Gibbs measures in hyperbolic geometry and dynamics

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Abstract

The goal of these lectures is to recall briefly the definition and a geometric construction of Gibbs measures, and to illustrate their importance in various geometrical / dynamical contexts.

These notes are my personal notes of the lectures given at IHES in july 2025. They are not accurate and full of mistakes, not intended for publication. If you do not understand something, it is maybe false, please ask me! They are available here `https://imag.umontpellier.fr/ schapira/recherche/IHES2025-Barbara.pdf` You can also see the more introductory lectures given at CIRM in april 2025 `https://imag.umontpellier.fr/ schapira/recherche/CIRM2025-Barbara.pdf`. My older notes of lectures at CIRM in april 2016 are maybe more elementary, see `https://imag.umontpellier.fr/ schapira/recherche/texteCIRM-Hasselblatt.pdf`

1 Construction of Gibbs measures

1.1 The geodesic flow on the hyperbolic disc

1.1.1 Hyperbolic plane / disc

The hyperbolic plane is defined as $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ and endowed with the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. The geodesics are the curves which minimize the distance.

Exercise 1.1 Check these classical facts. The hyperbolic geodesics are the vertical half-lines and the half-circles orthogonal to the boundary $\mathbb{R} \times \{0\}$. The isometries preserving orientation are the homographies $z \to \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with determinant 1.

The model of the disk is more natural geometrically. The hyperbolic disk is the open disk $D(0,1)$ in $\mathbb{C}$, endowed with the image metric from the hyperbolic metric of $\mathbb{H}$ through the map $z \mapsto \frac{z-i}{z+i}$. In the disk model, the geodesics are the diameters and the pieces of circles orthogonal to the boundary.

1.1.2 Geodesic flow

A vector v on $T^1\mathbb{D}$, the geodesic (at unique speed) $(c_v(t))$ determined by v (such that $c_v'(0) = v$).

Figure 1: Two conformal models of the hyperbolic plane and their geodesics

The geodesic flow g_t moves v along the geodesic that it determines. If $(c_v(t))$ is the geodesic such that $c'_v(0) = v$, then

$$g_t(v) = c'_v(t)\,.$$

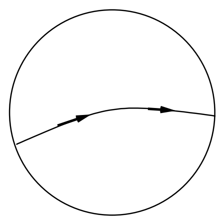

Figure 2: Geodesic flow

Exercise 1.2 *If c_1, c_2 are two geodesic rays such that $d(c_1(t), c_2(t)) \to 0$ when $t \to +\infty$, then show that for every $t \geq 0$,*

$$d(c_1(t), c_2(t)) \leq e^{-t} d(c_1(0), c_2(0))$$

Hint *Use the upper half plane model and come back to two vertical rays.*

The *Busemann cocycle* is the following map, defined on $S^1 \times \mathbb{D} \times \mathbb{D}$:

$$\beta_\xi(x, y) = \lim_{t \to +\infty} d(x, c_x(t)) - d(y, c_x(t)),$$

where (c_x) is a geodesic ray at unit speed from x to ξ. A level set of a function $x \to \beta_\xi(x, y)$ is a horocycle.

Denote by o the center of the disk $\mathbb{D}$. If $v \in T^1\mathbb{D}$, $\pi(v)$ is the basepoint of v. The Hopf coordinates are given by the homeomorphism

$$H : v \in T^1\mathbb{D} \mapsto \big(v^-, v^+, \beta_{v^+}(o, \pi(v))\big)\big)\,.$$

In these coordinates, the geodesic flow acts as follows. If $v \simeq (v^-, v^+, s)$, then

$$g_t(v) \simeq (v^-, v^+, s + t)\,.$$

Consider an isometry $\gamma \in PSL(2, \mathbb{R})$. In these coordinates it acts as follows

$$\gamma.v \simeq (\gamma v^-, \gamma v^+, s + \beta_{v^+}(\gamma^{-1} o, o)\,.$$

Exercise 1.3 *Check and prove the above formulas.*

On the unit tangent bundle $T^1\mathbb{D}$ of the disk $\mathbb{D}$, the dynamics is not interesting. Every orbit goes straight, from the horizon at infinity to the horizon at infinity. We will study the geodesic flow on quotients of $T^1\mathbb{D}$.

Consider a topological surface S, and its fundamental group $\pi_1(S)$ (cf lectures F Fanoni). Consider a discrete and faithful representation $\rho : \pi_1(S) \to PSL(2,\mathbb{R})$ of $\pi_1(S)$ as a discrete subgroup of $PSL(2,\mathbb{R})$. This allows to put a structure of Riemann surface on S, and consider S as $\mathbb{D}/\Gamma$, with $\Gamma = \pi_1(S)$.

We will study the geodesic flow on the unit tangent bundle $T^1S \simeq T^1\mathbb{D}/\Gamma$,.

Exercise 1.4 *There is a* $1-1$ *correspondance between Radon measures* m *invariant under the geodesic flow on* T^1S*, Radon measure* $\tilde{m}$ *invariant under the geodesic flow and the group* Γ *on* $T^1\mathbb{D}$*, and* geodesic currents*, i.e. Radon measures* $\mathcal{C}$ *on* $\partial^2\mathbb{D}$ *that are* Γ *invariant.*

1.2 Patterson Sullivan Gibbs construction

Here, we follow the conventions of [PPS15] and not the slightly different ones from [Led95]. The differences will be adressed in the last section of these lectures, if time allows it.

Exercise 1.5 *Show that if* Γ *contains at least two hyperbolic isometries with distinct axes, then there does not exist* Γ*-invariant probability measures on* S^1*.*

We shall construct a family of measures on T^1S by constructing first quasi-invariant measures on S^1, second geodesic currents on $S^1 \times S^1$.

