Measure of maximal entropy for H-flows on non-compact manifolds

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Abstract

In this work, we introduce a natural class of chaotic flows on non-compact manifolds, called *H-flows*, which includes geodesic flows on non-compact manifolds with pinched negative curvature. We show that, under the additional assumption, called *strong positive recurrence*, that their entropy at infinity is strictly smaller than the topological entropy, such flows admit an invariant probability measure maximizing entropy. In particular, we compare several notions of entropy in a non-compact setting.

Keywords— Hyperbolic dynamics, entropy, measure of maximal entropy, periodic orbits.

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Contents

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1 Introduction

1.1 Anosov flows on compact manifolds

Anosov flows are the archetype of chaotic dynamics. Defined by Anosov in [Ano69], an Anosov flow is a differentiable flow on a compact manifold such that the tangent bundle of the manifold splits into the direct sum of three invariant subbundles, the (one-dimensional) direction of the flow, the stable bundle, uniformly contracted in the future by the differential of the flow, and the unstable bundle, uniformly contracted in the past. Anosov flows exhibit many remarkable properties. The *shadowing property*, also known as the pseudo-orbit tracing property, tells us that any path not too far from an orbit is very well approximated by a true trajectory. These flows are *expansive*: two points on different orbits separate one from another at some time, in the future or in the past. They also satisfy the *closing lemma*: near any almost closed orbit, one can find a periodic one. As a consequence, there are infinitely many periodic orbits and they are dense in the non-wandering set of the flow. This is a manifestation of chaotic behaviour.

From the ergodic point of view, an Anosov flow admits infinitely many (ergodic) invariant probability measures, whose typical points describe infinitely many different typical behaviours. It is another manifestation of chaotic behaviour.

Anosov flows have *positive topological entropy*. Topological entropy is the exponential growth rate, when the time T goes to infinity, of the maximal number of different "behaviours" of orbits of period T. Therefore, positive entropy is another typical property of very chaotic dynamics.

The closing lemma allows to show that for Anosov flows, this topological entropy coincides with the *Gurevic entropy*, i.e., the exponential growth rate when $T \to +\infty$, of the number of periodic orbits with period at most T. This does not hold in general: there exist examples of minimal flows (so, with no periodic orbits) of positive topological entropy, see [Ree81, BCLR07].

The *variational principle* (see [Wal82, §8.2]) states that for any continuous dynamical system on a compact metric space, the topological entropy equals the *variational entropy*, that is the supremum of the measure-theoretic entropies of all invariant (ergodic) probability measures. When it exists, a measure that realizes the supremum, i.e., whose entropy equals the topological entropy, is called a *measure of maximal entropy.* Typical orbits of such a measure reflect the most chaotic behaviour of the dynamics.

For transitive Anosov flows (on a compact manifold), in [Bow72] Bowen showed that there exists a measure of maximal entropy, obtained as a limit of measures equidistributed on longer and longer periodic orbits. Bowen-Ruelle [BR75] obtain the uniqueness (in the more general context of equilibrium states) for transitive Axiom A flows. Bowen's proof crucially relies on the *specification property* of Anosov flows: for it, the compactness of the manifold is essential. Margulis [Mar69, Mar04] provided an alternative construction, using equidistribution of larger and larger pieces of unstable manifolds, where compactness is also used to guarantee the convergence of the construction. Sullivan [Sul79] proposed a geometric construction that holds only in the particular case of geodesic flows in negative curvature. His construction is very robust and extends to noncompact manifolds, see for example [OP04, PPS15, ST21], but only for geodesic flows.

1.2 Dynamics on non-compact manifolds, motivations

In the non-compact setting, the picture is less clear, for many reasons. Let us highlight some of the main difficulties and questions.

