0 > 0$ such that when $R \geq R_0$, for all $t \in [-1, 1]$, $d^{g_0}(\gamma_1(t), \gamma_0(t)) \leq \alpha/2$.

By Proposition 5.2, $d^{g_0}(\gamma_1(0), \gamma_\varepsilon) \leq 2R\sqrt{\varepsilon}$.

Therefore,

$$d^{g_0}(\gamma_0(0), \gamma_\varepsilon) \leq d^{g_0}(\gamma_0(0), \gamma_1(0)) + d^{g_0}(\gamma_1(0), \gamma_\varepsilon) \leq \alpha/2 + 2R\sqrt{\varepsilon}.$$

Choose $R \geq R_0$ and $\varepsilon > 0$ such that $2R_0\sqrt{\varepsilon} \leq \alpha/2$ to get $d^{g_0}(\gamma_0(0), \gamma_\varepsilon) \leq \alpha$. As the origin on γ_0 is arbitrary, the result follows. \square

Observe that, in the above proof, ε can be made relatively explicit. For $K = C_1(g_0, g_1)$, $T = 1$ and $\alpha/2$ we get $R_0 = 1 + \frac{2C_1(g_0, g_1)}{a} \ln \frac{2C_1(g_0, g_1)}{\alpha}$ and $\varepsilon = \frac{\alpha^2}{16R_0^2}$.

Moreover, our proof only uses $K_{g_0} \leq -a^2 < 0$ and that for all g_ε , the g_ε -geodesic between two points at infinity is unique. The negative upperbound on the K_{g_ε} does not need to be uniform.

Proposition 5.4. *Let $(g_\varepsilon)_{-1 \leq \varepsilon \leq 1}$ be a family of complete metrics on \widetilde{M} with $K_{g_\varepsilon} \leq -a^2$, such that for all $\varepsilon > 0$, at all $x \in \widetilde{M}$, $e^{-\varepsilon}g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$.*

Then the map $\mathcal{B}^{g_\varepsilon} : (x, y, \xi) \mapsto \mathcal{B}_\xi^{g_\varepsilon}(x, y)$ converges to \mathcal{B}^{g_0} as $\varepsilon \rightarrow 0$, uniformly on compact sets of $\widetilde{M} \times \widetilde{M} \times \partial \widetilde{M}$.

Proof. Any compact set $K \subset \widetilde{M}$ is contained in some (noncompact) set of the form $H_C = \{(x, y, \xi) \in \widetilde{M} \times \widetilde{M} \times \partial \widetilde{M}; d^{g_0}(x, y) \leq C\}$, for some $C > 0$. It is enough to show that $\mathcal{B}^{g_\varepsilon} \rightarrow \mathcal{B}^{g_0}$ as $\varepsilon \rightarrow 0$, uniformly on each H_C .

Let $C > 0$ be fixed. For all $\varepsilon \in (-1, 1)$ and all $(x, y, \xi) \in H_C$,

$$d^{g_\varepsilon}(x, y) \leq 2d^{g_0}(x, y) \leq 2C.$$

Let $\eta > 0$ be fixed. Choose x_t at distance t from $x = x_0$ on the g_0 -geodesic (x, ξ) , and let y_t be the point on the g_0 -geodesic (y, ξ) such that $\mathcal{B}_\xi^{g_0}(x, y) = 0$.

Let x_t^ε be the projection of x_t on the g_ε -geodesic from x to ξ . Proposition 5.3 ensures that $d^{g_\varepsilon}(x_t, x_t^\varepsilon) \leq \alpha(\varepsilon)$. Let us write

$$\begin{aligned} \left| \mathcal{B}_\xi^{g_\varepsilon}(x, y) - \mathcal{B}_\xi^{g_0}(x, y) \right| &\leq \left| \mathcal{B}_\xi^{g_\varepsilon}(x, y) - d^{g_\varepsilon}(x, x_t^\varepsilon) + d^{g_\varepsilon}(y, x_t^\varepsilon) \right| \\ &+ |d^{g_\varepsilon}(x, x_t^\varepsilon) - d^{g_\varepsilon}(y, x_t^\varepsilon) - d^{g_\varepsilon}(x, x_t) + d^{g_\varepsilon}(y, x_t)| \\ &+ |d^{g_\varepsilon}(x, x_t) - d^{g_0}(x, x_t) - d^{g_\varepsilon}(y, x_t) + d^{g_0}(y, x_t)| \\ &+ \left| \mathcal{B}_\xi^{g_0}(x, y) - d^{g_0}(x, x_t) + d^{g_0}(y, x_t) \right|. \end{aligned}$$

For $t \geq 2C$, by Lemma 5.1, the last term on the right hand side is bounded from the above by $4Ce^{-at}$. For $t \geq 2Ce^\varepsilon + \alpha(\varepsilon)$, we also have $d^{g_\varepsilon}(x, x_t^\varepsilon) \geq 2C$ so that again by Lemma 5.1, the first term is bounded from the above by $4Ce^{-ad^{g_\varepsilon}(x, x_t^\varepsilon)} \leq 4Ce^{\alpha(\varepsilon)}e^{-at/2}$. By triangular inequality, the second term is bounded from the above by $2\alpha(\varepsilon)$. The inequality $e^{-\varepsilon}g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$ allows to bound the third term by $2(e^\varepsilon - 1)(t + C)$.

At last, we get

$$\left| \mathcal{B}_\xi^{g_\varepsilon}(x, y) - \mathcal{B}_\xi^{g_0}(x, y) \right| \leq 4Ce^{\alpha(\varepsilon)}e^{-at/2} + 2\alpha(\varepsilon) + 2(e^\varepsilon - 1)(t + C) + 4Ce^{-at}.$$

Let $\eta > 0$ be fixed. Choose first ε_0 so that for $\varepsilon \leq \varepsilon_0$, $\alpha(\varepsilon) \leq 1$. Chose $t \geq 2C$ large enough to guarantee that the first and the last term are each bounded from the above by $\eta/4$. Choose $\varepsilon_1 \leq \varepsilon_0$ small enough to guarantee that for $\varepsilon \leq \varepsilon_1$, $\alpha(\varepsilon) \leq \eta/4$ and $2(e^\varepsilon - 1)(t + C) \leq \eta/4$. Thus, $|\mathcal{B}_\xi^{g_\varepsilon}(x, y) - \mathcal{B}_\xi^{g_0}(x, y)| \leq \eta$. This gives the desired result. \square

Remark 5.5. Eventhough this section is written in a Riemannian setting, all the previous proofs apply verbatim to a family of distances $(d_\varepsilon)_{-1 < \varepsilon < 1}$ on X such that for all $\varepsilon \in (-1, 1)$, the metric space (X, d_ε) is CAT(-1) and $e^{-\varepsilon}d_0 \leq d_\varepsilon \leq e^\varepsilon d_0$.

5.2 Higher regularity, Morse correspondances and geodesic stretch

In this section, we consider metrics $g_\varepsilon \rightarrow g_0$ in the C^1 -topology. To emphasize the necessity of this assumption, observe that $g_\varepsilon \rightarrow g_0$ in the C^0 -topology does not imply the convergence of the curvatures nor the convergence of the geodesic flow.

In particular, one can "add mushrooms" on a hyperbolic manifold, and make the mushrooms as small as we want, and build a sequence of manifolds with many points of nonnegative curvature converging to a hyperbolic manifold. The geodesic flow of such g_ε will not converge in general to the geodesic flow of g_0 .

In view of its importance in the sequel, recall the convergence that we shall use.

Definition 5.6. A family $(g_\varepsilon)_{-1 \leq \varepsilon \leq 1}$ of complete Riemannian metrics on \widetilde{M} (or M) converges in the C^1 -topology, uniformly on compact sets, to g_0 if:

1. (g_ε) converges to g_0 uniformly on compact sets, i.e. for all compact sets $K \subset \widetilde{TM}$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{v \in K} |g_\varepsilon(v, v) - g_0(v, v)| = 0;$$

2. the first derivatives of g_ε also converge uniformly on compact sets to those of g_0 .

By Theorem 2.79 of [GHL04], it implies for all fixed $T > 0$ the uniform convergence on compact sets of the geodesic flows $v \mapsto g_\varepsilon^T v$. As a consequence, we get the following result.

Theorem 5.7. *Let $(g_\varepsilon)_{-1 < \varepsilon < 1}$ be a family of metrics on \widetilde{M} with sectional curvatures satisfying $K_{g_\varepsilon} \leq -a^2$, such that for all $\varepsilon \in (-1, 1)$, at all $x \in \widetilde{M}$, $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$, and $g_\varepsilon \rightarrow g_0$ in the C^1 topology, uniformly on compact sets.*

Let $\Phi^{g_0 \rightarrow g_\varepsilon}$ and $\Psi^{g_0 \rightarrow g_\varepsilon}$ be the Morse correspondances between $S^{g_0} \widetilde{M}$ and $S^{g_\varepsilon} \widetilde{M}$ defined in section 2.4. Then $\Phi^{g_0 \rightarrow g_\varepsilon} \rightarrow \text{Id}$ and $\Psi^{g_0 \rightarrow g_\varepsilon} \rightarrow \text{Id}$ uniformly on all compact sets $K \subset S^{g_0} \widetilde{M}$ in the uniform topology of $C^0(K, T\widetilde{M})$.

Proof. Let K be a fixed compact set of $S^{g_0} \widetilde{M}$ and $v \in K$, with $v_\pm^{g_0}$ the endpoints of its g_0 -geodesic in $\partial \widetilde{M}$. Denote by $(\gamma_0(t))_{t \in \mathbb{R}}$ the parametrization of this geodesic such that $\gamma_0'(0) = v$. Let γ_ε be the parametrization of the g_ε -geodesic with same endpoints, with $v_\varepsilon = \gamma_\varepsilon'(0) = \Phi^{g_0 \rightarrow g_\varepsilon}(v)$.

By Proposition 5.4 and definitions from Section 2.4, uniform convergence of $\Psi^{g_0 \rightarrow g_\varepsilon}$ on compact sets will follow from the convergence of $\Phi^{g_0 \rightarrow g_\varepsilon}$. So let us prove the latter.

We will use the distance $d(w, w') = \sup_{t \in [0, 1]} d^{g_0}(\pi(g_0^t w), \pi(g_0^t w'))$ on $T\widetilde{M}$ and show that for all $\alpha > 0$, if ε is small enough, for all $v \in K$ and $t \in [0, 1]$, $d^{g_0}(\pi(g_0^t v), \pi(g_0^t v_\varepsilon)) \leq \alpha$.

Choose some $\alpha > 0$. By Propositions 5.3 and 5.4, for ε small enough, uniformly in $v \in K$, and $t \in [-1, 1]$, we know that γ_ε is in the $\alpha/2$ -neighbourhood of γ_0 , and $\gamma_\varepsilon(t)$ is uniformly close to $\gamma_0(t)$. It implies that $v_\varepsilon = \gamma_\varepsilon'(0)$ and $v_0 = \gamma_0'(0)$ are uniformly close. As $g_\varepsilon \rightarrow g_0$ in the C^1 -topology, uniformly on compact sets, it implies that for ε small enough, for all $t \in [-1, 1]$, $\pi(g_\varepsilon^t(v_\varepsilon))$ and $\pi(g_0^t(v_0))$ will stay $\alpha/2$ -close. In particular, $\pi(g_0^t(v_\varepsilon))$ will stay α -close from $\pi(g_0^t(v_0))$, for $t \in [-1, 1]$. That is the desired convergence. \square

Remark 5.8. Adapting Theorem 5.7 and the definition of the geodesic stretch in the setting of $\text{CAT}(-1)$ spaces would require a careful definition of the tangent bundle on such spaces with its topology, which we will not do here.

Let us conclude this section by a key technical ingredient.

Theorem 5.9. *Let $(g_\varepsilon)_{-1 < \varepsilon < 1}$ be a family of metrics on \widetilde{M} with sectional curvatures satisfying $K_{g_\varepsilon} \leq -a^2$, such that for all $\varepsilon \in (-1, 1)$, at all $x \in \widetilde{M}$, $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$, and $g_\varepsilon \rightarrow g_0$ in the C^1 -topology, uniformly on compact sets.*

Then uniformly on compact sets of $S^{g_0} \widetilde{M}$, we have

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}^{g_0 \rightarrow g_\varepsilon}(v) \leq 1.$$

Moreover,

$$\mathcal{E}^{g_0 \rightarrow g_\varepsilon} \rightarrow 1 \quad m_\mu^{g_0} - \text{almost surely.}$$

Proof. Observe that Lemma 2.5 gives the obvious upper bound $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}^{g_0 \rightarrow g_\varepsilon} \leq 1$, uniformly on $S^{g_0} \widetilde{M}$. For the same reason, $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}^{g_\varepsilon \rightarrow g_0} \leq 1$, uniformly on

$S^{g_\varepsilon} \widetilde{M}$. By Corollary 2.15, one easily deduces that

$$\frac{\|m_\mu^{g_\varepsilon}\|}{\|m_\mu^{g_0}\|} \rightarrow 1 \quad \text{when } \varepsilon \rightarrow 0 \quad (19)$$

Combined with the fact that $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}^{g_0 \rightarrow g_\varepsilon} \leq 1$, this implies in turn that $\mathcal{E}^{g_0 \rightarrow g_\varepsilon} \rightarrow 1$ $m_\mu^{g_0}$ -almost surely. \square

5.3 Narrow convergence of measures associated to a fixed geodesic current

Recall that if μ is a Γ -invariant geodesic current and g an admissible metric on M , we denote by m_μ^g the locally finite Radon measure on $S^g M$ whose lift to $S^g \widetilde{M}$ is given by

$$d\widetilde{m}_\mu^g(v) = (H^g)^*(\mu \times dt).$$

The results of the previous paragraph imply the following fact.

Proposition 5.10. *Let $(g_\varepsilon)_{-1 < \varepsilon < 1}$ be a family of metrics on \widetilde{M} whose sectional curvatures satisfy $K_{g_\varepsilon} \leq -a^2$, and such that for all $\varepsilon \in (-1, 1)$, at all $x \in \widetilde{M}$, $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$, and $g_\varepsilon \rightarrow g_0$ in the \mathcal{C}^1 -topology, uniformly on compact sets. Let μ be a Γ -invariant geodesic current. Then the measures $m_\mu^{g_\varepsilon}$ converge to $m_\mu^{g_0}$ in the dual of bounded continuous maps on TM (i.e. in the narrow topology).*

Proof. By definition, for all $\varepsilon \in (-1, 1)$ we have $m_\mu^{g_\varepsilon} = (\Psi^{g_0 \rightarrow g_\varepsilon})_* (\mathcal{E}^{g_0 \rightarrow g_\varepsilon} \times m_\mu^{g_0})$. Therefore the weak-* convergence (in the dual of continuous compactly supported functions) is an immediate consequence of Theorem 5.7 and dominated convergence Theorem.