1.2.1 Hölder maps, Poincare series

Consider a Hölder continuous map $f : T^1S \to \mathbb{R}$, and its Γ-invariant lift $\tilde{f}$ on $T^1\mathbb{D}$. If $a, b \in \mathbb{D}$, denote by $\int_a^b f$ the integral of f along the unique geodesic from a to b. More precisely, if $c : [0, d(a,b)] \to \mathbb{D}$ is this geodesic,

$$\int_a^b f := \int_0^{d(a,b)} f(c'(t))dt$$

[1] Define the Poincaré series associated with (Γ, f) as

$$P_{(\Gamma,f)}(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^\gamma \tilde{f}}$$

Set

$$\delta^f = \lim_{t\to\infty} \frac{1}{t} \log \sum_{\gamma\in\Gamma, d(o,\gamma o)\in[t,t+1]} e^{\int_o^{\gamma o} \tilde{f}} .$$

Exercise 1.6 *Show that this series converges for* $s > \delta^f$ *and diverges for* $s < \delta^f$*.*

Define a probability measure ν_f^s on $\mathbb{D} \subset \mathbb{D} \cup \partial\mathbb{D}$ by

$$\nu_s^f = \frac{1}{P_{(\Gamma,f)}(s)} \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^\gamma \tilde{f}} \mathcal{D}_{\gamma o}$$

where $\mathcal{D}_x$ is the Dirac measure at the point x.

By compactness of $\mathbb{D} \cup S^1$, one can find decreasing sequences $s_n \to \delta^f$ such that $\nu_{s_n}^f \to \nu^f$.

[1] If you are very new in the subject, feel free to consider $f \equiv 0$.

Theorem 1.1 (Patterson) *One can modify P slightly without changing δ^f and get that $P_{(\Gamma,f)}$ diverges at $s = \delta^f$.*

Exercise 1.7 *Read Patterson's trick in [Pat76].*

Exercise 1.8 *Deduce that ν^f is supported on S^1*

Exercise 1.9 *Define the* limit set $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o$.

- *Show that it is the smallest Γ-invariant set on S^1.*
- *Show that ν^f gives full measure to Λ_Γ.*

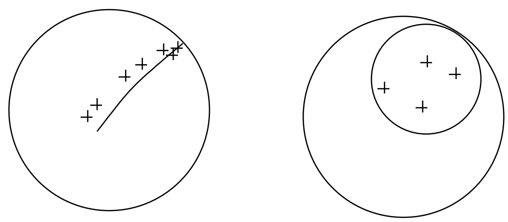

Figure 3: Limit set: a radial limit point, and a horospherical limit point

Exercise 1.10 *Denote by ρ^f and β^f the cocycles on $S^1 \times \mathbb{D} \times D$ defined by*

$$\rho^f_\xi(x,y) = \lim_{t\to\infty} \int_x^\xi \tilde{f} - \int_y^\xi \tilde{f} \quad \textit{and} \quad \beta^f = \delta^f \beta - \rho^f$$

Use the geodesic rays c_x and c_y from x (resp y) to ξ to give a rigorous meaning to the above expression.

Exercise 1.11 *Show that the measure ν^f is Γ quasi invariant and that for a.e. ξ and all $\gamma \in \Gamma$,*

$$\frac{d\gamma_*\nu^f}{d\nu^f}(\xi) = \exp = \exp(-\delta^f \beta_\xi(\gamma o, o) + \rho^f_\xi(\gamma o, o))\,.$$

A key property of the measure is the so-called Shadow lemma. A shadow $\mathcal{O}_x(B(y,R))$ is the set of points $\xi \in S^1$ such that $[x,\xi)$ intersects $B(y,R)$.

Picture

Theorem 1.2 (Sullivan, Hamenstadt, Ledrappier) *There exists R_0 such that for $R \geq R_0$, there exists $C > 0$ such that for every $\gamma \in \Gamma$,*

$$\frac{1}{C}\exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right) \leq \nu^f(\mathcal{O}_o(B(\gamma o, R)) \leq C \exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right)$$

Exercise 1.12 *Prove the above Theorem, by following the steps below.*

1. *Use the conformality of ν^f to get*
$$\nu^f(\mathcal{O}_o(B(\gamma o, R)) = \gamma_*\nu^f(\gamma^{-1}(\mathcal{O}_o(B(\gamma o, R)) = \gamma_*\nu^f(\mathcal{O}_{\gamma^{-1}o}(B(o,R)))\,.$$
2. *Show that on $\mathcal{O}_{\gamma^{-1}o}(B(o,R))$ the Radon Nikodym derivative $d\gamma_*\nu^f/d\nu^f$ is uniformly close to* $\exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right)$.
3. *Check that the measure on the right is bounded from above*
4. *Use the fact that ν^f is not a single Dirac measure to show that there exists some $\alpha > 0$, such that for every $y \in \mathbb{D} \cup S^1$, $\nu^f(\mathcal{O}_y(B(o,R)) \geq \alpha > 0$.*

1.2.2 Product measure

Exercise 1.13 *Show that the measure $\mathcal{C}^f$ on $S^1 \times S^1$ defined by*

$$d\mathcal{C}^f(\xi,\eta) = \exp\left(\beta^f_\eta(o,x) + \beta^f_\xi(o,x)\right) d\nu^f(\xi)\, d\nu^f(\eta),$$

(with $x \in (\xi\eta)$ an arbitrary point= is a geodesic current, i.e. a Γ-invariant measure. We admit that it gives zero measure to the diagonal of $S^1 \times S^1$.

It allows to define a measure $\tilde{m}^f$ on $T^1\mathbb{D}$, as

$$\tilde{m}^f = (H^{-1})_*(\tilde{C}^f \otimes dt)\,.$$

This measure $\tilde{m}^f$ is Γ-invariant and (g^t) invariant.Therefore it induces a measure m^f on $T^1S = T^1\mathbb{D}/\Gamma$, that is a Radon measure, i.e. gives finite mass to compact sets.

Exercise 1.14 *Show that the measure m^f is supported on*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma \subset T^1S$$

Exercise 1.15 *Show that the surface $S = \mathbb{D}/\Gamma$ is convex-cocompact if and only if $H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})$ is cocompact, i.e.*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma$$

is compact.