- The notion of Anosov flow does not generalize easily to the non-compact case. The definition and several resulting properties, closing lemma in particular, depend strongly on the compactness of the manifold.
- Under what kind of assumptions do topological entropy and Gurevic entropy coincide?
- There exist many different, useful notions of measure-theoretic entropy, such as Kolmogorov-Sinai entropy, Katok entropy, Brin-Katok upper and lower local entropies. Do they coincide in general?
- Thanks to [HK95a], the variational principle still holds for a non-compact manifold, and so we can still talk about measures of maximal entropy. Nevertheless, by lack of compactness, there is no easy argument to ensure neither the existence nor the uniqueness of an invariant probability measure maximizing entropy.

Although these questions do not have clear answers in general, there are two very important classes of (non-compact) dynamical systems for which some results exist: suspension flows over a symbolic dynamics with countable alphabet on the one hand, and geodesic flows on the unit tangent bundle of (non-compact) negatively curved manifolds on the other hand. In both cases, different relevant notions of entropy coincide. See for example [OP04, PPS15, ST21, GST23] for the geodesic flow, and [VJ67, Sar99, MU01, Sar03, BBG06, CS09, BBG14] for subshifts and suspension flows over infinite alphabets.

In the case of geodesic flows, under an assumption called *strong positive recurrence*, see [ST21, GST23], one can prove the existence of a (unique) invariant probability measure that maximizes entropy. This notion appeared in different ways in different contexts: for Markov shifts [VJ67, Sar03], for geodesic flows [ST21, GST23], for diffeomorphisms on closed manifolds [BCS25], in geometric group theory [ACT15, Yan19], for Hénon-like diffeomorphisms [Ber19], ...

Strong positive recurrence means that the exponential growth of the complexity of trajectories that spend most of the time close to infinity (or to any other bad zone, in different contexts), also called *entropy at infinity*, is strictly smaller than the global exponential growth of the dynamics, i.e., the *entropy.* In [GST23], the authors introduce different notions of entropy at infinity, in terms of periodic orbits, or invariant measures, or critical exponent, i.e., orbital growth of the fundamental group, and show that they coincide. However, as in [ST21], the core of the proof of the existence (and uniqueness) of a measure of maximal entropy relies on geometric arguments specific to geodesic flows, since it uses the *critical exponent* and the *critical exponent at infinity.* These critical exponents make sense only in this geometric setting, and the measure of maximal entropy is obtained in a geometric way, through the so-called Patterson-Sullivan construction.

The initial motivation of this work was to propose a general dynamical framework where this kind of results holds. More precisely, our initial goals were the following ones.

1. Propose a relevant definition of Anosov flow in the non-compact setting, that includes geodesic flows of non-compact hyperbolic manifolds.
2. Compare different notions of entropy in this context.
3. Give one or several definition(s) of entropy at infinity and compare them.
4. Under a *strongly positive recurrent* hypothesis, construct a measure of maximal entropy.
5. Study the uniqueness and ergodicity of this measure.
6. Construct new families of examples.

1.3 Our results

In the present work, we address points 1, 2, 3 and 4. We postpone point 6 to [FSV25].

The definition of an Anosov flow (on a compact manifold) strongly uses the Riemannian metric. It does not matter in the compact case, where all metrics are equivalent, but becomes an important choice in the noncompact setting.

In [DLRW13], the authors avoid this choice by proposing a definition of *topological Anosov flows*, relying on the idea that two points $(x, y) \in M^2$ are said *close* if they belong to a small neighbourhood of the diagonal in $M \times M$. A topological Anosov flow is then a topologically expansive flow, which satisfies a topological shadowing property. Nevertheless, they observe that this definition does not have clear relations with the usual one, even in the compact case.