We also showed that $\|m_\mu^{g_\varepsilon}\| \rightarrow \|m_\mu^{g_0}\|$, see Equation (19). It is classical that it implies the convergence of the above measures in the dual of bounded continuous functions. The result follows. \square

5.4 Weak convergence of Bowen-Margulis measures

We now show that, provided they are unique, the Bowen-Margulis measures are continuous in the weak-* topology under Lipschitz deformations of the metric.

Proposition 5.11. *Let $(g_\varepsilon)_{-1 \leq \varepsilon \leq 1}$ be a family of metrics on M with sectional curvatures satisfying $K_{g_\varepsilon} \leq -a^2$, such that for all $\varepsilon \in (-1, 1)$, (Γ, g_ε) is divergent and at all $x \in \widetilde{M}$, $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$.*

Then for all $x \in \widetilde{M}$, the Patterson-Sullivan measures for g_ε normalized at o converge: $\lim_{\varepsilon \rightarrow 0} \nu_x^{g_\varepsilon} = \nu_x^{g_0}$ in the weak- topology, uniformly in x on compact sets of \widetilde{M} .*

Proof. For all $\varepsilon \in (-1, 1) \setminus \{0\}$, the measure $\nu_o^{g_\varepsilon}$ is a probability measure on Λ_Γ . Let $\widetilde{\nu}_o = \lim_{\varepsilon_i \rightarrow 0} \nu_o^{g_{\varepsilon_i}}$ be any of its weak limits. Define for all $x \in \widetilde{M}$ a measure $\widetilde{\nu}_x$ on Λ_Γ by

$$\frac{d\widetilde{\nu}_x}{d\widetilde{\nu}_o}(\xi) = e^{-\delta(g_0) \mathcal{B}_\xi^{g_0}(o, x)}.$$

It is a Γ -invariant, $\delta(g_0)$ conformal family of measures, normalized at o . By uniqueness of such a family, it coincides with $(\nu_x^{g_0})_{x \in \widetilde{M}}$. \square

Recall that μ_{BM}^g denotes the g -Bowen-Margulis geodesic current on $\partial^2 \widetilde{M}$ given by

$$d\mu_{BM}^g(\eta, \xi) = d\nu_x^g(\eta) d\nu_x^g(\xi) = e^{-\delta(g)(\mathcal{B}_\xi^g(o,x) + \mathcal{B}_\eta^g(o,x))} d\nu_o^g(\eta) d\nu_o^g(\xi),$$

where x is any point on the g -geodesic with endpoints (η, ξ) . We get the immediate corollary of Propositions 5.4 and 5.11.

Corollary 5.12. *Under the same assumptions, in the weak- $*$ topology of $\partial^2 \widetilde{M}$, $\lim_{\varepsilon \rightarrow 0} \mu_{BM}^{g_\varepsilon} = \mu_{BM}^{g_0}$.*

Remark 5.13. Once again, Proposition 5.11 and Corollary 5.12 are still valid if we consider a family of Γ -invariant distances $(d_\varepsilon)_{\varepsilon \in (-1,1)}$ on \widetilde{M} such that $(\widetilde{M}, d_\varepsilon)$ is a CAT(-1) and $e^{-\varepsilon} d_0 \leq d_\varepsilon \leq e^\varepsilon d_0$ for all $\varepsilon \in (-1, 1)$.

We end this section by the convergence of Bowen-Margulis measures.

Theorem 5.14 (Convergence of Bowen-Margulis measures). *Let $(g_\varepsilon)_{-1 < \varepsilon < 1}$ be a family of metrics on \widetilde{M} with sectional curvatures satisfying $K_{g_\varepsilon} \leq -a^2$, such that for all $\varepsilon \in (-1, 1)$, at all $x \in \widetilde{M}$, $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$ and $g_\varepsilon \rightarrow g_0$ in the \mathcal{C}^1 -topology, uniformly on compact sets. Assume that Γ is divergent for all metrics g_ε . Then in the weak- $*$ topology of TM ,*

$$\lim_{\varepsilon \rightarrow 0} m_{BM}^{g_\varepsilon} = m_{BM}^{g_0}.$$

Proof. Let φ be a continuous map with compact support on TM . Write the difference $\int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon} - \int_{TM} \varphi dm_{\mu_{BM}^{g_0}}^{g_0}$ as

$$\left(\int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon} - \int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_0} \right) + \left(\int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_0} - \int_{TM} \varphi dm_{\mu_{BM}^{g_0}}^{g_0} \right)$$

By Corollary 5.12, the second difference converges to 0.

Proposition 2.13 allows to rewrite the first difference as

$$\int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon} - \int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_0} = \int_{TM} (\varphi \circ \Psi^{g_0 \rightarrow g_\varepsilon} \times \mathcal{E}^{g_0 \rightarrow g_\varepsilon} - \varphi) dm_{\mu_{BM}^{g_0}}^{g_0}.$$

By Corollary 5.12, $m_{\mu_{BM}^{g_\varepsilon}}^{g_0}$ converges weakly to $m_{\mu_{BM}^{g_0}}^{g_0}$ in the dual of continuous functions with compact support.

By Theorem 5.7, as $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}^{g_0 \rightarrow g_\varepsilon} \leq 1$, if $\varphi \geq 0$, we have $\limsup_{\varepsilon \rightarrow 0} \varphi \circ \Psi^{g_0 \rightarrow g_\varepsilon} \times \mathcal{E}^{g_0 \rightarrow g_\varepsilon} - \varphi \leq 0$. As the support of these maps is included in a fixed compact set, we deduce that

$$\limsup_{\varepsilon \rightarrow 0} \int_{TM} (\varphi \circ \Psi^{g_0 \rightarrow g_\varepsilon} \times \mathcal{E}^{g_0 \rightarrow g_\varepsilon} - \varphi) dm_{\mu_{BM}^{g_0}}^{g_0} \leq 0.$$

Now, rewrite this first difference as

$$- \left(\int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_0} - \int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon} \right) = - \int_{TM} (\varphi \circ \Psi^{g_\varepsilon \rightarrow g_0} \times \mathcal{E}^{g_\varepsilon \rightarrow g_0} - \varphi) dm_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon}.$$

Observe first that, by the same arguments used in the proof of Equation (19), the ratios of masses $\frac{\|m_{\mu_{BM}^{g_\varepsilon}}^{g_0}\|}{\|m_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon}\|}$ goes to 1 when $\varepsilon \rightarrow 0$.

For the same reason as above, $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}^{g_\varepsilon \rightarrow g_0} \leq 1$, so that for $\varphi \geq 0$, by Theorem 5.7, uniformly on TM , the limsup of $\varphi \circ \Psi^{g_\varepsilon \rightarrow g_0} \times \mathcal{E}^{g_\varepsilon \rightarrow g_0} - \varphi$ is nonpositive. By convergence of the ratio of masses mentioned above, and by convergence of $m_{\mu_{BM}^{g_\varepsilon}}^{g_0}$ to $m_{\mu_{BM}}^{g_0}$, its integral also has a nonpositive limsup, and the sign minus in the above expression gives

$$\liminf_{\varepsilon \rightarrow 0} \int_{TM} \varphi dm_{\mu_{BM}^{g_\varepsilon}}^{g_\varepsilon} - \int_{TM} \varphi dm_{\mu_{BM}}^{g_0} \geq 0.$$

The result follows. \square

6 Differentiability of the metric and topological entropies

In this section, we show differentiability of topological and measure theoretic entropies at $\varepsilon = 0$, when along a variation $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ of metrics of a negatively curved Riemannian manifold $(M = \widetilde{M}/\Gamma, g_0)$. We will focus on two distinct situations.

First, let μ be a Γ -invariant geodesic current on $\partial^2 \widetilde{M}$, and for all $\varepsilon \in (-1, 1)$, let $m_\mu^{g_\varepsilon}$ be the associated invariant measure for the geodesic flow (g_ε^t) (see Section 2). Assume that the total mass of $m_\mu^{g_0}$ is finite. We will show that the measure theoretic entropy $\varepsilon \mapsto h(m_\mu^{g_\varepsilon}, g_\varepsilon)$ is \mathcal{C}^1 , with explicit derivatives.

We then focus on the topological entropy. Provided that Bowen-Margulis measures of each geodesic flow (g_ε^t) are finite, and that their masses vary continuously, we show that the topological entropy is also \mathcal{C}^1 , with a similar formula for its derivative. The proofs are similar in both situations, and inspired from [KKW91] and [Tap11].

Definition 6.1. *Let M be a (non-compact) manifold. We say that a family of complete Riemannian metrics $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ on M converges to g_0 uniformly in the \mathcal{C}^1 -topology if:*

1. $g_\varepsilon \rightarrow g_0$ in the \mathcal{C}^1 topology, uniformly on compact sets, as in Definition 5.6;
2. there exists $\kappa > 0$ such that for all $\varepsilon \in (-1, 1)$ and all $v \in TM$ with $\|v\|_{g_0} \leq 1$,

$$\left| \frac{d}{ds} \Big|_{s=\varepsilon} g_s(v, v) \right| \leq \kappa;$$

A \mathcal{C}^2 variation of metric with compact support, or with non-compact support but uniformly bounded second derivative, is a typical example of such uniformly \mathcal{C}^1 family. If $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ is such a uniformly \mathcal{C}^1 family of complete metrics on M , one immediately see that there exists $B = B(C_1, \varepsilon) > 0$ such that at all $x \in M$ and for all $\varepsilon \in (-1, 1)$,

$$e^{-B\varepsilon} g_0 \leq g_\varepsilon \leq e^{B\varepsilon} g_0,$$

which allows us to apply the results shown in the previous section.

6.1 Variation of metric entropy

This paragraph is devoted to the proof of the following result, which seems to us new even in the compact case.

Theorem 6.2. *Let $b > a > 0$, $\varepsilon > 0$ and let $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ be a family of complete metrics on $M = \widetilde{M}/\Gamma$ whose curvatures and derivatives of curvatures are uniformly bounded, and moreover such that for all $\varepsilon \in (-1,1)$ and at all points, $-b^2 \leq K_{g_\varepsilon} \leq -a^2$. Assume that $g_\varepsilon \rightarrow g_0$ uniformly \mathcal{C}^1 . Let μ be a Γ -invariant geodesic current on $\partial^2 \widetilde{M}$ such that $m_\mu^{g_0}$ is finite.*

Then the local entropy $\varepsilon \mapsto h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon)$ of the (g_ε^t) -invariant measures $m_\mu^{g_\varepsilon}$ is differentiable at $\varepsilon = 0$ with derivative given by

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon) = -h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \times \int_{S^{g_0}M} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \|v\|^{g_\varepsilon} \frac{dm_\mu^{g_0}(v)}{\|m_\mu^{g_0}\|}.$$

Proof. Let μ be a Γ -invariant geodesic current on $\partial^2 \widetilde{M}$ such that $m_\mu^{g_0}$ is finite. It follows from Proposition 2.15 that for all $\varepsilon \in (-1,1)$, the measures $m_\mu^{g_\varepsilon}$ are finite. Moreover, by Corollary 5.10, $\lim_{\varepsilon \rightarrow 0} m_\mu^{g_\varepsilon} = m_\mu^{g_0}$ and $\lim_{\varepsilon \rightarrow 0} \frac{m_\mu^{g_\varepsilon}}{\|m_\mu^{g_\varepsilon}\|} = \frac{m_\mu^{g_0}}{\|m_\mu^{g_0}\|}$ in the narrow topology.

By Theorem 3.7, if g_1 and g_2 are admissible metrics on M , we know that

$$h_\Gamma^{loc}(m_\mu^{g_2}, g_2) = \int_{S^{g_2}M} \mathcal{E}^{g_1 \rightarrow g_2}(v) d\bar{m}_\mu^{g_2}(v) \cdot h_\Gamma^{loc}(m_\mu^{g_1}, g_1). \quad (20)$$

By Theorem 5.9, this implies that the local entropy $h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon)$ converges to $h_\Gamma^{loc}(m_\mu^{g_0}, g_0)$ when $\varepsilon \rightarrow 0$. Moreover, (20) and Lemma 2.5 also imply that

$$h_\Gamma^{loc}(m_\mu^{g_2}, g_2) \leq \int_{S^{g_2}M} \|v\|_{g_1} d\bar{m}_\mu^{g_2}(v) \cdot h_\Gamma^{loc}(m_\mu^{g_1}, g_1).$$

Applying it with $g_1 = g_0$ and $g_2 = g_\varepsilon$ first, and second with $g_1 = g_\varepsilon$ and $g_2 = g_0$, we get

$$\begin{aligned} h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \left(\frac{1}{\int_{S^{g_0}M} \|v\|^{g_\varepsilon} d\bar{m}_\mu^{g_0}(v)} - 1 \right) &\leq h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon) - h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \\ &\leq h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \left(\int_{S^{g_\varepsilon}M} \|v\|^{g_0} d\bar{m}_\mu^{g_\varepsilon}(v) - 1 \right), \end{aligned}$$

which yields to

$$\begin{aligned} h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \frac{\int_{S^{g_0}M} \frac{\|v\|^{g_0} - \|v\|^{g_\varepsilon}}{\varepsilon} d\bar{m}_\mu^{g_0}(v)}{\int_{S^{g_0}M} \|v\|^{g_\varepsilon} d\bar{m}_\mu^{g_0}(v)} &\leq \frac{h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon) - h_\Gamma^{loc}(m_\mu^{g_0}, g_0)}{\varepsilon} \\ &\leq h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \int_{S^{g_\varepsilon}M} \frac{\|v\|^{g_0} - \|v\|^{g_\varepsilon}}{\varepsilon} d\bar{m}_\mu^{g_\varepsilon}(v). \end{aligned}$$

Now, dominated convergence theorem, continuity of $\varepsilon \rightarrow h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon)$ at $\varepsilon = 0$, and narrow convergence of $\frac{m_\mu^{g_\varepsilon}}{\|m_\mu^{g_\varepsilon}\|}$ towards $\frac{m_\mu^{g_0}}{\|m_\mu^{g_0}\|}$ give

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h_\Gamma^{loc}(m_\mu^{g_\varepsilon}, g_\varepsilon) = -h_\Gamma^{loc}(m_\mu^{g_0}, g_0) \times \int_{S^{g_0}M} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \|v\|^{g_\varepsilon} d\bar{m}_\mu^{g_0}(v).$$

This is the desired result. \square

Remark 6.3. In the above approach, we used lemma 2.5 to bound from the above $\mathcal{E}^{g_1 \rightarrow g_2}$ by the smooth quantity $\|v\|^{g_2}$. A slightly more direct approach could have been to use in the above computations the fact that

$$\mathcal{E}^{g_0 \rightarrow g_\varepsilon}(v) = 1 + \varepsilon \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \|v\|^{g_\varepsilon} + \varepsilon \alpha(\varepsilon, v),$$

where α is uniformly bounded on $(-1, 1) \times S^{g_0}M$ and for all $v \in S^{g_0}M$, $\lim_{\varepsilon \rightarrow 0} \alpha(v, \varepsilon) = 0$. However, the proof of such estimate would be long and technical, and would require additional assumptions of regularity on the variation of metric.