Hint*: Recall that S is convex-cocompact, by definition, if the convex hull $\mathcal{C}^{core}$ of the limit set in $\mathbb{D}$ is cocompact. First observe that $\Omega \subset T^1\mathcal{C}^{core}/\Gamma$ and deduce that one direction of the equivalence is easy. For the other direction, use the fact that triangles are thin to show that any point of C^{core} is at uniformly bounded distance of a geodesic joining two points of Λ_Γ.*

Remark 1.3 If S is compact or convex cocompact, then m^f is finite.

Theorem 1.4 (Lifshits) *The measures m^f and m^g coincide iff $f = g + cste +$ coboundary, where "coboundary" means a function which is the derivative of another map in the direction of the flow.*

Said in other words, $m^f = m^g$ iff for every periodic orbit $p \in \mathcal{P}$, $\int_p(\delta^f - f) = \int_p(\delta^g - g)$.

Exercise 1.16 *Show the above Theorem, thanks to the transitivity and closing lemma / density of periodic orbits in the set of invariant probability measures.*

Remark 1.5 *If S is compact, T^1S is also compact, and therefore m^f is finite, and implicitely (here) renormalized as a probability measure.*

If S is not compact, there are criteria [**?**], and sufficient conditions [**?**], to ensure that m^f is finite, and examples where it is (or not) the case.

When it is finite, we assume that this measure is **normalized** into a probability measure.

This family of measures have many interesting features.

Exercise 1.17 *Define a dynamical ball as the set*

$$B(v,T,\epsilon) = \{w \in T^1S, \forall 0 \le t \le T, d(g^tv, g^tw) \le \epsilon\}$$

*Show that $B(v,T,\epsilon)$ is comparable (in Hopf coordinates) to $\mathcal{O}_{\pi(v)}(B(\pi(g^Tv),r)) \times \mathcal{O}_{\pi(g^Tv)}(B(v,r)) \times [-\rho,\rho]$ for r,ρ suitable constants. See [PPS15] (***Completer ref***)*

Proposition 1.6 *When finite, the measure m^f satisfies the Gibbs property: for* **every** $v \in T^1S$,

$$m^f(B(v,T,\epsilon)) \asymp \exp\left(-\delta^f T + \int_0^T f(g^t v)\right) dt$$

Exercise 1.18 *Prove the above proposition thanks to the Shadow Lemma and the above exercise.*

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END OF FIRST Lecture
Second lecture :
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RECALL DIAGRAMM $T^1\mathbb{D}, T^1S, S^1 \times S^1 \setminus$ **diag** $\times \mathbb{R}$, S^1, **measures** m^f, $\tilde{m}^f$, $\mathcal{C}^f$, ν^f, **invariances**

Definition 1.7 *The* local entropy *is "defined" as*

$$\overline{h}_{loc}(m) = \mathrm{supess}_{v\in T^1S} \limsup_{T\to\infty} -\frac{1}{T}\log m(B(v,T,\epsilon)) \quad or \quad \underline{h}_{loc}(m) = \mathrm{infess}_{v\in T^1S} \liminf_{T\to\infty} -\frac{1}{T}\log m(B(v,T,\epsilon))$$

Corollary 1.8 *In particular, we have*

$$h(m^f) = \delta^f - \int f\, dm^f$$

Exercise 1.19 *Check that the corollary is true.*

Theorem 1.9 (Hopf, Sullivan, Hamenstadt, Ledrappier, Babillot, Otal-Peigné, PPS...) *If m^f is finite, then it satisfies the following properties:*

- *it is ergodic (Hopf argument),*
- *it is mixing, (Babillot)*
- *when S is compact, it is exponentially mixing (Dolgopyat)([2])*
- *It satisfies the Gibbs property: for every $v \in T^1S$, denote by $B(v,T,\epsilon) = \{w \in T^1S, \forall 0 \le t \le T, d(g^t v, g^t w) \le \epsilon)\}$. Then for* **every** $v \in T^1S$,

$$m^f(B(v,T,\epsilon)) \asymp \exp\left(-\delta^f T + \int_0^T f(g^t v)\right) dt$$

- *Its entropy, "defined" as the a.s. limit of $-\frac{1}{T}\log m^f(B(v,T,\epsilon)$ when $T\to\infty$, satisfies $h(m^f) = \delta^f - \int f\, dm^f$.*
- *It is the unique measure maximizing the pressure, i.e. realizing the following supremum:*

$$\delta^f = \sup_{m\in\mathcal{M}^1}\left(h(m) + \int f\, dm\right)$$

- *When $f = 0$ we recover the so-called Bowen-Margulis-Sullivan measure.*
- *Weighted equidistribution of periodic orbits, counting (PPS, Schapira-Tapie) : If $\delta^f > 0$, then*

$$\sum_{p\in\mathcal{P}_K(T)} e^{\int_p f} \sim \frac{e^{\delta^f T}}{\delta^f T}.$$

[2] It is the only place in the statement, and in these lectures where being on a compact hyperbolic surface, and not an arbitrary nonnecessarily compact negatively curved manifold is important.

1.2.3 Interesting examples

Consider here a manifold M with variable negative curvature instead of a hyperbolic surface. The above construction works exactly in the same way.

- When $f \equiv 0$, one obtains the so-called BMS measure, measure that maximizes entropy as soon as it is finite.
- When S has finite volume and $f = \frac{d}{dt} J(g^t)_{|E^{su}}$ (in variable curvature) one recovers the Liouville/Lebesgue measure.
- For a suitable potential, one gets the harmonic measure.

Sullivan conjecture These measures, well defined on any compact negatively curved manifold, coincide iff the manifold is locally symmetric. Proven by Katok and Ledrappier in dimension 2.

Other examples : you want to create a bump or a hole. You consider a potential f that equals 0 everywhere except at the place where you want to create the bump/hole, where you can choose it positively/negatively large. Then the associated Gibbs measure will see mainly the trajectories that go through the bump/hole very often/rarely, but they will still have full support, with good ergodic properties.