In Section 2 we address point 1 with another approach, inspired by [CS10]. We propose an axiomatic definition of a *H-flow* on a (non compact) manifold, by requiring dynamical properties that mimic properties of compact uniformly hyperbolic flows in the non compact setting. These properties are stronger than those of [CS10]. Observe that this axiomatic definition finds echos in [BFM25], where the authors introduce the notion of Anosov-like group actions by asking for the occurrence of certain properties. More precisely, in Definition 2.9, we define a *H-flow* on a (non compact) Riemannian manifold M as a C^1 flow φ with lipschitz bounds that is *transitive*, *expansive*, satisfies a *closing lemma* and a suitable *shadowing property.* We observe in Theorem 2.11 that the geodesic flow of a negatively curved manifold with pinched negative curvature, a lower bound on the injectivity radius and a full nonwandering set is a H-flow.

Observe that we require a H-flow to be transitive on the whole manifold. It is likely that we could weaken this property, defining what could be thought of as a noncompact Axiom A flow, with good uniformly hyperbolic behaviour in restriction to a closed invariant attractor instead of a noncompact generalization of an Anosov flow on the whole manifold, and get the same kind of results. This should heuristically work but presents some technical difficulties that we hope to solve in the future.

The definition of H-flow uses a little bit the Riemannian metric and more deeply the induced distance d on M, but not so deeply as the usual definition of Anosov flow. A natural alternative approach would be to adapt the definition of Anosov flow in the noncompact setting, by requiring some contraction and expansion of stable/unstable bundles, with a uniform control on the angle between them. We refer to [FSV25] to compare this approach with the present one, and show that a noncompact Anosov flow with further natural uniform assumptions is a H-flow. In [FSV25] we will also construct new families of examples, besides geodesic flows, of H-flows on non compact manifolds, providing then answers to point 6. These examples will be non compact versions of the systems studied in [FH13] and [FHV21].

On the way towards the construction of a measure of maximal entropy for H-flows, we needed to define, clarify and compare different notions of topological and measure-theoretic entropies in this noncompact setting. We describe them briefly in this introduction and refer to sections 3, 4 5 for details.

Let φ be a flow on M and let μ be a φ-invariant probability measure. The classical *Kolmogorov-Sinai* entropy $h_{\mathrm{KS}}(\mu)$ measures the asymptotic growth of the average information of a finite measurable partition iterated under the dynamics (see Definition 3.2). We will also largely use the *Katok entropy* $h_{\mathrm{Kat}}(\mu)$, that uses Bowen's definition of dynamical balls. A dynamical ball $B(x, \varepsilon, T)$ is the set of points $y \in M$ whose trajectory follows the one of x at the precision ε during a time T. Katok entropy is the asymptotic growth rate of the minimal number of dynamical balls needed to cover a set of a given positive measure, see Definition 3.7. Given μ, we can also consider the asymptotic lower and upper exponential decay rates of the measure μ of a typical dynamical ball, called the *Brin-Katok lower and upper local entropies* $\underline{h}_{BK}(\mu)$ and $\overline{h}_{BK}(\mu)$, see Definition 3.8. Collecting those of the results of [Kat80, BK83, Riq18] that hold in the noncompact setting, we observe in Corollary 4.4 that when a measure μ is ergodic, all these notions of entropies coincide. Comparison between these entropies for a nonergodic measure is less clear, partial results are recalled in Theorem 4.2.

The well-known variational principle asserts (in the compact setting) that the *topological entropy* of a dynamical system coincides with the supremum of Kolmogorov-Sinai entropies over all invariant measures. In the noncompact setting, the historical notion of topological entropy through open covers, due to Adler-Konheim-Weiss [AKM65] is too often infinite, and therefore not relevant. Bowen's definition of topological entropy [Bow75] strongly relies on the metric on M. To bypass this dependence, the noncompact variational entropy proven by Handel and Kitchens [HK95b] shows that for a dynamical system on a noncompact manifold, the supremum of Kolmogorov-Sinai entropies over all invariant probability measures is equal to the infimum of Bowen metric entropies over all distances defining the topology. Therefore, as in [ST21, GST23], we call *variational entropy* of the flow, denoted by $h_{\mathrm{var}}(\varphi)$, this supremum of Kolmogorov-Sinai measured entropies over all φ-invariant probability measures, see Definition 3.3. When it exists, a measure realizing the supremum is called a *measure of maximal entropy.* The existence of such a measure allows to build orbits that achieve in some sense the maximal possible chaotic behaviour of the dynamics.