6.2 Variation of topological entropy

We now show differentiability of the topological entropy $h_{top}(g_\varepsilon)$ at $\varepsilon = 0$. It is not a corollary of Theorem 6.2 since we have to consider Bowen-Margulis geodesic currents $\mu_{BM}^{g_\varepsilon}$ depending on the metric g_ε . However, the strategy of proof is very similar, as by Theorem 5.14, $m_{BM}^{g_\varepsilon} \rightarrow m_{BM}^{g_0}$ in the weak-* topology. The only missing ingredient is the convergence of Bowen-Margulis measures in the dual of bounded continuous functions. It is therefore required in the assumptions of Theorem 6.4. We refer to Section 7 for the study of the large class of the so-called SPR manifolds, which will satisfy this assumption.

Theorem 6.4. *Let $b > a > 0$, and let $(g_\varepsilon)_{\varepsilon \in (-1, 1)}$ be a family of complete metrics on M such that*

1. *for all $\varepsilon \in (-1, 1)$ and at all point, $-b^2 \leq K_{g_\varepsilon} \leq -a^2$;*
2. *$g_\varepsilon \rightarrow g_0$ uniformly in the C^1 topology as in Definition 6.1 ;*
3. *for all $\varepsilon \in (-1, 1)$, the Bowen-Margulis measure $m_{BM}^{g_\varepsilon}$ of the geodesic flow $(g_\varepsilon^t)_{t \in \mathbb{R}}$ on $S^{g_\varepsilon}M$ has finite mass;*
4. *the map $\varepsilon \rightarrow \|m_{BM}^{g_\varepsilon}\|$ is continuous at $\varepsilon = 0$.*

Then the entropy $\varepsilon \mapsto h_{top}(g_\varepsilon)$ is C^1 at $\varepsilon = 0$ with derivative given by

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h_{top}(g_\varepsilon) = -h_{top}(g_0) \int_{S^{g_0}M} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \|v\|^{g_\varepsilon} \frac{dm_{BM}^{g_0}(v)}{\|m_{BM}^{g_0}\|}.$$

Proof. As the preceding one, our strategy of proof is inspired from [KKW91] and [Tap11]. Corollary 3.17 shows that if g_1 and g_2 are admissible metrics M with finite Bowen-Margulis measures, then

$$h_{top}(g_2) \leq \int_{S^{g_2}M} \mathcal{E}^{g_1 \rightarrow g_2}(v) \frac{dm_{BM}^{g_2}(v)}{\|m_{BM}^{g_2}\|} \cdot h_{top}(g_1) \leq \int_{S^{g_2}M} \|v\|_{g_1} \frac{dm_{BM}^{g_2}(v)}{\|m_{BM}^{g_2}\|} \cdot h_{top}(g_1),$$

where the last inequality follows from Lemma 2.5. Applying it to g_ε and g_0 on both sides, we get for all $\varepsilon \in (-1, 1)$,

$$\begin{aligned} h_{top}(g_0) \times \frac{\int_{S^{g_0}M} \frac{\|v\|^{g_0} - \|v\|^{g_\varepsilon}}{\lambda} \frac{dm_{BM}^{g_0}(v)}{\|m_{BM}^{g_0}\|}}{\int_{S^{g_0}M} \|v\|^{g_\varepsilon} \frac{dm_{BM}^{g_0}(v)}{\|m_{BM}^{g_0}\|}} &\leq \frac{h_{top}(g_\lambda) - h_{top}(g_0)}{\lambda} \\ &\leq h_{top}(g_0) \times \int_{S^{g_\varepsilon}M} \frac{\|v\|^{g_0} - \|v\|^{g_\varepsilon}}{\varepsilon} \frac{dm_{BM}^{g_\varepsilon}(v)}{\|m_{BM}^{g_\varepsilon}\|}. \end{aligned}$$

The assumptions of the Theorem are now exactly done to make the above integrals converge. We deduce that topological entropy is differentiable at $\varepsilon = 0$, with

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h_{top}(g_\varepsilon) = -h_{top}(g_0) \times \int_{S^{g_0}M} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \|v\|^{g_\varepsilon} d\bar{m}_{BM}^{g_0}(v).$$

□

7 Entropy at infinity and Strongly Positively Recurrent groups

In this section, our goal is to propose a wide class of manifolds and metrics to which Theorem 6.4 will apply. In view of this goal, proving differentiability of entropy, this section is apparently technical. However, the definition of this class of manifolds, and the related concepts studied here, is probably one of the main novelties in our paper. We refer to [ST] for further results on these manifolds.

We define the *entropy at infinity* $\delta_\infty(M, g)$ of a negatively curved manifold (M, g) (see Definition 7.12), as the maximal exponential growth of the dynamics away from any given (large) compact set. In particular, it is invariant under any \mathcal{C}^2 compact perturbation of a negatively curved metric.

We introduce the class of *strongly positively recurrent* manifolds (M, g) , defined as those negatively curved manifolds whose entropy at infinity is strictly smaller than the total topological entropy of the geodesic flow.

As said in the introduction, the notion of *strong positive recurrence* appeared in [Sar01] in the context of symbolic dynamics over an infinite alphabet, and has been used later by some other authors among which [BBG14]. A former terminology due to [GS98] was *stable positive recurrence*. This terminology could be more adapted to the kind of results that we prove here. In any case, as will be seen below and in [ST], the acronym SPR is perfectly adapted to the concept.

The simplest nontrivial examples are geometrically finite hyperbolic manifolds, but this class also includes most known examples of non-compact manifolds with negative curvature whose geodesic flow has a finite Bowen-Margulis measure, and many new ones (see section 7.3).

Our main result is the following.

Theorem 7.1. *Let (M, g_0) be a manifold with pinched negative curvature and bounded derivatives of the curvature.*

If (M, g_0) is a strongly positively recurrent manifold, then the Bowen-Margulis measure of its geodesic flow is finite.

Moreover, if $(g_\varepsilon)_{\varepsilon \in (-1, 1)}$ is a uniformly \mathcal{C}^1 -variation of smooth complete metrics on M with pinched negative curvature and bounded derivatives of metrics, satisfying all $K_{g_\varepsilon} \leq -a^2 < 0$, then

1. *For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ small enough, all metrics g_ε are strongly positively recurrent.*
2. *The mass of the associated (finite) Bowen-Margulis $m_{BM}^{g_\varepsilon}$ varies continuously on $(-\varepsilon_0, \varepsilon_0)$.*

The first part of this theorem (finiteness of Bowen-Margulis) has been proven independently and simultaneously by A. Velozo [Vel17] by a different approach.

As a corollary, all assumptions of Theorem 6.4 hold for such a variation of metrics, so that we get the following result, which answers positively the question at the origin of this work.

Corollary 7.2. *Let $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ be a uniformly C^1 family of complete metrics on the manifold M with pinched negative curvature and bounded derivatives of the curvature. Assume that for all $\varepsilon \in (-1,1)$, $-b^2 \leq K_{g_\varepsilon} \leq -a^2$ for some $b > a > 0$. Then the entropy $\varepsilon \mapsto h_{top}(g_\varepsilon)$ is C^1 around $\varepsilon = 0$, and its derivative is given by*

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} h_{top}(g_\varepsilon) = -h_{top}(g_0) \times \int_{S^{g_0}M} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \|v\|^{g_\varepsilon} \frac{dm_{BM}^{g_0}(v)}{\|m_{BM}^{g_0}\|}.$$

In view of the length of this section, let us present the strategy of the proof.

Heuristically, the SPR assumption allows to neglect the dynamical contribution of the complement of a large compact set to the dynamics. We develop this idea in two introductory parts 7.1 and 7.2, defining the growth of the fundamental group outside a compact set, the entropy at infinity and the class of Strongly Positively Recurrent manifolds.

In Subsection 7.3 we provide an illustration of this concept, by describing different families of examples of SPR manifolds.

A criterion of finiteness of the Bowen-Margulis measure from [PS18] is used to prove the first part of Theorem 7.1. Subsection 7.4 is devoted to this proof.

All entropies considered here are continuous for a negatively curved perturbation $(g_\varepsilon)_{-1 \leq \varepsilon \leq 1}$ satisfying $e^{-\varepsilon} g_0 \leq g_\varepsilon \leq e^\varepsilon g_0$. Thus, the SPR assumption, which is the existence of a critical gap between the entropy at infinity and the topological entropy is stable under such small perturbations. And the existence of a large compact set concentrating the most part of the dynamics allows to prove that its complement is of small Bowen-Margulis measure, uniformly in the perturbation. These ideas are developed in subsection 7.5, where we prove that for a variation of a SPR metric as above, the mass of the Bowen-Margulis measures varies continuously.

As said in the introduction, these results imply all Theorems stated in the introduction. Theorem 1.5 is an immediate consequence of section 7.3 and the first part of Theorem 7.1. Theorem 1.6 is a reformulation of the second part of Theorem 7.1. At last, our main result, Theorem 1.7, follows from Theorems 6.4 and 1.6 (or 7.1).

7.1 Fundamental group outside a given compact set

Let (M, g) be a complete Riemannian manifold with pinched negative curvature $-b^2 \leq K_g \leq -a^2 < 0$, whose fundamental group $\Gamma = \pi_1(M)$ is non-elementary. Let $p_\Gamma : \widetilde{M} \rightarrow M$ be the universal covering map. Let $o \in \widetilde{M}$ be a point, fixed once for all. For any set $W \subset M$, we will write $W^c = M \setminus W$.

Definition 7.3. *Let $W \subset M$ be a compact pathwise connected set which is the closure of its interior, and whose boundary is piecewise C^1 . A nice preimage of W is a compact set $\widetilde{W} \subset \widetilde{M}$ such that*

1. $p_\Gamma(\widetilde{W}) = W$ and the restriction of p_Γ to the interior of \widetilde{W} is injective ;

2. \widetilde{W} has a piecewise \mathcal{C}^1 boundary.

Remark 7.4. We will often refer to and use results of [PS18]. In this reference, \mathcal{W} is a subset of $S^g M$ and $\widetilde{\mathcal{W}}$ is an open set inside $p_\Gamma^{-1}(\mathcal{W})$ such that $p_\Gamma : \widetilde{\mathcal{W}} \rightarrow \mathcal{W}$ is onto. As we deal with several metrics and several unit tangent bundle, it is better here to work with $W \subset M$. The reader can think to \mathcal{W} as $S^g W$. The fact that W is compact here, and \mathcal{W} open in [PS18] is just a matter of taste in some arguments.

We gather in the following lemma elementary useful facts.

Lemma 7.5. *Let W be a compact pathwise connected set with piecewise \mathcal{C}^1 boundary, which is the closure of its interior.*

1. A nice preimage \widetilde{W} of W exists.
2. If $W_2 \supset W_1$, then they admit nice preimages $\widetilde{W}_2 \supset \widetilde{W}_1$.
3. If $\gamma \neq \text{id}$ then $\gamma \cdot \overset{\circ}{\widetilde{W}} \cap \overset{\circ}{\widetilde{W}} = \emptyset$
4. The set $\{\gamma \in \Gamma ; \gamma \cdot \widetilde{W} \cap \widetilde{W} \neq \emptyset\}$ is finite. We call such $\gamma \widetilde{W}$ the adjacent elements of \widetilde{W} .

Proof. Choose some $w \in W$, lift it to $\widetilde{w} \in p_\Gamma^{-1}(W)$ and construct the Dirichlet domain

$$\widetilde{W} = \{z \in p_\Gamma^{-1}(W), \forall \gamma \in \Gamma, d^g(z, \widetilde{w}) \leq d^g(z, \gamma \widetilde{w})\}.$$

It is a compact set with \mathcal{C}^1 -boundary which satisfies the properties stated in the lemma. If $W_1 \subset W_2$, choose some $w \in p_\Gamma^{-1}(W_1) \subset p_\Gamma^{-1}(W_2)$. For $i = 1, 2$ the Dirichlet domains $\widetilde{W}_1 \subset \widetilde{W}_2 \subset p_\Gamma^{-1}(W_i)$ satisfy Fact 2. \square

The following notion was introduced in [PS18].

Definition 7.6. *Let $W \subset M$ be a compact set and \widetilde{W} a nice preimage of W . The fundamental group of M out of \widetilde{W} is the set $\Gamma_{\widetilde{W}}^g$ of elements $\gamma \in \Gamma$ such that there exists $x, y \in \widetilde{W}$ and a g -geodesic segment c_γ joining x to γy such that for all $h \in \Gamma$,*

$$c_\gamma \cap p_\Gamma^{-1}W = c_\gamma \cap \Gamma \cdot \widetilde{W} \subset \widetilde{W} \cup \gamma \cdot \widetilde{W}.$$

By compactness of \widetilde{W} we will always assume that $x, y \in \partial \widetilde{W}$.

Heuristically, as explained in [PS18], $\Gamma_{\widetilde{W}}^g$ represents loops $p_\Gamma([x, \gamma y])$ which go outside W at the beginning, and come back to W only at the end. This heuristics does not work perfectly, depending on the topology of W , for example when it has holes.

The set $\Gamma_{\widetilde{W}}^g$ will help controlling what happens far at infinity. In particular it follows immediately from the definition that it is not sensitive to small compact perturbations of the metric g , as stated in the proposition below.

Proposition 7.7. *Let (M, g_0) be a complete negatively curved metric and $W \subset M$ be a compact set, with nice preimage \widetilde{W} . For any proper compact subset $K \subset \overset{\circ}{\widetilde{W}}$ and any metric g such that $g_1 = g_2$ outside K , we have $\Gamma_{\widetilde{W}}^g(g_1) = \Gamma_{\widetilde{W}}^g(g_2)$.*

By definition, $id_\Gamma \in \Gamma_{\widetilde{W}}^g$, and $\gamma \in \Gamma_{\widetilde{W}}^g$ iff $\gamma^{-1} \in \Gamma_{\widetilde{W}}^g$. When (M, g) is a geometrically finite manifold, for suitable choice of W , $\Gamma_{\widetilde{W}}^g$ is a union of groups. But in general, $\Gamma_{\widetilde{W}}^g$ is not a group at all, as shown in the following proposition.