2 Patterson-Sullivan Gibbs construction twisted by a representation and amenanility of covers

Topic added after listening to the first lecture of Roman Sauer on amenability and (T).

For $f \equiv 0$ we set δ_Γ instead of δ^0 for the associated critical exponent.

Goal prove the following theorem.

Theorem 2.1 (Cohen, Brooks,..., Stadlbauer, Coulon-dal'bo-Sambusetti, Coulon-Dougall-schapira-Ta *Let $\Gamma < \Gamma_0$ be a subgroup of a discrete group of isometries of X proper hyperbolic space. Assume the action Γ_O on X is SPR (for example cocompact but much more general).*

Then $\delta_\Gamma = \delta_{\Gamma_0}$ iff Γ_0/Γ is amenable.

2.1 Strategy of the proof

We will give details here on the implication $\delta_\Gamma = \delta_{\Gamma_0}$ implies Γ_0/Γ amenable.

(For the other implication, we use Kesten criterion. We construct measures on spheres of increasing radius of Γ_0/Γ, and compute the spectral radius of random walks associated with the uniform measure on these spheres, in terms of δ_Γ and δ_{Γ_0}. Kesten criterion implies that the spectral radius is 1 and therefore, that $\delta_\Gamma = \delta_{\Gamma_0}$.)

For the direct implication, consider the regular representation $\rho : \Gamma_0 \to \mathcal{U}(\ell^2(\Gamma_0/\Gamma))$. We want to show that this representation almost has invariant vectors.

1. (Twisted Poincaré series) Consider $A(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}\rho(\gamma)$. It is a countable (weighted) sum of unitary representations. Therefore, it is maybe a bounded operator, but not unitary a priori. Define the critical exponent δ_ρ as the infimum $\delta_\rho = \inf\{s \in \mathbb{R}, \|A(s)\| < \infty\}$. Check that $\delta_\Gamma \le \delta_\rho \le \delta_{\Gamma_0}$.

2. (Twisted PS-Measure on the boundary) Define $a(s) = \frac{1}{\|A(s)\|}\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}\rho(\gamma)\,\mathcal{D}_{\gamma o}$. It is a measure on $\mathbb{D} \cup \partial\mathbb{D}$, with values in the set of bounded representations. By taking subsequences, consider a measure a^ρ on $\partial\mathbb{D}$, with values bounded operators.

3. (Absolute continuity) Using $\delta_\Gamma = \delta_{\Gamma_0}$ and therefore $\delta_\rho = \delta_{\Gamma_0}$, show that $a^\rho << \nu$, where ν is the classical PS measure (for $f \equiv 0$)

4. (Ergodicity) By ergodicity show that $a^\rho = D\nu$, where D is the Radon-Nikodym derivative, and is a.s. constant.

5. (Conclusion) The quasi-invariance properties of a^ρ and ν show that the almost sure value of D produces the desired 'almost invariant vectors".

2.2 details on each step

See the beamer of my talk here
`https://imag.umontpellier.fr/ schapira/recherche/beamer-Cetraro.pdf`

2.2.1 The twisted Poincaré series

Study $A(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}\rho(\gamma)$. The Hilbert space $\mathcal{H} = \ell^2(\Gamma_0/\Gamma)$ admits a partial order : $\Phi \ge 0$ iff for every $y \in \Gamma_0/\Gamma$, $\Phi(y) \ge 0$. The partial order allows to define a positive cone $\mathcal{H}^+ = \{\Phi \in \mathcal{H}, \Phi \ge 0\}$.

Say that $A(s)$ is bounded if there exists $M < \infty$ st for every finte set $S \subset \Gamma$, $\|\sum_{\gamma\in S} e^{-sd(o,\gamma o)}\rho(\gamma)\| \le M$.

Thanks to positivity arguments, there exists δ_ρ st $A(s)$ bounded for every $s > \delta_\rho$.
Easy to show that $\delta_\Gamma \leq \delta_\rho \leq \delta_{\Gamma_0}$.
The assumption $\delta_\Gamma = \delta_{\Gamma_0}$ is only - but crucially - used to show that $\delta_\rho = \delta_{\Gamma_0}$

2.2.2 The twisted PS measure

See the beamer of my talk here
`https://imag.umontpellier.fr/ schapira/recherche/beamer-Cetraro.pdf`

2.2.3 Absolute continuity

See the beamer of my talk here
`https://imag.umontpellier.fr/ schapira/recherche/beamer-Cetraro.pdf`

See the beamer of my talk here
`https://imag.umontpellier.fr/ schapira/recherche/beamer-Cetraro.pdf`

2.2.4 Ergodicity

See the beamer of my talk here
`https://imag.umontpellier.fr/ schapira/recherche/beamer-Cetraro.pdf`

2.2.5 Conclusion

See the beamer of my talk here
`https://imag.umontpellier.fr/ schapira/recherche/beamer-Cetraro.pdf`

3 Regularity of entropy under perturbations

Assume in this section that S is compact, and see [?] for the noncompact case.

3.1 Boundaries

Consider a same topological surface with two (hyperbolic) metrics (S, g_1) and (S, g_2)

PICTURE

Consider the universal cover as the same differentiable manifold with two distinct metrics $(\tilde{S}, g_1)$ and $(\tilde{S}, g_2)$.

The boundaries at infinity are a priori distinct, but when S is compact for example, or when the two metrics are equivalent, then (Morse) the g_1-geodesics are at bounded distance of a g_2 geodesic, so that we can identify the boundaries $\partial_{g_1}\tilde{S} \simeq \partial_{g_2}\tilde{S}$.

3.2 Identifying the sets of invariant measures

Denote by $\mathcal{M}^1_{g_i}$ the set of invariant probability measures on $T^1_{g_i}S$, and by $Curr(\Gamma)$ the set of projective geodesic currents on $\partial^2\tilde{S}$ (i.e. geodesic currents up to normalization).