In hyperbolic dynamics, topological entropy is often measured through the so-called *Gurevic entropy* $h_{\mathrm{Gur}}(\varphi)$, i.e., the exponential growth rate of the number of periodic orbits of period at most T, when $T \to +\infty$. We study its basic properties in the context of H-flows in Theorem 3.10. In Theorem 4.3, we show that for a H-flow φ, the Katok entropy of a φ-invariant probability measure is always bounded from above by the Gurevic entropy:

$$h_{\mathrm{Kat}}(\mu) \leq h_{\mathrm{G}ur}(\varphi)\,,$$

which implies immediately

$$h_{\mathrm{var}}(\varphi) \leq h_{\mathrm{G}ur}(\varphi)\,.$$

In Section 5, we introduce a notion of entropy that has not been used yet, to our knowledge. The *chord entropy* $h_{\mathcal{C}}(\varphi)$ of a H-flow is the exponential growth rate of the maximal cardinality of a separated set of chords from (the neighbourhood of) a point to (the neighbourhood of) another point. We prove in Theorem 5.12 that this entropy coincides with Gurevic entropy :

$$h_{\mathcal{C}}(\varphi) = h_{\mathrm{G}ur}(\varphi)\,.$$

It turns out that this definition through chords is much more flexible, and therefore more convenient to use in several arguments.

The results mentioned above answer question 2. In Sections 3.4, 5.4 and 5.5 we address the different possible definitions of entropies at infinity, as proposed in point 3. We are particularly concerned with two of them:

- the *Gurevic entropy at infinity* $h^{\infty}_{\mathrm{Gur}}(\varphi)$ is the exponential growth rate of the number of periodic orbits that spend most of their time outside a large compact set (see Definition 3.16);

- the *chord entropy at infinity* $h^\infty_{\mathcal{C}}(\varphi)$ is the exponential growth rate of the number of separated paths which remain outside a large compact set.

In Theorem 5.22, we prove that, for a H-flow, these two notions of entropy at infinity coincide:

$$h^\infty_{\mathrm{Gur}}(\varphi) = h^\infty_{\mathcal{C}}(\varphi)\,.$$

The heart of the paper is the construction of a measure of maximal entropy, as asked in the above point 4. As in [ST21, GST23] for geodesic flows, we say that a H-flow φ is *strongly positively recurrent* if its Gurevic entropy at infinity is strictly smaller than the Gurevic entropy:

$$h^\infty_{Gur}(\varphi) < h_{Gur}(\varphi)\,.$$

All results mentioned above are interesting and useful. However, the main result of the paper is the following Theorem.

Theorem 1.1. *Let $\varphi\colon M \to M$ be a H-flow on a Riemannian manifold (M, g) such that $h^\infty_{\mathrm{Gur}}(\varphi) < h_{\mathrm{Gur}}(\varphi)$. Then, there exists a φ-invariant probability measure $m_{\max}$ on M maximizing entropies:*

$$h_{\mathrm{KS}}(m_{\max}) = \underline{h}_{\mathrm{BK}}(m_{\max}) = \overline{h}_{\mathrm{BK}}(m_{\max}) = h_{\mathrm{Kat}}(m_{\max}) = h_{\mathrm{Gur}}(\varphi) = h_{\mathrm{var}}(\varphi)\,.$$

Note that we are interested in H-flows, but we could have defined a very similar notion of H-diffeomorphisms and proven the same result. We let the verification to the interested reader. Moreover, as said above and suggested to us by L. Flaminio, it is likely that this Theorem could be extended to flows that satisfy transitivity, expansivity, closing lemma, and shadowing on a closed invariant subset of the Riemannian manifold. This should be checked.