Proposition 7.8. *With the previous notations, let $W \subset M$ be a compact path-wise connected set with piecewise C^1 boundary and \widetilde{W} be a nice preimage of W . If $\gamma \in \Gamma_{\widetilde{W}}^g$ is a hyperbolic element whose axis A_γ intersects the interior of \widetilde{W} , then there exists $N = N(\gamma) > 0$ such that for all $n \geq N$, $\gamma^n \notin \Gamma_{\widetilde{W}}^g$.*

Proof. Let $\gamma \in \Gamma_{\widetilde{W}}^g$ be such an hyperbolic element. Its axis A_γ intersects \widetilde{W} , and therefore also $\gamma\widetilde{W}$ and all iterates $\gamma^n\widetilde{W}$. Choose some $x_0 \in A_\gamma \cap \overset{\circ}{\widetilde{W}}$ and let $d_0 = d^g(x_0, \partial\widetilde{W}) > 0$. Let $x, y \in \widetilde{W}$. By lemma 5.1 (2), with $K = \text{diam}(\widetilde{W})$, $\alpha = d_0/2$, we know that if $d^g(x, \gamma^n y) = n\ell^g(\gamma) \pm 2\text{diam}\widetilde{W} \geq 2R_0$, all points in the middle interval of length $2T = \ell^g(\gamma)$ of the g -geodesic segment from x to $\gamma^n y$ would be at distance less than $d_0/2$ from A_γ , and therefore some of them would be inside $\gamma^k\widetilde{W}$, for some $1 \leq k \leq n-1$. This implies $n\ell^g(\gamma) \leq 2R_0 + 2\text{diam}(\widetilde{W})$, which proves the proposition. \square

The set $\Gamma_{\widetilde{W}}^g$ depends on W and the choice of its preimage \widetilde{W} , but not too strongly as illustrated by the following proposition.

Proposition 7.9. *1. Let $W \subset M$ be a compact set (with piecewise C^1 boundary), and \widetilde{W} be a nice preimage. let $\alpha \in \Gamma$. Then $\Gamma_{\alpha\widetilde{W}}^g = \alpha\Gamma_{\widetilde{W}}^g\alpha^{-1}$.*
2. If W_1 and W_2 are compact sets of M (with piecewise C^1 boundary) such that $W_1 \subset \overset{\circ}{W_2}$ with respective nice preimages $\widetilde{W}_1 \subset \widetilde{W}_2$, there exists $k \geq 1$ and $\alpha_1, \dots, \alpha_k \in \Gamma$ such that

$$\Gamma_{\widetilde{W}_2}^g \subset \bigcup_{i,j=1}^k \alpha_i \Gamma_{\widetilde{W}_1}^g (\alpha_j)^{-1}.$$

3. If \widetilde{W}_1 and \widetilde{W}_2 are nice preimages of W , then there exists a finite set $\{\alpha_1, \dots, \alpha_k\} \subset \Gamma$ such that

$$\Gamma_{\widetilde{W}_2}^g \subset \bigcup_{i,j=1}^p \alpha_i \Gamma_{\widetilde{W}_1}^g (\alpha_j)^{-1}.$$

Proof. The first item of the proposition is obvious. Let us show 2. Set

$$D = 2\text{diam}(W_2) \quad \text{and} \quad \eta = \inf \{d^g(w, \partial W_2) ; w \in W_1\} > 0.$$

Let $\gamma \in \Gamma_{\widetilde{W}_2}^g$. There exist $x_2, y_2 \in \partial\widetilde{W}_2$ such that the g -geodesic segment $[x_2, \gamma y_2]$ intersects $\Gamma\widetilde{W}_2$ only in $\widetilde{W}_2 \cup \gamma\widetilde{W}_2$. Now, choose some $x_1, y_1 \in \partial\widetilde{W}_1$.

By Lemma 5.1, there exists $L = L(D, \eta) > 0$ and $R = R(D, \eta) > 2L$ such that for all $x_1, y_1, x_2, y_2 \in \widetilde{M}$ with $d^g(x_1, x_2) \leq D$, $d^g(y_1, y_2) \leq D$ and $d^g(x_2, y_2) \geq R$, the g -geodesic segment (x_1, y_1) is contained in the $\frac{\eta}{2}$ -neighbourhood of (x_2, y_2) except inside the balls $B^g(x_1, L)$ and $B^g(y_1, L)$.

Let $\alpha_1, \dots, \alpha_k \in \Gamma$ be the (finitely many) elements such that $d^g(\widetilde{W}_1, \alpha_i \widetilde{W}_1) = \inf\{d^g(a, b), a \in \widetilde{W}_1, b \in \alpha_i \widetilde{W}_1\} \leq L$.

Let $\widetilde{W}_1 \subset \widetilde{W}_2$ be included nice preimages of W_1 and W_2 , and let $\gamma \in \Gamma_{\widetilde{W}_2}$ such that $d^g(o, \gamma o) \geq R + 2D$. Then there exists $x_2, y_2 \in \partial \widetilde{W}_2$ such that $(x_2, \gamma y_2)$ does not intersect $p_{\Gamma}^{-1}(W_2)$. By construction there exists $x_1, y_1 \in \widetilde{W}_1$ such that $d^g(x_1, x_2) \leq D$ and $d^g(y_1, y_2) \leq D$. The geodesic $(x_1, \gamma \cdot y_1)$ is $\frac{\eta}{2}$ -close to the geodesic $(x_2, \gamma \cdot y_2)$ outside the balls $B^g(x_1, L)$ and $B^g(\gamma \cdot y_1, L)$, hence does not intersect $p_{\Gamma}^{-1}(W_1)$ except maybe in these balls. Thus, there exist α_i, α_j in the above finite set, such that the geodesic segment $(x_1, \gamma y_1)$ does not intersect $\Gamma \widetilde{W}_1$ between $\alpha_i \widetilde{W}_1$ and $\gamma \alpha_j \widetilde{W}_1$. Therefore, $\alpha_i^{-1} \gamma \alpha_j \in \Gamma_{\widetilde{W}_1}$ or in other words,

$$\gamma \in \alpha_i \Gamma_{\widetilde{W}_1} \alpha_j^{-1}.$$

The proof of the last item is similar, and we let it to the reader. \square

7.2 Entropy at infinity

Proposition 7.10. *Let $W \subset M$ be a compact set and \widetilde{W} a nice preimage of W . The critical exponent $\delta_{W^c}(g)$ of the Poincaré series $\sum_{\gamma \in \Gamma_{\widetilde{W}}} e^{-s d^g(o, \gamma o)}$ is equal to*

$$\delta_{W^c}(g) = \limsup_{R \rightarrow \infty} \frac{\log \# \{ \gamma \in \Gamma_{\widetilde{W}}, d^g(o, \gamma o) \leq R \}}{R}$$

and does not depend on the choice of a nice preimage $\widetilde{W} \subset \widetilde{M}$ of W nor $o \in \widetilde{M}$. We call it the entropy out of W of (M, g) .

Proof. It follows from the triangular inequality that $\delta_{W^c}(g)$ does not depend on the choice of o . Let us show that it does not depend on the choice of preimage. Let \widetilde{W}_1 and \widetilde{W}_2 be two nice preimages of W . By Proposition 7.9, there exists $k \geq 0$ and $\alpha_1, \dots, \alpha_k \in \Gamma$ such that

$$\Gamma_{\widetilde{W}_2} \subset \bigcup_{i,j=1}^k \alpha_i \Gamma_{\widetilde{W}_1} \alpha_j^{-1}. \quad (21)$$

Set

$$D = \max \left\{ d^g(w, o); w \in \widetilde{W}_2 \cup \bigcup_{i,j=1}^k \alpha_i \Gamma_{\widetilde{W}_1} (\alpha_j)^{-1} \right\}.$$

Define for $i = 1, 2$ and $R > 0$,

$$\Gamma_{\widetilde{W}_i}(R) = \left\{ \gamma \in \Gamma_{\widetilde{W}_i}; d^g(o, \gamma o) \leq R \right\}.$$

It follows from (21) and triangular inequality that for all $R > 0$,

$$\Gamma_{\widetilde{W}_2}(R) \subset \bigcup_{i,j=1}^k \alpha_i \Gamma_{\widetilde{W}_1}(R + 2D) (\alpha_j)^{-1},$$

and therefore $\#\Gamma_{\widetilde{W}_2}(R) \leq k^2 \#\Gamma_{\widetilde{W}_1}(R + 2D)$. This gives immediately

$$\limsup_{R \rightarrow +\infty} \frac{1}{R} \log \#\Gamma_{\widetilde{W}_2}(R) \leq \limsup_{R \rightarrow +\infty} \frac{1}{R} \log \#\Gamma_{\widetilde{W}_1}(R).$$

By symmetry, the reverse inequality also holds, and the result follows. \square

Proposition 7.11. *Let (M, g) be a complete negatively curved metric.*

1. *For any proper compact subset $K \subset \overset{\circ}{W}$ and any metric g_2 such that $g_1 = g_2$ outside K , we have $\delta_{W^c}(g_1) = \delta_{W^c}(g_2)$.*
2. *For all compact sets W_1, W_2 such that $W_1 \subset \overset{\circ}{W}_2 \subset M$, we have $\delta_{W_1^c} \geq \delta_{W_2^c}$.*

Proof. Item 1 follows from Proposition 7.7. Item 2 can be proven similarly to Proposition 7.10, thanks to Proposition 7.9. \square

For a global variation of the metric (i.e. beyond W), even small, the behaviour of $\delta_{W^c}(g)$ is not clear since the set $\Gamma_{\widetilde{W}}$ depends on the metric.

Definition 7.12. *The entropy at infinity of (M, g) is*

$$\delta_\infty(g) = \inf \{ \delta_W(g), W \subset M \text{ compact set} \}.$$

Proposition 7.11 implies the following natural characterization of the entropy at infinity.

Proposition 7.13. *Let (M, g) be a complete negatively curved manifold and $(W_i)_{i \in \mathbb{N}}$ be an increasing exhaustion of M by compact sets. Then*

$$\delta_\infty(g) = \lim_{i \rightarrow \infty} \delta_{W_i^c}(g).$$

Moreover, it is invariant under any negatively curved perturbation of the metric with compact support.

This entropy at infinity is a dynamical analogous to the bottom of the essential spectrum of the Laplacian in spectral geometry. We will use this fact in some of the examples given in Section 7.3.

Definition 7.14. *The complete manifold (M, g) is called strongly/stably positively recurrent (SPR), if $\delta_\infty(g) < \delta_\Gamma(g)$. We will also call this property a critical gap at infinity.*

By definition, if (M, g) is strongly positively recurrent, there exists a compact set $W \subset M$ such that $\delta_W < \delta_\Gamma$.

Remark 7.15. *The reader may have noticed that the definition of $\Gamma_{\widetilde{U}}$ given in [PS18] p.4 is slightly different to ours, since it is written for an open set \widetilde{U} which projects onto $U = \overset{\circ}{W}$. Nevertheless, these definitions almost coincide in the following sense. Let $W \subset M$ be a compact set with nice preimage $\widetilde{W} \subset \widetilde{M}$, let $\widetilde{U} \subset \widetilde{M}$ be an open set which projects onto $U = \overset{\circ}{W}$. Let $\Gamma_{\widetilde{U}}$ be defined as in [PS18], and $\Gamma_{\widetilde{W}}$ be defined as above. Then there exists $\alpha_1, \dots, \alpha_k \in \Gamma$ such that*

$$\Gamma_{\widetilde{U}} \subset \bigcup_{i,j=1}^k \alpha_i \Gamma_{\widetilde{W}}(\alpha_j)^{-1} \quad \text{and} \quad \Gamma_{\widetilde{W}} \subset \bigcup_{i,j=1}^k \alpha_i \Gamma_{\widetilde{U}}(\alpha_j)^{-1}.$$

Therefore $\Gamma_{\tilde{U}}$ and $\Gamma_{\tilde{W}}$ have the same critical exponent and all results stated in [PS18] to characterize the finiteness of Gibbs measures in terms of $\Gamma_{\tilde{U}}$ are also valid for our definition of $\Gamma_{\tilde{W}}$.

7.3 Examples of SPR manifolds

We present here three classes of SPR manifolds. The first examples are geometrically finite manifolds with critical gap studied in [DOP00]. Schottky products furnish also plenty of examples, generalizing the examples of [Pei03]. At last, we describe examples inspired by Ancona's examples in [Anc].

These examples are almost the only known examples of non-compact manifolds with finite Bowen-Margulis measure. To our knowledge, the only exception is the construction of Peigné [Pei11] of geometrically finite manifolds with finite Bowen-Margulis measure but without critical gap, see [Pei11].

7.3.1 Geometrically finite manifolds with critical gap

The convex core $CC(M) \subset M$ is the image on M of the convex hull of the limit set Λ_Γ inside \tilde{M} . The nonwandering set $\Omega \subset S^g M$ of the geodesic flow is the set of vectors $v \in S^g M$ such that $v^\pm \in \Lambda_\Gamma$. By definition, $\Omega \subset S^g CC(M)$. A parabolic subgroup \mathcal{P} of Γ is a subgroup which fixes a point at infinity, and therefore stabilizes any horoball \mathcal{H} centered at this point.

A cusp is the image on M of such a horoball.

The manifold M is *geometrically finite* if its convex core can be written as a finite union

$$CC(M) = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_K,$$

where C_0 is a compact set and the C_i are finitely many cusps, images through p_Γ of horoballs \mathcal{H}_i stabilized by parabolic subgroups \mathcal{P}_i of Γ . The complete reference on such manifolds is [Bow95]. Parabolic subgroups have a positive critical exponent. The preimage on \tilde{M} of a cusp C_i is the orbit of an horoball \mathcal{H}_i , and the stabilizer of any horoball $\gamma\mathcal{H}_i$ is conjugated to the stabilizer \mathcal{P}_i of \mathcal{H}_i in Γ .

A *convex-cocompact manifold* is a geometrically finite manifold without cusps; in other words, it is a manifold whose convex core is compact.