The Hopf correspondance $H^{g_i} : T^1_{g_i}\tilde{S} \to \partial^2\tilde{S} \times \mathbb{R}$ leads to a bijection between $\mathcal{M}^1_{g_i}$ and $Curr(\Gamma)$, and therefore a bijection between $\mathcal{M}^1_{g_1}$ and $\mathcal{M}^1_{g_2}$. (It is necessary to take care of the normalization)

DIAGRAMME

Given a geodesic current $\mathcal{C}$, denote by $m^{g_i}_{\mathcal{C}}$ the normalized invariant probability measure associated with $\mathcal{C}$ on $T^1_{g_i}S$.

Exercise 3.1 *Show the following properties*

- $m^{g_1}_{\mathcal{C}}$ *has full support iff* $m^{g_2}_{\mathcal{C}}$ *has full support.*
- $m^{g_1}_{\mathcal{C}}$ *is ergodic iff* $m^{g_2}_{\mathcal{C}}$ *is ergodic.*
- $m^{g_1}_{\mathcal{C}}$ *is supported on a periodic orbit iff* $m^{g_2}_{\mathcal{C}}$ *is supported on a periodic orbit.*
- $m^{g_1}_{\mathcal{C}}$ *is a quasi product measure (i.e.* $\mathcal{C}$ *is equivalent to a product measure) iff* $m^{g_2}_{\mathcal{C}}$ *is a quasi product measure.*

I will give the flavour of the proof of

Theorem 3.1 (Schapira-Tapie) *Assume that* S *is a compact surface (manifold). (Compactness not necessary).*

- $m^{g_1}_{\mathcal{C}}$ *is a Gibbs measure wrt the potential* f_1 *iff* $m^{g_2}_{\mathcal{C}}$ *is a Gibbs measure wrt the potential* f_2 *(details below).*
- $m^{g_1}_{\mathcal{C}}$ *has positive entropy iff* $m^{g_2}_{\mathcal{C}}$ *has positive entropy.*

3.3 Geodesic stretch

Consider (S, g_1) and (S, g_2) two hyperbolic (neg. curved) metrics on the same differentiable surface (manifold), consider $\tilde{S}$ the universal cover as a differentiable surface.

As said above, one can identify the boundaries, denoted by $\partial\tilde{S}$.

Given a vector $v \in T^1_{g_1}\tilde{S}$, we can consider the g_1 geodesic $(g_1^t v)_{t\geq 0}$, the g_1-endpoint $v^{g_1}_+ \in \partial\tilde{S}$, and the g_2-Busemann cocycle $\beta^{g_2}_{v^{g_1}_+}(\pi(v), \pi(g_1^t v))$.

PICTURE

Proposition 3.2 (Ledrappier) *The map* $t \mapsto \beta^{g_2}_{v^{g_1}_+}(\pi(v), \pi(g_1^t v))$ *is differentiable at* $t = 0$.

We introduce ([?]) the *infinitesimal geodesic stretch* as

$$\mathcal{E}^{g_1\to g_2}(v) = \frac{d}{dt}_{t=0}\beta^{g_2}_{v^{g_1}_+}(\pi(v), \pi(g_1^t v))$$

Exercise 3.2 *Use the fact that the* g_1 *and* g_2*- geodesics from* $\pi(v)$ *to* $v^{g_1}_+$ *are at bounded distance one another to prove that the ergodic average of the geodesic stretch satisfies*

$$\frac{1}{T}\int_0^T \mathcal{E}^{g_1\to g_2}(g_1^t v)dt \asymp \frac{d^{g_2}(\pi(v), \pi(g_1^T v))}{d^{g_1}(\pi(v), \pi(g_1^T v))}$$

3.4 Morse correspondance

Hopf coordinates give a homeomorphism

$$T^1_{g_1}\tilde{S} \longleftrightarrow \partial^2\tilde{S} \times \mathbb{R} \longleftrightarrow T^1_{g_2}\tilde{S}$$

However, the homeomorphism $\Phi^{g_1\to g_2} = (H^{g_2})^{-1} \circ H^{g_1}$ is **NOT** Γ-invariant, and therefore does **NOT** induce a map from $T^1_{g_1}S$ to $T^1_{g_2}S$.

That is a good news in some sense, because if it were true, it would mean that the geodesic flows (g_1^t) and (g_2^t) are conjugated, which is false in general.

Indeed, by construction, we have the following :

Exercise 3.3 *Check from the definition that the Hopf coordinates, and therefore the homeomorphism $\Phi^{g_1\to g_2}$ commute with the geodesic flow :*

$$\Phi^{g_1\to g_2} \circ g_1^t = g_2^t \circ \Phi^{g_1\to g_2}$$

The point is that the two Γ-actions

$$\gamma.v \simeq (\gamma v^-, \gamma v^+, s + \beta^{\mathbf{g_i}}_{v^+}(\gamma^{-1}o, o)$$

on $\partial^2\tilde{S} \times \mathbb{R}$ differ on the third (real) coordinate.

The good news is that one can build an orbit equivalence, i.e. a homeomorphism from $T^1_{g_1}S$ to $T^1_{g_2}S$, that sends orbits to orbits, by modifying slightly $\Phi^{g_1\to g_2}$. There are several ways to do that. For example, define $\tilde{\Psi}^{g_1\to g_2}$ as the map from $T^1_{g_1}\tilde{S}$ to $T^1_{g_2}\tilde{S}$ that associates to v the unique vector $\tilde{\Psi}^{g_1\to g_2}(v) \in T^1_{g_2}\tilde{S}$ on the g_2 geodesic from $v_-^{g_1}$ to $v_+^{g_1}$ that satisfies $\beta^{g_2}_{v_+^{g_1}}(\pi(v), \pi(w)) = 0$.