The main idea for the proof of Theorem 1.1 is inspired by the approach of Bowen in [Bow72]. We construct a sequence of φ-invariant probability measures, each one obtained by normalizing the sum of all measures supported on periodic orbits of (almost) given period. Up to extracting a subsequence, we consider a weak limit of this (sub)sequence. One of the main difficulties on non compact spaces arises here: the sequence could loose its whole mass at infinity and converge to the zero measure. Strong positive recurrence prevents a total loss of mass: the limit measure m_∞ is non zero. Therefore it can be renormalized into a probability measure $m_{\max}$, and the latter is the good candidate to be a measure of maximal entropy. It remains to compute carefully its entropy. This is done through a strong uniform control of the $m_{\max}$ measure of all dynamical balls of the manifold. More precisely, in Theorem 7.6, we prove a rigorous version of the following heuristics : on every compact set K, up to uniform constants, for every $x \in K$, the measure of every dynamical ball $B(x, \varepsilon, T)$ such that $\varphi_T(x)$ belongs in K satisfies

$$m_{\max}(B(x, \varepsilon, T)) \asymp \frac{1}{\mathcal{N}_{\mathcal{C}}(x, T, \delta)}\,,$$

where the denominator is the cardinality of a δ-separated set of chords of time-length T from a neighbourhood of x to itself. This strong uniform statement allows easily to compute the entropy of the measure and deduce Theorem 1.1.

Theorem 7.6 is proven through subtle subadditivity statements: Propositions 6.4 and 6.5. These subadditivity statements are quite classical for hyperbolic flows on compact manifolds, or in a symbolic context and say essentially that the number of periodic orbits of period L is essentially comparable to the product of the number of periodic orbits of period $L - T$ times the number of periodic orbits of period T. This is done by cutting / concatenating periodic orbits at good points. The main difficulty that we have to deal with is that a very long periodic orbit does not necessarily come back often in a given compact set, so that it is very hard to play the usual game and cut it into two smaller periodic orbits intersecting the same compact set. Indeed, to compare the number of long periodic orbits intersecting a given compact set with the number of shorter periodic orbits, we need to show that most periodic orbits come back almost regularly in a compact set. This is done in Proposition 6.9.

Our approach does not allow us yet to explore whether such a maximal entropy measure is unique or ergodic. We would like to address this question in the future. Moreover, as in the case of geodesic flow, it is likely that the strong positive recurrence assumption is not necessary for the existence of a measure of maximal entropy. This should also be investigated.

The structure of the paper is the following. In Section 2, we define H-flows. In Section 3, we define different entropies of a φ-invariant probability measure. In Section 4, we compare them. In Section 5, we introduce the notion of chord entropy and compare it to the Gurevic entropy. Section 6 is the technical heart of the paper. Under the strong positive recurrence assumption, we construct the measure that is the natural candidate to be the measure of maximal entropy, and we deduce from its existence subtle subadditivity properties on the number of periodic orbits of a given period. These properties are classical in a compact setting, but absolutely non trivial without compactness. In the last Section 7, we deduce that the measure that we constructed is the required measure of maximal entropy.

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2 H-flows on non-compact manifolds

We propose here a definition of hyperbolic flows on non-compact Riemannian manifolds through some important dynamical properties that they satisfy. The definition involves the distance, but not the Riemannian structure. We call them *H-flows*, thinking in particular to *hyperbolic flows*, but this word has too many significations so that we prefer to avoid it.

We first describe the dynamical properties used in the description of H-flows.

2.1 Notations

Let (M, g) be a complete connected Riemannian manifold and denote by d the associated distance. Let $(\varphi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^1$-flow on M generated by a vector field X.