Proposition 7.16. *Let (M, g) be a manifold with pinched negative curvature. If (M, g) is convex-cocompact, its entropy at infinity vanishes. If (M, g) is geometrically finite with k cusps represented by parabolic subgroups $\mathcal{P}_1, \dots, \mathcal{P}_k \subset \Gamma$, then*

$$\delta_\infty(g) = \max \{ \delta_{\mathcal{P}_1}(g), \dots, \delta_{\mathcal{P}_k}(g) \} .$$

In particular, a geometrically finite manifold is strongly positively recurrent if and only if

$$\max \{ \delta_{\mathcal{P}_1}(g), \dots, \delta_{\mathcal{P}_k}(g) \} < \delta_\Gamma .$$

This condition is precisely the *critical gap* criterion introduced by Dalbo, Otal and Peigné in [DOP00]. It is satisfied in particular by locally symmetric geometrically finite manifolds and their small compact C^2 perturbations. The notion of SPR manifold allows to generalize many results of [DOP00] and others on geometrically finite manifolds to all strongly positively recurrent manifolds.

The result follows immediately from Proposition 7.17 below.

Proposition 7.17. *Let (M, g) be a manifold with pinched negative curvature.*

If (M, g) is convex-cocompact and W is a compact set such that $CC(M) \subset \overset{\circ}{W}$, then $\Gamma_{\widetilde{W}}$ is finite.

If (M, g) is geometrically finite with k cusps, then there exists a compact set $W \subset M$ with nice preimage \widetilde{W} , a finite set $\Gamma_{\widetilde{W}}^0$, finitely many elements $\alpha_1, \dots, \alpha_N \in \Gamma$, and parabolic subgroups $\mathcal{P}_1, \dots, \mathcal{P}_k \subset \Gamma$ such that

$$\Gamma_{\widetilde{W}}(g) = \Gamma_0(\widetilde{W}) \cup \bigcup_{i,j} \alpha_i (\mathcal{P}_1 \cup \dots \cup \mathcal{P}_k) \alpha_j^{-1}.$$

Proof. Assume first (M, g) be convex-cocompact and $CC(M) \subset \overset{\circ}{W}$. Let D be the diameter of W and $\eta = \inf \{d^g(w, \partial W) ; w \in CC(M)\} > 0$. Let $\gamma \in \Gamma_{\widetilde{W}}$, $x, y \in \partial \widetilde{W}$ and choose $x_1, y_1 \in CC(M)$ such that $d(x, x_1) \leq D$ and $d(y, y_1) \leq D$. By Lemma 5.1, there exists some R_0 depending on D, η such that if $\ell^g(\gamma) \geq R_0$, there exists some $z \in (x, \gamma y)$, $z_1 \in (x_1, \gamma y_1)$ such that $d^g(z, z_1) \leq \eta/2$. But $CC(M)$ is convex, so that $z_1 \in CC(M)$ and z is at distance $\eta/2$ of $CC(M)$ and therefore inside \widetilde{W} . Thus, $\gamma \notin \Gamma_{\widetilde{W}}$. Therefore, all elements of $\Gamma_{\widetilde{W}}$ have bounded length less than R_0 , so that $\Gamma_{\widetilde{W}}$ is finite, included in $\{\gamma \in \Gamma, \ell^g(\gamma) \leq R_0\}$.

Assume now that M is geometrically finite with cusps, and let $CC(M) = C_0 \sqcup (\sqcup_{i=1}^k)$ be a decomposition of the convex core into a compact part and finitely many disjoint cusps. Let $W \subseteq M$ be a compact set such that $\overset{\circ}{W} \supset CC(M)$. Choose some nice preimage \widetilde{W} and disjoint horoballs \mathcal{H}_i , $1 \leq i \leq k$ whose boundary intersect \widetilde{W} . Let \mathcal{P}_i be the stabilizer of \mathcal{H}_i in Γ .

Let $\gamma \in \Gamma_{\widetilde{W}}$ be such that $\ell^g(\gamma) \geq R_0$ and $x, y \in \partial \widetilde{W}$. As noticed above, by Lemma 5.1, the geodesic segment $(x, \gamma y)$ is (except at the beginning and the end, inside balls $B^g(x, L)$ and $B^g(\gamma y, L)$) in the $\eta/2$ neighbourhood of $CC(M)$. As already said in [PS18], if $\gamma \in \Gamma_{\widetilde{W}}$, except for a bounded amount of time at the beginning and the end, the geodesic segment $p_\Gamma(x, \gamma y)$ has to leave the compact part C_0 and enter in some cusp C_i . Therefore, there exists a finite set $\{\alpha_1, \dots, \alpha_N\}$ such that for some $1 \leq i, j \leq N$, the geodesic segment $(\alpha_i x, \gamma \alpha_j y)$ stays in some horoball \mathcal{H}_i . As in the proof of Proposition 7.9, one deduces that $\Gamma_{\widetilde{W}} \subset \Gamma_{\widetilde{W}}^0 \cup \cup_{i,j,l} \alpha_i \mathcal{P}_i \alpha_j^{-1}$ with $\Gamma_{\widetilde{W}}^0 \subset \{\gamma \in \Gamma, \ell^g(\gamma) \leq R_0\}$ as in the convex-cocompact case. \square

7.3.2 Schottky products

We present now a family of geometrically infinite examples first studied in [Pei03]. Let G and H be discrete groups of isometries of a complete manifolds (\widetilde{M}, g) with pinched negative curvature. They are *in Schottky position* if there exists disjoint compact sets $U_G, U_H \subset \widetilde{M} \cup \partial \widetilde{M}$ such that for all $g \in G \setminus \{\text{id}\}$ and all $h \in H \setminus \{\text{id}\}$, we have

$$g \left((\widetilde{M} \cup \partial \widetilde{M}) \setminus U_G \right) \subset U_G \quad \text{and} \quad h \left((\widetilde{M} \cup \partial \widetilde{M}) \setminus U_H \right) \subset U_H.$$

In particular, by Klein's ping-pong argument, they generate a free product: $\Gamma = \langle G, H \rangle = G * H$. The entropy at infinity behaves nicely under Schottky products, as shown by the following theorem.

Theorem 7.18. *Let G and H be discrete groups of isometries of a complete manifold (\widetilde{M}, g) with pinched negative curvature which are in Schottky position. Let $\Gamma = \langle G, H \rangle = G * H$. Denote respectively by $M_\Gamma = \widetilde{M}/\Gamma$, $M_G = \widetilde{M}/G$ and $M_H = \widetilde{M}/H$ the associated quotient manifolds endowed with the quotient metric induced by g . Then*

$$\delta_\infty(M_\Gamma) = \max \{ \delta_\infty(M_G), \delta_\infty(M_H) \} .$$

As an immediate corollary, we get the following result.

Corollary 7.19. *Let G and H be discrete groups of isometries of a complete manifold (\widetilde{M}, g) with pinched negative curvature which are in Schottky position. Let M_G, M_H , and M_{G*H} be the quotient manifolds. Their critical exponents satisfy*

$$\delta_{G*H} \geq \max \{ \delta_G, \delta_H \} \geq \max \{ \delta_\infty(M_G), \delta_\infty(M_H) \} = \delta_\infty(M_{G*H}) \quad (22)$$

In particular,

1. if G and H are Strongly Positively Recurrent, then $G * H$ is also.
2. if $\delta_{G*H} > \max \{ \delta_\infty(M_G), \delta_\infty(M_H) \}$, then $G * H$ is strongly positively recurrent.

In both cases $(\widetilde{M}/\Gamma, g)$ has a finite Bowen-Margulis measure.

It was originally shown by M. Peigné in [Pei11] that if $\delta_\Gamma > \max \{ \delta_G, \delta_H \}$ then $(\widetilde{M}/\Gamma, g)$ has a finite Bowen-Margulis measure. The above corollary with Theorem 7.1 guarantees this finiteness under a weaker condition.

It was shown in [DOP00] that if $G \subset \Gamma$ is a *divergent* subgroup, then $\delta_G < \delta_\Gamma$. We get therefore the following corollary.

Corollary 7.20. *Let G, H be discrete divergent groups of isometries of a complete manifold (\widetilde{M}, g) with pinched negative curvature which are in Schottky position. Then $\Gamma = \langle G, H \rangle = G * H$ is strongly positively recurrent.*

This last corollary allows a lot of topologically infinite examples. For instance, if G and H are discrete subgroups of the group of isometries of the hyperbolic space, whose limit set are not the whole boundary, they can be settled in Schottky position by taking suitable conjugation with hyperbolic elements. If G and H are \mathbb{Z} -covers of convex-cocompact groups, they are divergent and their Schottky product gives a SPR manifold, hence with finite Bowen-Margulis measure, whose fundamental group is not even finitely generated.

Proof. Our proof relies on the ideas of Section 9 of [PS18]. Let U_G and U_H be the sets ensuring the Schottky position of G and H . Since they are compact in $\widetilde{M} \cup \partial\widetilde{M}$ and since $K_g \leq -a^2 < 0$, a key point is that there exists $\rho > 0$ such that all geodesics from U_G to U_H intersect the ball $B^g(o, \rho)$. Moreover, without loss of generality, we can assume that the point o is neither in U_G nor in U_H .

Let $M_\Gamma = \widetilde{M}/\Gamma$, $M_G = \widetilde{M}/G$ and $M_H = \widetilde{M}/H$. Let $p_\Gamma : \widetilde{M} \rightarrow M_\Gamma$, $p_G : \widetilde{M} \rightarrow M_G$ and $p_H : \widetilde{M} \rightarrow M_H$ be the associated covering maps.

For all $R \geq \rho$, define $W_\Gamma^R = p_\Gamma(B^g(o, R)) \subset M_\Gamma$, $W_G^R = p_G(B^g(o, R)) \subset M_G$ and $W_H^R = p_H(B^g(o, R)) \subset M_H$. Let $\widetilde{W}_\Gamma^R, \widetilde{W}_G^R, \widetilde{W}_H^R \subset \widetilde{M}$ be nice preimages

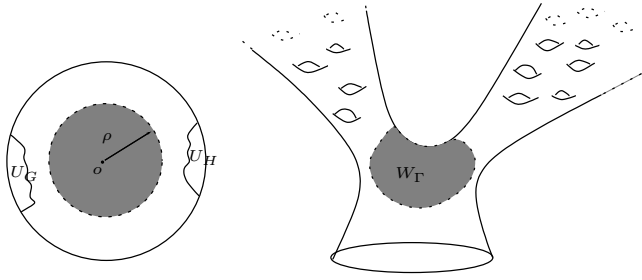


Figure 6: Schottky manifold

(Dirichlet domains viewed from o) of W_Γ^R, W_G^R, W_H^R , respectively for the actions of Γ, G, H . By definition of W_Γ^R, W_G^R, W_H^R and of a Dirichlet domain, one easily checks that they all lie inside $B^g(o, R)$. Moreover, as p_G and p_H are intermediate covers between \widetilde{M} and M_Γ , we have $o \in \widetilde{W}_\Gamma^R \subset \widetilde{W}_G^R \cap \widetilde{W}_H^R \subset B^g(o, R)$.

Let $\Gamma_{\widetilde{W}_\Gamma^R} \subset \Gamma, G_{\widetilde{W}_G^R} \subset G$ and $H_{\widetilde{W}_H^R} \subset H$ be the fundamental groups respectively of Γ, G and H respectively out of $\widetilde{W}_\Gamma^R, \widetilde{W}_G^R, \widetilde{W}_H^R$, according to Definition 7.6.

A key fact is the following.

Lemma 7.21 (Pit-Schapira, [PS18]). *For all $R > 0$, there exists a finite set $S \subset \Gamma$ such that*

$$\Gamma_{\widetilde{W}_G^R} \subset G \cup H \cup S.$$

It implies that $\delta_\infty(M_\Gamma) \leq \delta_{(W_\Gamma^R)^c}(\Gamma) \leq \max\{\delta_G, \delta_H\}$, and therefore, if $\delta_\Gamma > \max\{\delta_G, \delta_H\}$ then Γ is strongly positively recurrent.

We precise this inclusion in the following lemma, which implies immediately Theorem 7.18.

Lemma 7.22. *With the previous notations, for all $R \geq \rho + 1$, there exist a finite set F such that*

$$\Gamma_{\widetilde{W}_\Gamma^{2R}} \subset S \cup G_{\widetilde{W}_G^{2R}} \cup H_{\widetilde{W}_H^{2R}} \subset S \cup F \cup \bigcup_{\alpha, \beta \in F} \alpha \Gamma_{\widetilde{W}_\Gamma^R} \beta.$$

Proof. Let us first show the left inclusion. It follows from the previous lemma that $\Gamma_{\widetilde{W}_\Gamma^{2R}} \subset G \cup H \cup S$. Moreover,

$$G \cdot \widetilde{W}_G^{2R} = G \cdot B^g(o, 2R) \subset \Gamma \cdot \widetilde{W}_\Gamma^{2R} = \Gamma \cdot B^g(o, 2R).$$

For each $\gamma \in \Gamma_{\widetilde{W}_\Gamma^{2R}} \cap G$, there exist $x, y \in \widetilde{W}_\Gamma^{2R} \subset \widetilde{W}_G^{2R}$ such that

$$[x, \gamma y] \cap \Gamma \cdot \widetilde{W}_\Gamma^{2R} \subset \widetilde{W}_\Gamma^{2R} \cup \gamma \cdot \widetilde{W}_\Gamma^{2R}, \quad \text{whence} \quad [x, \gamma y] \cap G \cdot \widetilde{W}_G^{2R} \subset \widetilde{W}_G^{2R} \cup \gamma \widetilde{W}_G^{2R},$$

so that $\gamma \in G_{\widetilde{W}_G^{2R}}$. It shows that $\Gamma_{\widetilde{W}_\Gamma^{2R}} \cap G \subset G_{\widetilde{W}_G^{2R}}$.

Similarly, $\Gamma_{\widetilde{W}_\Gamma^{2R}} \cap H \subset H_{\widetilde{W}_H^{2R}}$.

Let us now prove the right inclusion. We want to show that there exists a finite set $F \subset \Gamma$ such that $G_{\widetilde{W}_G^{2R}} \subset F \cup \bigcup_{\alpha, \beta \in F} \alpha \Gamma_{\widetilde{W}_\Gamma^R} \beta$, the case of $H_{\widetilde{W}_H^{2R}}$ being similar.

Define $F_{\lambda R}$ as $F_{\lambda R} = \{\gamma \in \Gamma, \gamma B^g(o, \lambda R) \cap B^g(o, \lambda R) \neq \emptyset\}$.