Exercise 3.4 *Show that $\tilde{\Psi}^{g_1\to g_2}$ is Γ-equivariant, and induces therefore an orbit equivalence from $T^1_{g_1}S$ to $T^1_{g_2}S$.*

Exercise 3.5 *Let $G : T^1_{g_2}S \to \mathbb{R}$ be a continuous map, and $m^{g_i}_{\mathcal{C}}$ be two invariant probability measures on $T^1_{g_i}S$ associated with the same geodesic current $\mathcal{C}$ on $\partial^2\tilde{S}$. Show that*

$$\int_{T^1_{g_2}S} G(w)\, dm^{g_2}_{\mathcal{C}}(w) = \int_{T^1_{g_1}S} G \circ \Psi^{g_1\to g_2}(v) \times \mathcal{E}^{g_1\to g_2}(v)\, dm^{g_1}_{\mathcal{C}}(v)\,.$$

If you succeeded to prove that, you forgot the normalization. Indeed, $m^{g_i}_{\mathcal{C}}$ are the normalized probability measures on $T^1_{g_i}S$ associated through quotient by Γ and Hopf coordinates H^{g_i} to the measure $\mathcal{C} \otimes dt$. Show that the correct formula is

$$\int_{T^1_{g_2}S} G(w)\, dm^{g_2}_{\mathcal{C}}(w) = \frac{\int_{T^1_{g_1}S} G \circ \Psi^{g_1\to g_2}(v) \times \mathcal{E}^{g_1\to g_2}(v)\, dm^{g_1}_{\mathcal{C}}(v)}{\int_{T^1_{g_1}S} \mathcal{E}^{g_1\to g_2}(v)\, dm^{g_1}_{\mathcal{C}}(v)}$$

3.5 Shadows are shadows

Let (S, g_1) and (S, g_2) be two hyperbolic metrics on the same surface. Let $C > 0$ be a constant such that any g_1 geodesic of $\tilde{S}$ is at distance at most C from a g_2 geodesic with the same endpoints and conversely.

Exercise 3.6 *Show that*

$$\mathcal{O}^{g_1}_x(B^{g_1}(y, r)) \subset \mathcal{O}^{g_2}_x(B^{g_2}(y, r + C))\,.$$

As a corollary we get

Exercise 3.7 *Prove that* $m_{\mathcal{C}}^{g_2}$ *is a Gibbs measure wrt the potential* G *iff* $m_{\mathcal{C}}^{g_1}$ *is a Gibbs measure wrt the potential* $G \circ \mathcal{E}^{g_1 \to g_2}$.

Exercise 3.8 *Prove that*

$$h(m_{\mathcal{C}}^{g_2}) = \frac{1}{\int_{T^1_{g_1} S} \mathcal{E}^{g_1 \to g_2} dm_{\mathcal{C}}^{g_1}} \times h(m_{\mathcal{C}}^{g_1})$$

Using the above, we proved in [**?**] a generalization in a noncompact setting of a result of [**?**]:

Theorem 3.3 (Katok-Knieper-Weiss, Schapira-Tapie) *Let* (g_ϵ) *be a* C^1*-uniform variation of the hyperbolic metric* g_0 *on* S *(compact or SPR). The entropy* $h_{top}(g_\epsilon)$ *is* C^1 *at* 0. *More precisely*

$$\frac{d}{dt}_{|t=0} h(g_\epsilon) = -h(g_0) \int \frac{d\|v\|^{g_\epsilon}}{d\epsilon} \, dm^{g^0}_{BMS}$$

4 Invariant measures under the horocyclic flow on abelian covers, after Babillot-Ledrappier [?]

4.1 Horocyclic flow, invariant measures

The horocyclic flow is defined as follows : **Picture**

Identify $T^1\mathbb{D}$ with $T^1\mathbb{H}$ through the bihol map $z \in \mathbb{H} \mapsto \frac{z-i}{z+i} \in \mathbb{D}$, and $T^1\mathbb{H}$ with $PSL(2, \mathbb{R})$, through the classical map

$$g \in PSL(2, \mathbb{R}) \mapsto g.(i, i)$$

where (i, i) denotes the vertical vector tangent to i in the upper direction.

Exercise 4.1 *In this identification, check that the action of the geodesic flow* (g^t) *on* $T^1\mathbb{D}$ *or* $T^1\mathbb{H}$ *corresponds to the right multiplication by diagonal matrices* $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

Definition 4.1 *In this identification, the stable horocycle flow* $(h^s)_{s\in\mathbb{R}}$ *is defined by the right multiplication by unipotent matrices* $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

PICTURE

Exercise 4.2 *Check that on* $T^1\mathbb{D}$, *the orbits of the horocyclic flow are the stable manifolds of the geodesic flow.*

$$g^t \circ h^s = h^{se^{-t}} \circ g^t$$

Picture

As consequences of this deep relation, properties of (g^t) and (h^s) are intricated. It is classical (and not the topic of these lectures) that one has the following implications (on a compact hyperbolic surface).

- Product structure $\longrightarrow$ ergodicity of the geodesic flow wrt any quasi product measure.

- Ergodicity of the geod flow wrt the Liouville (BMS) measure $\longrightarrow$ ergodicity of the horocyclic flow wrt to the Liouville (BR) measure.
- Ergodicity of the horocyclic flow $\longrightarrow$ mixing of the geodesic flow.
- Mixing of the geodesic flow $\longrightarrow$ unique ergodicity of the horocyclic flow.

The ergodic behaviour of the horocyclic flow reflects the different possible asymptotic behaviours of the geodesic flow.

Theorem 4.2 (Hedlund 30',Furstenberg, Dani, Dani-Smillie 70's) *When S is compact, the horocyclic flow is minimal and uniquely ergodic. All orbits are equidistributed towards the Liouville measure: for all $\phi : T^1S \to \mathbb{R}$ and every $v \in T^1S$, we have*

$$\frac{1}{T}\int_0^T \phi \circ h^s(v)ds \to \int \phi d\mathcal{L}$$

When S has finite volume, all orbits are periodic or dense, and the Liouville measure is the unique nonperiodic ergodic measure, all nonperiodic orbits are equidistributed toward the Liouville measure.