If $c : [a, b] \to M$ is a piece of orbit of the flow φ, i.e. $c(t) = \varphi_t(x)$ for some $x \in M$, and $t \in [a, b]$, denote by $\ell(c) = b - a$ its "length". By extension, if γ is a periodic orbit (possibly with multiplicity), we denote by $l(\gamma)$ its period (with multiplicity); we will sometimes refer to $\ell(\gamma)$ also as the "length" of the periodic orbit. For $A \subset M$ any measurable subset, and γ a periodic orbit, we define also $\ell(\gamma \cap A) = \{t \in [0, \ell(\gamma)], \gamma(t) \in A\}$.

2.2 Bounds on the flow

We require that for every $\tau \in [-1, 1]$, the time τ-map φ_τ of the flow is *Lipschitz continuous* with uniform Lipschitz constant: there exists a constant $lip(\varphi)$ such that for all $x, y \in M$, and $\tau \in [-1, 1]$,

$$d(\varphi_\tau(x), \varphi_\tau(y)) \leq lip(\varphi)\, d(x, y)\,. \tag{1}$$

We also assume that there exist $0 < a \leq b$ such that for every $x \in M$, there exists $\rho > 0$ such that for every $t \in [-\rho, \rho]$, we have

$$a|t| \leq d(x, \varphi_t(x)) \leq b|t| \tag{2}$$

We could probably weaken the constraint on the lower bound, with the existence of such a constant a on every compact set. On the contrary, we really need b to be uniform. Notice that the lower bound is true locally for $t \in [-\rho, \rho]$, whereas triangular inequality implies that the upper bound $d(x, \varphi_t(x)) \leq b|t|$ is satisfied for every $t \in \mathbb{R}$.

Property (2) can be obtained by a uniform control on the norm of the infinitesimal generator X of the flow as stated in the following lemma.

Lemma 2.1. *If there exist $a', b' > 0$ such that for every $x \in M$,*

$$0 < a' \leq \|X(x)\| \leq b' < \infty\,, \tag{3}$$

then there exist $0 < a < b$ such that property (2) is satisfied.

Proof. Let $x \in M$. Consider the exponential map $\exp_x : U \subset T_xM \to V \subset M$ and let $Y = \exp_x^* X$. Choose an orthonormal basis $(e_1, \cdots, e_n)$ on T_xM endowed with the scalar product g_x given by the metric. This allows to identify T_xM with the canonical euclidean space $\mathbb{R}^n$. Without loss of generality, we may assume $Y(0) = \|X(x)\|\frac{\partial}{\partial x_1}$. By definition of the exponential map, for every point $y \in V$, $d(x, y) = \|\exp_x^{-1}(y)\|_{\text{eucl}}$ where $\|\cdot\|_{\text{eucl}}$ is the Euclidean norm on $\mathbb{R}^n$. Write $Y = (Y_1, \ldots, Y_n)$. One may shrink U to ensure that, for all $v \in U$, $\|Y(v)\|_{\text{eucl}} \leq 2b'$ and $Y_1(v) \geq a'/2$. Therefore, for t small enough

$$\frac{a'}{2}|t| \leq \left|\int_0^t Y_1(\exp_x^{-1}(\varphi_s(x)))ds\right| \leq d(x, \varphi_t(x)) = \left\|\int_0^t Y(\exp_x^{-1}(\varphi_s(x)))ds\right\|_{\text{eucl}} \leq 2b'|t|\,.$$

□

2.3 Topological transitivity

Definition 2.2 (Transitivity)**.** *The flow φ is* topologically transitive *on M if for all open sets $U, V \subset M$ and every $T > 0$, there exists $t \geq T$ such that $\varphi_t(U) \cap V \neq \emptyset$.*

Most of the time, we will use the stronger property of Lemma 2.14, where t can be made almost constant for open sets U, V inside a given compact set K.