First observe that for $\lambda \geq 2$, we have

$$\widetilde{W}_\Gamma^{2R} \subset \widetilde{W}_G^{2R} \subset \bigcup_{\alpha \in F_{\lambda R}} \alpha \cdot \widetilde{W}_\Gamma^R. \quad (23)$$

Let $g \in G_{\widetilde{W}_G^{2R}}$, $g \notin F$. By definition, there exist $x, y \in \widetilde{W}_G^{2R}$ such that (x, gy) intersects $G_{\widetilde{W}_G^{2R}}$ only in \widetilde{W}_G^{2R} and $g\widetilde{W}_G^{2R}$. We will show that (x, gy) intersects $\Gamma \cdot \widetilde{W}_\Gamma^R = \Gamma \cdot \widetilde{W}_G^R$ only inside \widetilde{W}_G^{2R} and $g\widetilde{W}_G^{2R}$. By equation (23), as in the proof of Proposition 7.9, it will imply that $g \in \bigcup_{\alpha, \beta \in F_{\lambda R}} \alpha \Gamma_{\widetilde{W}_\Gamma^R} \beta$. In fact, we will show that if (x, gy) intersects some $\gamma \cdot \widetilde{W}_\Gamma^R$, then either γ or $g^{-1}\gamma$ is in the finite set F , so that by the same argument, $g \in \bigcup_{\alpha, \beta \in F_{\lambda R}} \alpha \Gamma_{\widetilde{W}_\Gamma^R} \beta$.

By contradiction, assume that the geodesic segment (x, gy) intersects $\gamma \widetilde{W}_\Gamma^R$, with $\gamma \neq id, g$, and $\gamma, g^{-1}\gamma \notin F_{\lambda R}$. In particular, $d(o, \gamma o) > 2\lambda R$ and $d(go, \gamma o) > 2\lambda R$. As $g \in G_{\widetilde{W}_G^{2R}}$, we know that $\gamma \notin G$. Denote by $z_\gamma \in (x, gy)$ the closest point to γo in $(x, y) \cap \widetilde{W}_\Gamma^R$. By the above, we have $d(x, z_\gamma) \geq d(o, \gamma o) - 3R \geq (2\lambda - 3)R$.

By definition of a Schottky product, as $o \notin U_G \cup U_H$, either $\gamma o \in U_G$ or $\gamma o \in U_H$. Assume first that $\gamma o \in U_H$. Recall that $go \in U_G$. Therefore, the geodesic segment $(\gamma o, go)$ intersects the ball $B(o, \rho)$. As $d(\gamma o, go) \geq 2\lambda R$ and $d(\gamma o, z_\gamma) \leq R$, $d(go, gy) \leq 2R$, the geodesic segment (z_γ, gy) intersects the ball $B(o, \rho + 2R)$. Let w_γ be a point in this intersection. Therefore, we get $d(x, w_\gamma) \leq d(x, o) + d(o, w_\gamma) \leq 4R + \rho \leq 5R$. However, $d(x, w_\gamma) \geq d(x, z_\gamma) > (2\lambda - 3)R$, which leads to a contradiction as soon as $\lambda \geq 4$.

Therefore, the first case holds, $\gamma o \in U_G$, so that γ has a reduced form as $\gamma = g'h'\gamma'$, with $g' \in G \setminus \{id\}$, $h' \in H \setminus \{id\}$, $\gamma' \in \Gamma$. We will distinguish the cases $g' \in F$ and $g' \notin F$.

If $g' \notin F$, consider the segment $[(g')^{-1}o, h'\gamma'o]$. It goes from U_G to U_H so that it intersects the ball $B^g(o, \rho)$. It follows that $[o, \gamma o]$ intersects $g' \cdot B(o, \rho)$ at a point y with $d(o, y) \geq 2\lambda R - \rho$. By Lemma 5.1, for λ large enough, the point y is at distance less than ρ from the geodesic segment $(x, \gamma o)$, and therefore at distance less than $R + 1$ from the geodesic segment (x, z_γ) . Thus, we deduce that (x, z_γ) intersects the ball $g'B(o, \rho + R + 1)$. As we assumed $R \geq \rho + 1$, this ball is included in $g'B(o, 2R) \subset G \cdot \widetilde{W}_G^{2R}$. Moreover, as $\gamma' \notin F$, this intersection $(x, z_\gamma) \cap g'B(o, 2R)$ is disjoint from \widetilde{W}_G^{2R} , and as $\gamma \notin F$, and the intersection is between x and z_γ , this intersection is also disjoint from $g \cdot \widetilde{W}_G^{2R}$. This is a contradiction with the hypothesis $g \in G_{\widetilde{W}_G^{2R}}$.

It remains the case $g' \in F$, which implies in particular $g' \neq g$. Consider in this case the geodesic segment $[h'\gamma'o, (g')^{-1}go]$. It goes from U_H to U_G , so that it intersects the ball $B^g(o, \rho)$. It follows that $[\gamma o, go]$ intersects $g'B(o, \rho)$. The same arguments on $[z_\gamma, gy]$ instead of $[x, z_\gamma]$ lead once again to a contradiction with the hypothesis $g \in G_{\widetilde{W}_G^{2R}}$.

It concludes the proof, for $F = F_{\lambda R}$, for some $\lambda \geq 4$ determined by the use of Lemma 5.1. \square

□

7.3.3 Ancona-like examples

On hyperbolic manifolds, the dynamics is strongly related to the spectrum of the Laplacian. In particular, a well-known theorem of Patterson and Sullivan relates the entropy δ_Γ of $M = \mathbb{H}^{n+1}/\Gamma$ with the bottom of the spectrum of the Laplacian $\lambda_0(M)$:

Theorem 7.23 (Patterson 1976, Sullivan 1979 - 1987). *Let $M = \mathbb{H}^{n+1}/\Gamma$ be a complete hyperbolic manifold. If $\delta_\Gamma < \frac{n}{2}$, then $\lambda_0(M) = \frac{n^2}{4}$. If $\delta_\Gamma > \frac{n}{2}$, then $\lambda_0(M) = \delta_\Gamma(n - \delta_\Gamma)$.*

We present now a family of surfaces inspired by examples of Ancona [Anc], which is particularly easy to handle using the entropy at infinity introduced before.

Theorem 7.24. *There are plenty of examples of geometrically infinite hyperbolic surfaces $S = \mathbb{H}^2/\Gamma$ with $\delta_\infty(S) < \delta_\Gamma(S)$.*

By Theorem 7.1, all these examples have finite Bowen-Margulis measure. We postpone a more detailed study of these examples and relations between SPR property and spectrum of the Laplacian to our future paper [ST].

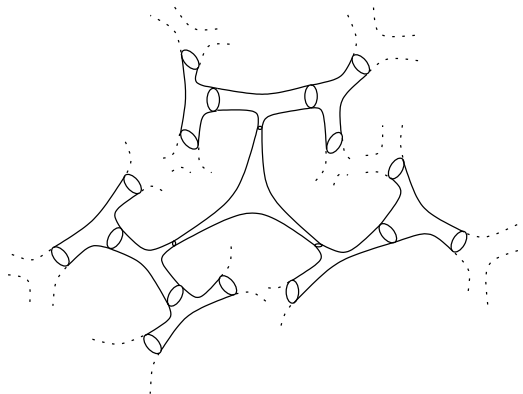


Figure 7: SPR surface

Proof. Let $N = \mathbb{H}^2/\Gamma$ be a complete hyperbolic surface with $\frac{n}{2} < \delta_\Gamma < n$. Denote by g_0 its metric. For example, N can be build as a nonamenable regular cover of a compact hyperbolic surface N_0 . In any pair of pants decomposition of N , choose finitely many pairs of pants P_1, \dots, P_K . Change the metric of N to a metric g_ε , which is equal to g_0 far from the pants P_i , and modified in the neighbourhood of the P_i by shrinking the lengths of the boundary geodesics of the pants P_i to a length ε . Let Γ_ε be a discrete group such that the new hyperbolic surface (N, g_ε) is isometric to $\mathbb{H}^2/\Gamma_\varepsilon$.

As the perturbation is compact, for all $\varepsilon > 0$, $\delta_\infty(g_\varepsilon) = \delta_\infty(g_0) < n$. An elementary computation (see for example [CCdV88, Prop. II.2 (ii)]) gives

$\lim_{\varepsilon \rightarrow 0} \lambda_0(N, g_\varepsilon) = 0$, therefore $\lim_{\varepsilon \rightarrow 0} \delta_{\Gamma_\varepsilon} = n$. This implies that for $\varepsilon > 0$ small enough, (N, g_ε) has a critical gap at infinity : $\delta_{\Gamma_\varepsilon} > \delta_{\Gamma_0} \geq \delta_\infty(g_\varepsilon)$. \square

7.4 SPR manifolds have finite Bowen-Margulis measure

This paragraph is devoted to the proof of the first part of Theorem 7.1 : if (M, g) is a strongly positively recurrent manifold, then the Bowen-Margulis measure of its geodesic flow has finite mass.

This finiteness result had been shown in [DOP00] on geometrically finite manifolds, under the assumption that $\max\{\delta_{\mathcal{P}_1}(g), \dots, \delta_{\mathcal{P}_k}(g)\} < \delta_\Gamma$, which is exactly the SPR assumption in the geometrically finite context, although they did not introduce this concept.

As said earlier, this result (finiteness of Bowen-Margulis measure) has been obtained independently, by a different approach, in [Vel17].

Our proof will rely on the following theorem shown in [PS18].

Theorem 7.25 (Pit-Schapira [PS18]). *Let (M, g) be a complete manifold with negative curvatures. Then the Bowen-Margulis measure of (M, g) is finite if and only if $\Gamma = \pi_1(M)$ is divergent and there exists a compact set $W \subset M$ with nice preimage \widetilde{W} such that $\Gamma_{\widetilde{W}}$ satisfies*

$$\sum_{\gamma \in \Gamma_{\widetilde{W}}} d(o, \gamma o) e^{-\delta_\Gamma d(o, \gamma o)} < +\infty.$$

Let (M, g) be a complete strongly positively recurrent manifold: there exists a compact set $W \subset M$ such that $\delta_{W^c}(g) < \delta_\Gamma(g)$. The second condition $\sum_{\gamma \in \Gamma_{\widetilde{W}}} d(o, \gamma o) e^{-\delta_\Gamma d(o, \gamma o)} < +\infty$ is then automatically satisfied for \widetilde{W} a nice lift of W . Therefore, Theorem 7.1 follows immediately from the following.

Theorem 7.26. *Let (M, g) be a strongly positively recurrent manifold. Then its fundamental group Γ is divergent.*

We give first the strategy of the proof. Let (M, g) be a SPR manifold, with $\Gamma = \pi_1(M)$. It follows from Hopf-Tsuji-Sullivan theorem (see [Rob03, p.18]) that Γ is divergent if and only if any Patterson-Sullivan measure ν_o^g (cf Section 5.4) gives full measure to the radial limit set Λ_Γ^r .

Theorem 7.26 follows from a careful study of Λ_Γ^r . More precisely, if W is a nice set with $\delta_{W^c} < \delta_\Gamma$ and nice lift \widetilde{W} , we introduce a kind of limit set \mathcal{L}_{W^c} of the subset $\Gamma_{\widetilde{W}}$ of Γ , see Definition 7.27 and Proposition 7.28. We show in Proposition 7.31 that $\nu_o^g(\mathcal{L}_{W^c}) = 0$. By definition, $\partial \widetilde{M} \setminus \mathcal{L}_{W^c}$ consists in asymptotic directions of geodesics returning infinitely often in the compact set W . In particular, it is included in the radial limit set. We deduce that $\nu_o^g(\Lambda_\Gamma^r) = 1$, which implies that Γ is divergent by Hopf-Tsuji-Sullivan Theorem.

Definition 7.27. *Let $W \subset M$ be a compact subset and $\widetilde{W} \subset \widetilde{M}$ a nice preimage of W . Introduce the set*

$$\Lambda_{\widetilde{W}} = \left\{ \xi \in \Lambda_\Gamma \text{ s.t. } \exists x \in \widetilde{W}, [x, \xi] \cap \Gamma \cdot \widetilde{W} \subset \widetilde{W} \right\}.$$

We call limit set of Γ out of W the set $\mathcal{L}_{W^c} = \Gamma \cdot \Lambda_{\widetilde{W}}$.

The following proposition shows that all elements of $\Lambda_{\widetilde{W}}$ are limit points of $\Gamma_{\widetilde{W}} \cdot o$ in the boundary at infinity, and that the only limit points of $\Gamma_{\widetilde{W}} \cdot o$ which are not in $\Lambda_{\widetilde{W}}$ are endpoints of geodesic rays which do not come back inside the interior $\Gamma \cdot \widetilde{W}$, after leaving \widetilde{W} but touch the boundary $\partial(\Gamma \cdot \widetilde{W})$.

Proposition 7.28. *Let $W \subset M$ be a compact subset and $\widetilde{W} \subset \widetilde{M}$ a nice preimage of W . Then*

$$\Lambda_{\widetilde{W}} \subset \overline{\Gamma_{\widetilde{W}} \cdot o} \setminus \Gamma_{\widetilde{W}} \cdot o \subset \left\{ \xi \in \Lambda_{\Gamma} \text{ s.t. } \exists x \in \widetilde{W}, [x, \xi] \cap \Gamma \cdot \overset{\circ}{\widetilde{W}} \subset \widetilde{W} \right\}.$$

Proof. Without loss of generality, assume $o \in \widetilde{W}$. We show first the left inclusion. Let $\xi \in \Lambda_{\widetilde{W}} \subset \Lambda_{\Gamma}$. There exists a sequence (γ_n) of elements of Γ such that $\gamma_n o \rightarrow \xi$. Moreover, by definition of $\Lambda_{\widetilde{W}}$, there exists $x \in \widetilde{W}$ such that the geodesic $[x, \xi]$ does not intersect $\Gamma \cdot \widetilde{W}$ after leaving \widetilde{W} . Thus, for n large enough, the geodesic segment $[x, \gamma_n o]$ also leaves \widetilde{W} before returning to $\gamma_n \widetilde{W}$. Let $\widetilde{\gamma}_n \cdot \widetilde{W}$ be the first image of \widetilde{W} crossed by the geodesic segment $[x, \gamma_n o]$ after leaving \widetilde{W} . By construction, $\widetilde{\gamma}_n \in \Gamma_{\widetilde{W}}$. Moreover, we have

$$\lim_{n \rightarrow +\infty} d^g(x, \widetilde{\gamma}_n o) = +\infty. \quad (24)$$

Indeed, for all $R > 0$, there exists $\eta > 0$ such that inside the (compact) ball $B^g(x, R)$, the distance between $[x, \xi] \cap B^g(x, R)$ and $\Gamma \cdot \widetilde{W} \setminus \widetilde{W}$ is at least η . Moreover, it follows from Lemma 5.1 that the sequence of geodesic segments $([x, \gamma_n o])_{n \in \mathbb{N}}$ converges to the half geodesic $[x, \xi]$ uniformly on $B^g(x, R)$. Thus, for all n large enough, $[x, \gamma_n o] \cap B^g(x, R)$ and $[x, \xi] \cap B^g(x, R)$ are $\frac{\eta}{2}$ -close, so that $[x, \gamma_n \cdot o]$ does not meet $\Gamma \cdot \widetilde{W} \setminus \widetilde{W}$ on $B^g(x, R)$, whence $d^g(x, \widetilde{\gamma}_n o) \geq R$.