Remark 4.3 When S is geometrically finite, similar results, see Burger, Roblin, Schapira

4.2 Abelian covers

Abelian covers are another very interesting situation where the ergodic components of the horocyclic flow are completely understood. Babillot-Ledrappier exhibited an infinite family of ergodic invariant measures. Sarig proved that there are no other ergodic invariant measures, and Sarig-Schapira proved an associated equidistribution result. [**?**] (**Completer ref**)

The case of nilpotent covers is partially understood (Babillot), see also Bispo-Stadlbauer or Ofer Schwartz for general covers.

The framework is the following.

$S_0 = \mathbb{D}/\Gamma_0$ compact surface. $S = \mathbb{D}/\Gamma$ (abelian) cover. i.e. $\Gamma \lhd \Gamma_0$ and $\Gamma_0/\Gamma = G \simeq \mathbb{Z}^d$.

Theorem 4.4 (Babillot 04) *$S \to S_0$ nilpotent cover of a compact hyperbolic surface, $G = \Gamma_0/\Gamma$ nilpotent. There is a $1-1$ correspondance between*

1. *Characters of G*
2. *Cohomology classes of S_0 that vanish on Γ*
3. *Γ-conformal ergodic measures ν on S^1 (up to multiplicative constants)*
4. *Radon measures M that are Γ-invariant and ergodic on $S^1 \times \mathbb{R}$ and quasi invariant under (g^t) ($dM = d\nu e^{\delta t}dt$*
5. *(h^s)-invariant ergodic measures on T^1S that are quasi invariant under (g^t) (of the form $ds(v^-)d\nu(v^+)e^{\delta t}dt$*

Theorem 4.5 (Ledrappier-Sarig 04, Sarig 00) *$S \to S_0$ Regular cover. All (h^s)-invariant measures ergodic are also quasi invariant under the geodesic flow*

Theorem 4.6 (Babillot-Ledrappier, 98) *$S \to S_0$, $G = \Gamma_0/\Gamma \simeq \mathbb{Z}^d$. For every $\vec{V} \in \mathbb{R}^d$ there exists a unique (up to normalization) (h^s) invariant ergodic measure $\lambda_{\vec{V}}$ on T^1S such that for every $a \in \mathbb{Z}^d$,*

$$a_* \lambda_{\vec{V}} = e^{<v|a>} \lambda_{\vec{V}}$$

It is an effective (older) version of the above Theorem. The measures $\lambda_{\vec{V}}$ are given by the thermodynamical formalism. The same should be true in the case of nilpotent covers. Not clear why it does not work.

Remark 4.7 The characters of G : same as the caracters of $G/[G,G]$. So it is too small when $G \simeq \mathbb{F}_2$ for example, or when $[G,G]$ is very close to G, i.e. G very far to be abelian. (

Our Goal : understand the construction of the measures in the case of abelian covers. More precisely, understand the objects appearing in the Theorem below :

Theorem 4.8 (Babillot-Ledrappier 98) *There is a $1-1$-correspondance between*

- *Characters, i.e. morphisms $\chi : G \to \mathbb{R}_+^*$*
- *de Rahm cohomology classes of 1-forms ω that vanish on loops of Γ*
- *ergodic (δ, Γ)-conformal probability measures on S^1, i.e. that satisfy*

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = \exp(\delta \beta_\xi(o, \gamma o)) \,.$$

- *(h^s)-Invariant ergodic measures on T^1S (up to multiplicative constants*
- *extremal eigenfunctions of Δ_S for nonnegative eigenvalues, where Δ_S is the Laplace operator of S*

4.2.1 Characters and cohomology classes

First a notation. Fix D_0 a fundamental domain for G action on T^1S. You can think of T^1S as $T^1D_0 \times G$. Given such an identification, denote by $|w] \in \mathbb{Z}^d$ the element $g \in G$ such that $w \in gD_0$. Coordinate in $\mathbb{Z}^d$.

Given $w \in T^1S_0$ and $\hat{w} \in T^1S$ a lift, the following limit exists m a.s., for any m invariant proba measure on T^1S_0 :

$$e(w) = \lim \frac{1}{t}[g^t(\hat{w})]_{\mathbb{Z}^d} \in \mathbb{R}^d$$

It is the *drift* of $\hat{w} \in \mathbb{Z}^d$.

Proposition 4.9 *For every $\chi : G \to \mathbb{R}_+^*$ character , there exists $\vec{V} \in \mathbb{R}^d$ such that for every $g \in G(= \mathbb{Z}^d)$,*

$$\chi(g) = e^{<\vec{V}, g>}$$

A character $\xi : G \to \mathbb{R}_+^* \leftrightarrow$ a morphism $\Gamma_0 \to \mathbb{R}_+^*$ that vanishes on Γ
As $\mathbb{R}_+^*$ is abelian, it is equivalent to have a linear form $H_1(S_0) \to \mathbb{R}$ that vanishes on homology classes of elements of Γ.

By Poincaré duality, it is the same as having a de Rahm colhomology class of a 1- form $\alpha \in H^1_{dR}(S_0)$ that vanishes on loops of Γ. Given α such a 1-form, let χ_α

the associated character, and $\vec{V}_\alpha$ the associated vector in $\mathbb{R}^d$. As a consequence, for every $\gamma \in \Gamma_0$, one has

$$\chi(g) = \exp(\int_{[\gamma]\alpha} = \exp(< \vec{V}_\alpha, [g^t w]_{\mathbb{Z}^d} >)$$

Exercise 4.3 *If $\hat{w} \in T^1 D_0$, then*

$$\log \chi([g^t w]_{\mathbb{Z}^d}) - \int_0^t \alpha(g^s w)\, ds$$

is bounded independently of w and t.

in particular, when the limits exist, we get :

$$\lim \frac{1}{t}\int_0^t \alpha(g^s w) ds = \lim_{t\to\infty} \frac{1}{t} \log \chi_\alpha([g^t w]_{\mathbb{Z}^d}) = \lim \frac{1}{t} < \vec{V}_\alpha, [g^t w]_{\mathbb{Z}^d} > = < \vec{V}_\alpha, e(w) >$$

4.2.2 Babillot-Ledrappier measures

If α is a closed 1-form, i.e. a section $\alpha : x \in S \to \alpha_x \in (T_x S)^*$, then it induces a smooth potential $f_\alpha : v \in T^1 M \to \alpha_{\pi(v)}(v)$.