2.4 Closing lemma

Definition 2.3 (Closing lemma)**.** *The flow satisfies the* closing lemma *if for every $\varepsilon > 0$ and $x \in M$ there exists $\delta > 0$ and $T_{\min} > 0$ such that for every $y \in B(x, \varepsilon)$ and $t \geq T_{\min}$ with $d(\varphi_t(y), y) \leq \delta$, there exist $z \in B(y, \varepsilon)$ and $\tau \in [t - \varepsilon, t + \varepsilon]$ such that $\varphi_\tau(z) = z$ and $d(\varphi_s(z), \varphi_s(y)) \leq \varepsilon$ for every $0 \leq s \leq t$.*

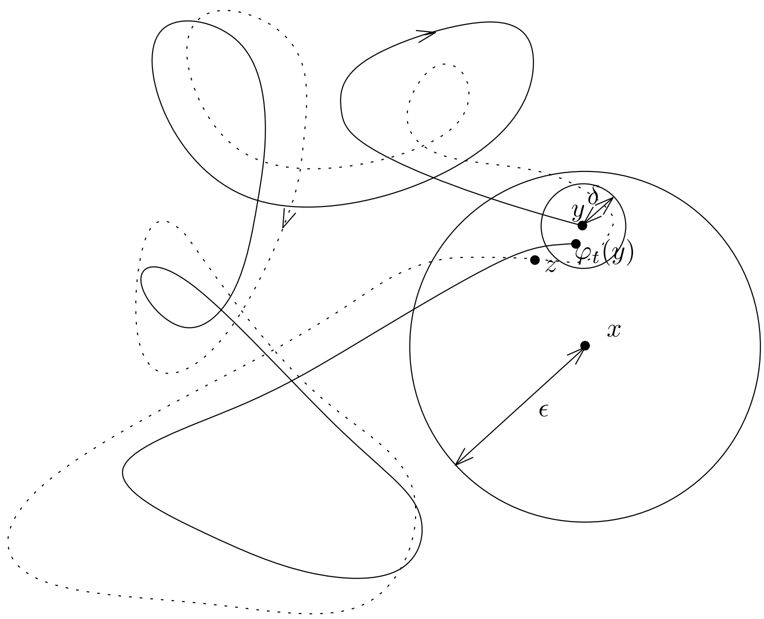

Figure 2.1: Closing Lemma

2.5 Expansivity

We follow [FH19, Definition 1.7.2].

Definition 2.4 (Expansivity)**.** *The flow φ is said expansive if for every $\nu > 0$ there exists $\varepsilon > 0$ such that for all $x, y \in M$, if there exists a continuous map $s : \mathbb{R} \to \mathbb{R}$ such that $s(0) = 0$ and for every $t \in \mathbb{R}$, $d(\varphi_t(x), \varphi_{s(t)}(y)) \leq \varepsilon$, then there exists $\tau \in [-\nu, \nu]$ such that $y = \varphi_\tau(x)$.*

Proposition 2.5. *The geodesic flow $(g_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $M = T^1N$ of a (not necessarily compact) pinched negatively curved manifold N whose injectivity radius is bounded from below by a positive constant is expansive with respect to the distance on T^1N induced by the Sasaki metric on TTN.*

We refer for example to [PPS15, p.18-19] for definitions and elementary useful facts on the Sasaki metric.

Proof. Let $\rho > 0$ be the positive infimum of all injectivity radii $\rho_{inj}(x)$ at all points $x \in N$. Choose $\varepsilon < \rho/2$. Consider two vectors $v, w \in M = T^1N$ and a map $s : \mathbb{R} \to \mathbb{R}$ that satisfy $d(g_t v, g_{s(t)} w) < \varepsilon$ for every $t \in \mathbb{R}$.