It follows from the above that the sequence of geodesic segments $([x, \widetilde{\gamma}_n o])_{n \in \mathbb{N}}$ also converges to the half geodesic $[x, \xi]$, so that

$$\xi = \lim_{n \rightarrow +\infty} \widetilde{\gamma}_n o \in \overline{\Gamma_{\widetilde{W}} \cdot o} \setminus \Gamma_{\widetilde{W}} \cdot o.$$

Let us now show that $\overline{\Gamma_{\widetilde{W}} \cdot o} \setminus \Gamma_{\widetilde{W}} \cdot o \subset \left\{ \xi \in \Lambda_{\Gamma} \text{ s.t. } \exists x \in \widetilde{W}, [x, \xi] \cap \Gamma \cdot \overset{\circ}{\widetilde{W}} \subset \widetilde{W} \right\}$.

Let $\xi \in \overline{\Gamma_{\widetilde{W}} \cdot o} \setminus \Gamma_{\widetilde{W}} \cdot o$. There exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of elements of $\Gamma_{\widetilde{W}}$ such that $\gamma_n o \rightarrow \xi$ and $d^g(o, \gamma_n \cdot o) \rightarrow +\infty$. By definition of $\Gamma_{\widetilde{W}}$, for all $n \geq 0$ there exist $x_n, y_n \in \widetilde{W}$ such that the geodesic segment $[x_n, \gamma_n y_n]$ intersects $\Gamma \cdot \widetilde{W}$ only in \widetilde{W} and $\gamma_n \cdot \widetilde{W}$. Up to taking a subsequence, we can assume that $x_n \rightarrow x_\infty \in \widetilde{W}$ and $y_n \rightarrow y_\infty \in \widetilde{W}$ as $n \rightarrow +\infty$. Once again, it follows from the compactness of \widetilde{W} and Lemma 5.1 that the sequence of geodesic segments $([x_\infty, \gamma_n y_\infty])_{n \in \mathbb{N}}$ converges to $[x_\infty, \xi]$ uniformly on compact sets. Therefore $[x_\infty, \xi]$ cannot intersect the interior of $\Gamma \cdot \widetilde{W}$. \square

We gather in the following proposition elementary properties of the sets $\Lambda_{\widetilde{W}}$ and \mathcal{L}_{W^c} .

Proposition 7.29. *Let (M, g) be a manifold with pinched negative curvature. Let $W \subset M$ be a nice compact set, and \widetilde{W} a nice preimage. With the above notations,*

- *the set $\mathcal{L}_{W^c} = \Gamma \cdot \Lambda_{\widetilde{W}}$ is the set of endpoints of geodesics which eventually leave $\Gamma \cdot \widetilde{W}$:*

$$\mathcal{L}_{W^c} = \Gamma \cdot \Lambda_{\widetilde{W}} = \left\{ v_+ \in \Lambda_\Gamma ; \exists v \in S^g \widetilde{M}, \exists T > 0 \text{ s.t. } \forall t \geq T, \pi g^t v \notin \Gamma \cdot \widetilde{W} \right\}.$$
- *The limit set out of \widetilde{W} , $\mathcal{L}_{W^c} = \Gamma \cdot \Lambda_{\widetilde{W}}$ does not depend on the choice of nice preimage \widetilde{W} .*
- *If $W_1 \subset W_2$, we have $\mathcal{L}_{W_2^c} \subset \mathcal{L}_{W_1^c}$.*
- *$\Lambda_\Gamma \setminus (\mathcal{L}_{W^c}) \subset \Lambda_\Gamma^r$, where Λ_Γ^r is the radial limit set.*

Proof. The first property is left to the reader.

The set $\left\{ v_+ \in \Lambda_\Gamma ; \exists v \in S^g \widetilde{M}, \exists T > 0 \text{ s.t. } \forall t \geq T, \pi g^t v \notin \Gamma \cdot \widetilde{W} \right\}$ only depends on $\Gamma \cdot \widetilde{W} = p_\Gamma^{-1}(W)$, which is independent of the choice of \widetilde{W} .

If $W_1 \subset W_2$, then for all nice preimages \widetilde{W}_1 and \widetilde{W}_2 , we have

$$\Gamma \cdot \widetilde{W}_1 = p_\Gamma^{-1}(W_1) \subset p_\Gamma^{-1}(W_2) = \Gamma \cdot \widetilde{W}_2,$$

which shows the third point.

The radial limit set is the set of $\xi \in \Lambda_\Gamma$ such that there exists $x \in \widetilde{M}$ and a compact set $K \subset M$ such that the geodesic ray $[x, \xi]$ intersects infinitely often the preimage $p_\Gamma^{-1}(K)$. If $\xi \in \Lambda_\Gamma \setminus (\Lambda_{\widetilde{W}})$, by the above proposition, the geodesic ray $[x, \xi]$ intersects infinitely often the set $\Gamma \cdot \widetilde{W} = p_\Gamma^{-1}(W)$, which shows the last claim. \square

As seen in Section 7.3, basic examples are given by geometrically finite manifolds. The following proposition is an immediate consequence of Propositions 7.17 and 7.28.

Proposition 7.30. *Let (M, g) be a geometrically finite manifold with pinched negative curvature, with k cusps C_1, \dots, C_k . Let $W = B^g(x, R)$ be a large ball. It admits a nice preimage \widetilde{W} such that*

$$\Lambda_{\widetilde{W}} = \{\xi_1, \dots, \xi_k\},$$

each point $\xi_1, \dots, \xi_k \in \Lambda_\Gamma$ being a parabolic point fixed by a parabolic group $\mathcal{P}_i < \Gamma$ representing the cusp C_i .

The following proposition is a detailed version of Theorem 7.26, with additional properties which will be useful in section 7.5.

Proposition 7.31. *Let $(M = \widetilde{M}/\Gamma, g)$ be a complete manifold with $K_g \leq -a^2$, with $\Gamma = \pi_1(M)$ its fundamental group. Assume that (M, g) is SPR. Then Γ is divergent.*

Moreover, for all compact sets $W \subset M$ such that $\delta_{W^c} < \delta_\Gamma$, for all $\eta \in (0, \delta_\Gamma - \delta_{W^c})$, there exists $C = C(g, W, \eta, a) > 0$ such that for all nice preimage \widetilde{W} of W and all $T \geq 4\text{diam}_g(W)$, if

$$U_T = U_T(\widetilde{W}, g) = \left\{ \xi \in \widetilde{M} \cup \partial \widetilde{M} ; \exists x \in \widetilde{W} \text{ s.t. } \forall t \in [0, T], [x, \xi]_T \cap \Gamma W \subset \widetilde{W} \right\},$$

then the unique Patterson-Sullivan density $(\nu_x^g)_{x \in \widetilde{M}}$ on Λ_Γ such that $\nu_o^g(\Lambda_\Gamma^r) = 1$ satisfies

$$\nu_o^g(U_T) \leq C e^{-(\delta_\Gamma - \delta_{W^c} - \eta)T}.$$

In particular,

$$\nu_o^g(\Lambda_{\widetilde{W}}) = \nu_o^g(\cap_{T>0} U_T) = 0.$$

Proof. We start with any Patterson-Sullivan density (ν_x^g) on Λ_Γ obtained as a weak limit of an average as in section 3.4. We will show that there exists $C > 0$ such that for all $T > 0$ large enough,

$$\nu_o^g(U_T) \leq C e^{-(\delta_\Gamma - \delta_{W^c} - \eta)T}. \quad (25)$$

By definition, U_T is the set of points joined by a geodesic from \widetilde{W} which, after exiting \widetilde{W} , does not enter $\Gamma \cdot \widetilde{W}$ before time T , so that

$$\Lambda_{\widetilde{W}} = \bigcap_{T>0} U_T.$$

Therefore, (25) implies

$$\nu_o^g(\Lambda_{\widetilde{W}}) = 0 \quad \text{so that} \quad \nu_o^g(\mathcal{L}_{W^c}) = 0 \quad \text{and} \quad \nu_o^g(\Lambda_\Gamma^r) = 1.$$

By Hopf-Tsuji-Sullivan Theorem, it will imply that Γ is divergent and the Patterson-Sullivan density is unique.

If $x \in \widetilde{M}$ and $\xi \in \widetilde{M} \cup \partial \widetilde{M}$, set $[x, \xi]_T = (\pi g^t v)_{t \in [0, T]}$, where $v \in S_x^g \widetilde{M}$ is the tangent vector at x of the geodesic $[x, \xi]$.

Recall notations from Section 3.4. We omit the mention of the metric g here. As in [Pat76], choose a positive increasing map $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $\eta > 0$, there exist $C_\eta > 0$ and $r_\eta > 0$ such that

$$\forall r \geq r_\eta, \quad \forall t \geq 0, \quad h(t+r) \leq C_\eta e^{\eta t} h(r), \quad (26)$$

and the series $\widetilde{P}_\Gamma(s) = \sum_{\gamma \in \Gamma} h(d(o, \gamma o)) e^{-sd(o, \gamma o)}$ diverges at the critical exponent

δ_Γ . Construct a Patterson-Sullivan density (ν_x) s.t. for all $x \in \widetilde{M}$, the measure ν_x is a weak limit as $s \rightarrow \delta_\Gamma^+$ of the positive finite measures

$$\nu_x^s = \frac{1}{\widetilde{P}_\Gamma(o, s)} \sum_{\gamma \in \Gamma} h(d(x, \gamma o)) e^{-sd(x, \gamma o)} \delta_{\gamma o}.$$

For all $\gamma \in \Gamma_{\widetilde{W}}$, define $\mathcal{O}_{\widetilde{W}}(\gamma \cdot \widetilde{W})$ as the set of $y \in \widetilde{M} \cup \partial \widetilde{M}$ such that there exists $v \in S^g \widetilde{W}$ such that the first intersection of the geodesic ray $(\pi g^t v)_{t \geq 0}$ with $\Gamma \cdot \widetilde{W}$, after the first exit of \widetilde{W} is in $\gamma \cdot \widetilde{W}$, and the point y belongs to $(\pi g^t v)_{t \geq 0}$.

By definition of U_T and $\Gamma_{\widetilde{W}}$, and triangular inequality, for all $T > 0$ and $\alpha \in \Gamma$, if $\alpha o \in U_T$, there exists $\gamma \in \Gamma_{\widetilde{W}}$ such that $\alpha o \in \mathcal{O}_{\widetilde{W}}(\gamma \cdot \widetilde{W})$ and $d(o, \gamma o) \geq T - 2D$, with $D = \text{diam}(\widetilde{W})$. Indeed, choose γ so that $\gamma \widetilde{W}$ is the first copy of \widetilde{W} intersected by all geodesic segments from \widetilde{W} to αo after exiting \widetilde{W} inside $\Gamma \widetilde{W}$.

In other words, we have

$$\Gamma \cdot o \cap U_T \subset \bigcup_{\gamma \in \Gamma_{\widetilde{W}}, d(o, \gamma \cdot o) \geq T-2D} \mathcal{O}_{\widetilde{W}}(\gamma \cdot \widetilde{W}). \quad (27)$$

Fix $s > \delta_\Gamma$ and recall from section 3.4 that for all $x, y \in \widetilde{M}$ and $\xi \in \Gamma \cdot o$,

$$\frac{d\nu_y^s}{d\nu_x^s}(\xi) = e^{-s(d(y, \xi) - d(x, \xi))} \frac{h(d(y, \xi))}{h(d(x, \xi))}.$$

Therefore, for all $\gamma \in \Gamma_{\widetilde{W}}$,

$$\begin{aligned} \nu_o^s(\mathcal{O}_{\widetilde{W}}(\gamma \cdot \widetilde{W})) &= \nu_{\gamma^{-1}o}^s(\mathcal{O}_{\gamma^{-1}\widetilde{W}}(\widetilde{W})) \\ &= \int_{\mathcal{O}_{\gamma^{-1}\widetilde{W}}(\widetilde{W})} e^{-s(d(\gamma^{-1}o, \xi) - d(o, \xi))} \frac{h(d(\gamma^{-1}o, \xi))}{h(d(o, \xi))} d\nu_o^s(\xi). \end{aligned}$$

Moreover, there exists $C > 0$ such that as soon as $d(o, \gamma o) > 2D$, for all $\xi \in \mathcal{O}_{\gamma^{-1}\widetilde{W}}(\widetilde{W})$, $d(\gamma^{-1}o, \xi) \geq d(\gamma^{-1}o, o) + d(o, \xi) - C$, which implies by (26) that

$$e^{-s(d(\gamma^{-1}o, \xi) - d(o, \xi))} \frac{h(d(\gamma^{-1}o, \xi))}{h(d(o, \xi))} \leq e^{sC} C_\eta e^{-s(d(\gamma^{-1}o, o) + \eta d(\gamma^{-1}o, o))}.$$

Therefore, as $\nu_o(\partial\widetilde{M}) = 1$, there exists $C_\eta > 0$ such that for all $\gamma \in \Gamma_{\widetilde{W}}$ with $d(o, \gamma o) > 2D$ and $s > \delta_\Gamma$,

$$\nu_o^s(\mathcal{O}_{\widetilde{W}}(\gamma \cdot \widetilde{W})) \leq C_\eta e^{(-s+\eta)d(o, \gamma \cdot o)}.$$

By (27), for all $T > 4D$, we get

$$\nu_o^s(U_T) \leq C_\eta \sum_{\substack{\gamma \in \Gamma_{\widetilde{W}} \\ d(o, \gamma \cdot o) \geq T-2D}} e^{(-s+\eta)d(o, \gamma \cdot o)}.$$

Taking any weak limit as $s \rightarrow \delta_\Gamma^+$, we obtain

$$\nu_o^g(U_T) \leq C_\eta \sum_{\substack{\gamma \in \Gamma_{\widetilde{W}} \\ d(o, \gamma \cdot o) \geq T-2D}} e^{(-\delta_\Gamma + \eta)d(o, \gamma \cdot o)}.$$

As $\delta_\Gamma - \eta > \delta_{W^c}$, the right hand side decreases exponentially fast as $T \rightarrow +\infty$. As mentioned at the beginning of the proof, by Hopf-Tsuji-Sullivan, we deduce that Γ is divergent so that Theorem 7.26 is proven.