We can do the Patterson Sullivan Gibbs construction of the measure m_α associated with (f_α or) α.

The measure ν_α is quasi invariant in the sense that for all $\gamma \in \Gamma_0$, we have

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{-\delta_\alpha \beta_\xi(o,\gamma o) + \alpha([\gamma])}$$

As α vanishes on Γ, it follows that ν_α is (δ_α, Γ) conformal.

Therefore, the measure $ds(v^-) d\nu_\alpha(v^+) e^{\delta_\alpha t} dt$ is (h^s)-invariant and Γ-invariant and descends to a (h^s) inv measure on $T^1 S$.

On the other hand, the measure $\mathcal{C}_\alpha \otimes dt$ induces a (g^t) invariant measure m_α on $T^1 S_0$, that is the Gibbs measure associated with α. It realizes the supremum

$$P(\alpha) = P(\vec{V}_\alpha) = \sup_{m \in \mathcal{M}^1(T^1 S_0)} \left(h(m) + \int \alpha dm \right) = \sup_m h(m) + < \vec{V}_\alpha, \int_{T^1 S_0} e(w) dm >$$

It can be rewritten

$$P(\vec{V}) = \sup h(m) + < \vec{V}, e(m) >$$

where $e(m)$ is the average drift wrt to m.

We get

Proposition 4.10 (Babillot Ledrappier) *$P : \mathbb{R}^d \to \mathbb{R}$ is analytic and $\nabla P : \mathbb{R}^d \to \mathbb{R}^d$ realizes a diffeomorphism $\mathbb{R}^d \to \int C$ where $C = \{e(w), w \in T^1 D_0\} \subset \mathbb{R}^d$.*

Now, the ergodicity of λ_α comes from the mixing property of m_α, and more precisely of

Theorem 4.11 (BL98) *m_α a.s.*

$$\int_{-1}^1 \phi \circ g^{-T} \circ h^s w ds \sim c e^{T h(m_\alpha)} T^{-d/2} \int \phi d\lambda_\alpha$$

4.2.3 From (δ, Γ)-conformal measures to (h^s)-invariant measures

(h^s)-invariant measures on T^1S are in $1-1$ correspondance with (h^s)- invariant and Γ invariant measures on $T^1\tilde{S}$

Observe that in the Hopf coordinates, a (h^s) orbit is a set $S^1 \setminus \{v^+\} \times \{v^+\} \times \{\tau(v)\} \simeq \mathbb{R} \times \{v^+\} \times \{\tau(v)\}$. If the first real factor is parametrized accordingly to the (h^s) action, a (h^s-invariant measure is of the form $ds \times dM$, where M is a Γ-invariant measure on $S^1 \times \mathbb{R}$. Therefore, (h^s)-invariant measures are in $1-1$ correspondance with Γ-invariant measures on $S^1 \times \mathbb{R}$. Here, the Γ action on $S^1 \times \mathbb{R}$ is the natural action induced by the action on $\partial^2\tilde{S} \times \mathbb{R}$:

$$\gamma.(v^-, v^+, \tau) = (\gamma v^-, \gamma v^+, \tau + \beta_{v^+}(\gamma^{-1}o, o)$$

so that

$$\gamma.(\xi, \tau) = (\gamma\xi, \tau + \beta_\xi(\gamma^{-1}o, o)$$

Exercise 4.4 *If ν is a (δ, Γ) conformal measure, observe that $dM(\xi, t) := d\nu(\xi)e^{\delta t}dt$ is Γ-invariant on $S^1 \times \mathbb{R}$.*

Moreover, show that if M is ergodic, then ν is ergodic. The converse is not obvious a priori.

Check that M induces a (h^s)-invariant measure defined locally as $ds(v^-)dM(v^+, t)$.

5 Positivity and renormalization

Ledrappier, normalization

$s_0 = \limsup \frac{1}{n} \log \#\{\gamma \in \Gamma, \int_\gamma f \in [n, n+1[\}$.

Exercise 5.1 *Assume δ^f is finite. Show that $\delta(-s_0 f) = 0$.*

hint : you can consider separately those $\gamma \in \Gamma$, such that $\int_\gamma f \leq CT$ and the $\gamma \in \Gamma$, such that $\int_\gamma f \geq CT$.

Theorem 5.1 (Gouezel, Schapira, Tapie) *As soon as they are finite, Gibbs measures have positive entropy.*

6 Gibbs measures and higher Teichmuller theory

6.1 Ledrappier's Theorem

Potentials / cocycles / measures / ...

Theorem 6.1 (Ledrappier [Led95]) *Let S be a compact negagatively curved surface (manifold). There is a $1-1$ correspondance between*

- *(Cohomology classes) of Hölder potentials on T^1S*
- *(Cohomology classes of) Hölder cocycles on $\Gamma \times \partial\tilde{S}$*
- *(Cohomology classes of) quasi invariant measures on $\partial\tilde{S}$ (**completer formule**)*
- *... (**voir ce dont je me sers**)*

6.2 Representations and cocycles

S compact hyperbolic surface. $\rho : \Gamma = \pi_1(S) \to SL(n, \mathbb{R})$ Anosov representation.

Iwasawa-Busemann cocycle

Real-valued cocycle

Ledrappier Theorem $\to$ Hölder continuous potential

Periods of the cocycle are positive $\to$ reparametrization of the flow

New Anosov flow, with unique measure of maximal entropy, equidistribution of periodic orbits, ...

Examples of consequences (Sambarino and coauthors)

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