Let us prove now the end of the statement of Proposition 7.31. The Patterson-Sullivan ν_o^g is the weak limit as $s \rightarrow \delta_\Gamma^+$ of $\nu_o^s = \frac{1}{P_\Gamma(o, s)} \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}$. Repeating

exactly the same computations, setting $h \equiv 1$, we get that there exists $C_a > 0$, depending only on the curvature upperbound, such that for all $T \geq 4D$

$$\nu_o^g(U_T) \leq e^{\delta_\Gamma C_a} \sum_{\substack{\gamma \in \Gamma_{\widetilde{W}} \\ d(o, \gamma o) \geq T-2D}} e^{(-\delta_\Gamma)d(o, \gamma o)}.$$

We get therefore that for all $T \geq 4D$,

$$\nu_o^g(U_T) \leq e^{\delta_\Gamma(C_a+2D)} e^{-(\delta_\Gamma-\delta_{W^c}-\eta)T} \sum_{\gamma \in \Gamma_{\widetilde{W}}} e^{-(\delta_{W^c}+\eta)d(o,\gamma o)},$$

which is precisely the desired estimate with

$$C(\eta, g, W, a) = e^{\delta_\Gamma(C_a+2D)} \sum_{\gamma \in \Gamma_{\widetilde{W}}} e^{-(\delta_{W^c}+\eta)d(o,\gamma o)}. \quad (28)$$

□

Under the above assumption, the Patterson-Sullivan measure ν_o^g gives full mass to the set of endpoints of lifts of geodesics of (M, g) which come back infinitely often in W . This set is in general strictly smaller than the radial limit set. The product structure of the Bowen-Margulis measure (see section 3.4) implies the following useful fact.

Corollary 7.32. *Under the same assumptions, let $W \subset M$ be any compact set such that $\delta_{W^c}(g) < \delta_\Gamma(g)$. Then the Bowen-Margulis measure of $S^g M$ is finite and gives full mass to the set of bi-infinite geodesics which intersect infinitely often W in the past and in the future.*

7.5 Entropy variation for SPR manifolds

As mentioned earlier, the original motivation of this article was to find reasonable geometric assumptions on non-compact manifolds with negative curvature such that the entropy is regular under a small variation of the metric. In this subsection, our aim is to finish the proof of Theorem 7.1.

Let $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ be a uniformly \mathcal{C}^1 family of complete metrics on the manifold M such that for all $\varepsilon \in (-1,1)$, $-b^2 \leq K_{g_\varepsilon} \leq -a^2$ for some $b > a > 0$, and (M, g_0) is SPR.

Let $W \subset M$ be a compact subset such that $\delta_{W^c}(g_0) < \delta_\Gamma(g_0)$, and let \widetilde{W} be a nice preimage of W . For $r > 0$, denote by $W_r = \{x \in M; d^{g_0}(x, W) \leq r\}$ the (g_0, r) -neighbourhood of W . Note that $\delta_{W_r^c}(g_0) \leq \delta_{W^c}(g_0) \leq \delta_\infty(g_0) < \delta_\Gamma(g_0)$. Denote by \widetilde{W}_r a nice preimage of W_r such that $\widetilde{W} \subset \widetilde{W}_r$. Observe that $\gamma \cdot \widetilde{W}_r$ is the (g_0, r) -neighbourhood of $\gamma \cdot \widetilde{W}$.

Lemma 7.33. *For all $r > 0$, there exists a finite set $F \subset \Gamma$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have*

$$\Gamma_{\widetilde{W}_{2r}}(g_0) \subset \bigcup_{\alpha, \beta \in F} \alpha \Gamma_{\widetilde{W}_r}(g_\varepsilon) \beta \quad \text{and} \quad \Gamma_{\widetilde{W}_r}(g_\varepsilon) \subset \bigcup_{\alpha, \beta \in F} \alpha \Gamma_{\widetilde{W}}(g_0) \beta.$$

Proof. We prove the right inclusion, the left one is proved similarly.

Let $D = \text{diam}_{g_0}(\widetilde{W})$ and $D' = e^1(D+1)$, so that for all $\varepsilon \in (-1,1)$, $\text{diam}_{g_\varepsilon}(\widetilde{W}_r) \leq D'$. It follows from Section 5 that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $x, y \in \widetilde{W}_r$, and $\gamma \in \Gamma_{\widetilde{W}_r}$, the g_ε -geodesic between x and γy is at distance less than r to the g_0 -geodesic between x and y . Reasoning as in Proposition 7.9 leads to the desired result. □

This lemma leads to the following corollary, which implies the first item of Theorem 7.1.

Corollary 7.34. *Let $(g_\varepsilon)_{\varepsilon \in (-1,1)}$ be a uniformly \mathcal{C}^1 family of complete metrics on the manifold M such that for all $\varepsilon \in (-1,1)$, $-b^2 \leq K_{g_\varepsilon} \leq -a^2$ for some $b > a > 0$, and (M, g_0) is SPR. Then for all $\alpha > 0$ and $r > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have*

$$e^{-\alpha} \delta_{W_{2r}}(g_0) \leq \delta_{W_r}(g_\varepsilon) \leq e^\alpha \delta_{W^c}(g_0).$$

In particular, the entropy at infinity $\varepsilon \mapsto \delta_\infty(g_\varepsilon)$ is continuous at $\varepsilon = 0$, and if $\alpha > 0$ is small enough, g_ε is SPR for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Proof. Let $r, \alpha > 0$ be fixed, and choose ε_0 as in Lemma 7.33. For all $\gamma \in \Gamma$, we have $d^{g_\varepsilon}(o, \gamma o) \geq e^{-\varepsilon/2} d^{g_0}(o, \gamma o)$. Therefore, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we get $\delta_{W_r}(g_\varepsilon) \leq e^{\varepsilon/2} \delta_{W^c}(g_0) \leq e^\alpha \delta_{W^c}(g_0)$ up to reducing ε_0 . The other inequality is proved similarly. \square

Let us show now the last item of Theorem 7.1, that is that the mass of the Bowen-Margulis measure of g_ε varies continuously. This will rely on the following estimate, which is a uniform version of Proposition 7.31.

Lemma 7.35. *For all $\delta_0 \in (0, \delta_\Gamma(g_0) - \delta_\infty(g_0))$ and $\beta \in (0, \delta_0)$, there exists a compact set $W \subset M$ with nice preimage \widetilde{W} , $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have $\delta_\Gamma(g_\varepsilon) - \delta_{W^c}(g_\varepsilon) \geq \delta_0$ and*

$$\nu_o^{g_\varepsilon}(U_T(\widetilde{W}, g_\varepsilon)) \leq C e^{-\beta T},$$

where $U_T(\widetilde{W}, g_\varepsilon)$ is defined as in Proposition 7.31.

Proof. Let $\delta_0 \in (0, \delta_\Gamma(g_0) - \delta_\infty(g_0))$ be fixed. By the above corollary, for $|\varepsilon|$ small enough, (M, g_ε) is SPR and has therefore a finite Bowen-Margulis. Choose $\alpha > 0$ small enough and a large enough compact set $W \subset M$ so that $\delta_\infty(g_0) \leq \delta_{W^c}(g_0) \leq e^\alpha \delta_\infty(g_0)$. Let $r > 0$ small enough and $\varepsilon_0 > 0$ given by Corollary 7.34 be such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$e^{-\alpha} \delta_{W^c}(g_0) \leq \delta_{W_r}(g_\varepsilon) \leq e^\alpha \delta_{W^c}(g_0).$$

Up to decreasing $\alpha > 0$, we can therefore assume that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$\delta_\Gamma(g_\varepsilon) - \delta_{W_r}(g_\varepsilon) \geq \delta_0 > 0.$$

Let $\beta \in (0, \delta_0)$ and \widetilde{W}_r nice preimage of W_r be fixed. Define $D > 0$ as $D = \sup_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)} \text{diam}(\widetilde{W}_r)$. For all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let $U_T^\varepsilon = U_T(\widetilde{W}_r, g_\varepsilon)$ be defined as in Proposition 7.31. By the last estimate in the proof of Proposition 7.31, there exists $C_a > 0$, only depending on the curvature upperbound of the metrics g_ε , such that for all $T > 4D$,

$$\nu_o^{g_\varepsilon}(U_T^\varepsilon) \leq e^{\delta_\Gamma(g_\varepsilon)(C_a + 2D)} e^{-\beta T} \sum_{\gamma \in \Gamma_{\widetilde{W}_r(g_\varepsilon)}} e^{-(\delta_{W_r}(g_\varepsilon) + \beta) d^{g_\varepsilon}(o, \gamma o)}.$$

Therefore,

$$\nu_o^{g_\varepsilon}(U_T^\varepsilon) \leq e^K e^{-\beta T} \sum_{\gamma \in \Gamma_{\widetilde{W}(g_0)}} e^{-(e^{-\varepsilon} \delta_W(g_0) + \alpha) e^{-\varepsilon} d^{g_0}(o, \gamma o)},$$

where $K \in \mathbb{R}$ is independent of ε . Up to reducing $\alpha > 0$ and $\varepsilon_0 > 0$, we can suppose that $e^{-\alpha} \delta_W(g_0) + \beta e^{-\varepsilon} > \delta_W(g_0) + \frac{\beta}{2}$. Therefore, we get that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$\nu_0^{g_\varepsilon}(U_T^\varepsilon) \leq C e^{-\beta T}, \quad (29)$$

with $C > 0$ being independent of ε . This concludes the proof of Lemma 7.35, the compact set W of the statement being the set W_r of the proof. \square

Let us now conclude the proof of Theorem 7.1. Let $W \subset M$, $\widetilde{W} \subset \widetilde{M}$ and $\beta, \varepsilon_0, C > 0$ satisfy the conclusion of Lemma 7.35. For all $R > 0$, set as usual $W_R = \{x \in M; d_{g_0}(x, W) \leq R\}$. We have shown in Theorem 5.14 that, under our current hypotheses, the Bowen-Margulis measure $\varepsilon \mapsto m_{BM}^{g_\varepsilon}$ varies continuously for the weak-* convergence, i.e. on the dual of compactly supported maps. In particular, for all *fixed compact set* $K \subset M$ with $m_{BM}^{g_0}(\partial S^{g_0} K) = 0$, the map $\varepsilon \mapsto m_{BM}^{g_\varepsilon}(S^{g_\varepsilon} K)$ is continuous at $\varepsilon = 0$. Therefore the following lemma will imply Theorem 7.1.

Lemma 7.36. *With the above notations, for all $\alpha > 0$, there exists $R_0 > 0$ such that for all $R \geq R_0$ and all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have*

$$m_{BM}^{g_\varepsilon}(S^{g_\varepsilon}(M \setminus W_R)) \leq \alpha.$$

Proof. Let $R > 8 \text{diam}(W)$ be fixed and let $O_R = M \setminus W_R$. By Corollary 7.32, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, since $\delta_{W^c}(g_\varepsilon) < \delta_\Gamma(g_\varepsilon)$, the Bowen-Margulis measure $m_{BM}^{g_\varepsilon}$ gives full mass to the set of vectors which hit infinitely often W in the past and in the future. In particular,

$$m_{BM}^{g_\varepsilon}(S^{g_\varepsilon} O_R) = m_{BM}^{g_\varepsilon} \left(\prod_{n \geq R-1} O_n^\varepsilon \right),$$

where O_n^ε is defined for all integers $n \geq R-1$ by

$$O_n^\varepsilon = \{v \in S^{g_\varepsilon} O_R; \exists t \in [n, n+1[\text{ s.t. } \forall s \in [0, t), \pi g^{-s} v \notin W \text{ and } \pi g^{-t} v \in W\}.$$

Therefore, since the Bowen-Margulis measure $m_{BM}^{g_\varepsilon}$ is invariant under the geodesic flow (g_ε^t) ,

$$m_{BM}^{g_\varepsilon}(S^{g_\varepsilon} O_R) = \sum_{n \geq R-1} m_{BM}^{g_\varepsilon}(O_n^\varepsilon) = \sum_{n \geq R-1} m_{BM}^{g_\varepsilon}(g_\varepsilon^{-n}(O_n^\varepsilon))$$

Now, by definition for all $v \in g_\varepsilon^{-n}(O_n^\varepsilon)$, there exists $t \in [0, 1)$ such that $w = g_\varepsilon^{-n-t} v \in S^{g_\varepsilon} W$ and for all $s \in [0, n]$, we have $\pi g^s w \notin W$.

Let us write

$$\widetilde{A}_n^\varepsilon = \left\{ v \in S^{g_\varepsilon} \widetilde{W}; \exists t \in [n, n+1) \text{ s.t. } \forall s \in (0, t), \pi g_\varepsilon^s v \notin \gamma \cdot \widetilde{W} \text{ and } \pi g^t v \in \gamma \cdot \widetilde{W} \right\}.$$

The reader will easily check that $\bigcup_{s \in [0, 1)} g_\varepsilon^s A_n^\varepsilon \subset S^{g_\varepsilon} \widetilde{M}$ projects onto $g_\varepsilon^{-n}(O_n^\varepsilon)$.

Moreover, as soon as ε_0 is small enough, since $g_\varepsilon \geq e^{-\varepsilon} g_0 \geq \frac{1}{4} g_0$, all vectors $v \in \widetilde{A}_n^\varepsilon$ have a point at infinity v^+ which satisfies $v_+ \in U_{n/2}(\widetilde{W}, g_\varepsilon)$. As the map

$$v \mapsto e^{\delta_\Gamma(g_\varepsilon)(\mathcal{B}_{v^+}(o, \pi v) + \mathcal{B}_{v^-}(o, \pi v))}$$

is uniformly bounded in $v \in W$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the product structure of the Bowen-Margulis measure (see section 3.4) implies

$$m_{BM}^{g_\varepsilon}(O_n^\varepsilon) = m_{BM}^{g_\varepsilon}(g_\varepsilon^{-n}(O_n^\varepsilon)) \leq 2K\nu_o^{g_\varepsilon}(\Lambda_\Gamma) \times \nu_o^{g_\varepsilon}(U_{n/2}^\varepsilon),$$

which eventually gives by Lemma 7.35

$$m_{BM}^{g_\varepsilon}(O_n^\varepsilon) \leq 2KCe^{-\frac{\alpha}{2}n},$$

where C and α do not depend on ε . Therefore, we get

$$m_{BM}^{g_\varepsilon}(S^{g_\varepsilon}O_R) \leq 2KC \sum_{n \geq R-1} e^{-\frac{\alpha}{2}n} \leq \varepsilon$$

as soon as R is large enough. □

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