Denote by $\widetilde{N}$ the universal covering of N. We claim that there exist $\widetilde{v}, \widetilde{w} \in T^1\widetilde{N}$ such that $d(g_t\widetilde{v}, g_{s(t)}\widetilde{w}) < \varepsilon$ for every $t \in \mathbb{R}$. Indeed, choose $\widetilde{v}$ and $\widetilde{w}$ two lifts of v and w such that $d(\widetilde{v}, \widetilde{w}) < \varepsilon$. Assume by contradiction that there exists t such that $d(g_t\widetilde{v}, g_{s(t)}\widetilde{w}) = \varepsilon$. Then, as $d(g_t v, g_{s(t)} w) < \varepsilon$, we can find a lift $\widetilde{w}_t$ of $g_{s(t)}w$ such that $d(g_t\widetilde{v}, \widetilde{w}_t) < \varepsilon$. Therefore, the two lifts $g_{s(t)}\widetilde{w}$ and $\widetilde{w}_t$ of $g_{s(t)}w$ satisfy $d(g_{s(t)}\widetilde{w}, \widetilde{w}_t) < 2\varepsilon < \rho$. This is in contradiction with the definition of injectivity radius.

Now, recall that $\widetilde{N}$ is a Hadamard manifold with pinched negative curvature. It implies that there are no distinct parallel geodesics. Therefore, as $d(g_t\widetilde{v}, g_{s(t)}\widetilde{w}) < \varepsilon$ for every $t \in \mathbb{R}$, there exists $\tau \in \mathbb{R}$ such that $w = g_\tau v$. Moreover, we have $|\tau| < \varepsilon$ by assumption. □

This condition of positive lower bound on the injectivity radius can fail for different reasons. When it fails, the expansivity property can still hold or it can also fail. If the manifold admits one cusp, and no other end, expansivity still holds. If there are infinitely many periodic orbits with lengths arbitrarily small, then one can build pairs of distinct orbits that stay arbitrarily close, one turning around a small periodic orbit and not the other, so that expansivity fails.

2.6 Shadowing properties

Definition 2.6 (Finite exact shadowing)**.** *The flow $\varphi\colon M \to M$ satisfies the finite exact shadowing property if for every compact subset $K \subset M$, every $\delta > 0$ and every integer $N \in \mathbb{N}^*$, there exists $\eta > 0$ such that the following shadowing holds. Given N orbits $(\varphi_t(x_i))_{0\leq t\leq t_i} \in K$ for $i = 1, \dots, N$ starting in K that satisfy*

$$d(\varphi_{t_i}(x_i), x_{i+1}) < \eta \,, \forall i = 1, \dots, N-1 \,,$$

there exists $y \in M$ such that for every $1 \leq i \leq N$, and $0 \leq s \leq t_i$,

$$d(\varphi_{s+\sum_{j=1}^{i-1} t_j}(y), \varphi_s(x_i)) < \delta \,.$$

See Figure 2.2. Here, the word *finite* refers to the finite number of orbits defined on finite intervals of time and the word *exact* to the fact that the orbit of y follows exactly the orbits of the (x_i), without reparametrization of time, by contrast with the usual definitions of shadowing, see for example [FH19, Definition 1.5.29] and [FH19, Definition 5.3.1]. For the sake of clarity, let us compare this shadowing property with the assumptions of [CS10]. In [CS10], the authors work with flows satisfying transitivity, closing lemma and a local product structure. This property, defined below, is stronger than the finite exact shadowing of Definition 2.6, as proven in Lemma 2.8 below.

Definition 2.7 (Local product structure)**.** *We say that φ satisfies the local product structure if for every $x_0 \in X$ there exits $\varepsilon_0 > 0$ such that, for every $\delta > 0$ there exists $\eta > 0$ satisfying the following property. For all $x, y \in B(x_0, \varepsilon_0)$ such that $d(x, y) \leq \eta$, there exists $z \in B(x_0, \varepsilon_0)$ and $|\tau| < \delta$ such that $d(\varphi_t x, \varphi_t z) \leq \delta$ for all $t \leq 0$ and $d(\varphi_{t+\tau} y, \varphi_t z) \leq \delta$ for all $t \geq 0$. We will use the standard notation $z = \langle x, y\rangle$.*

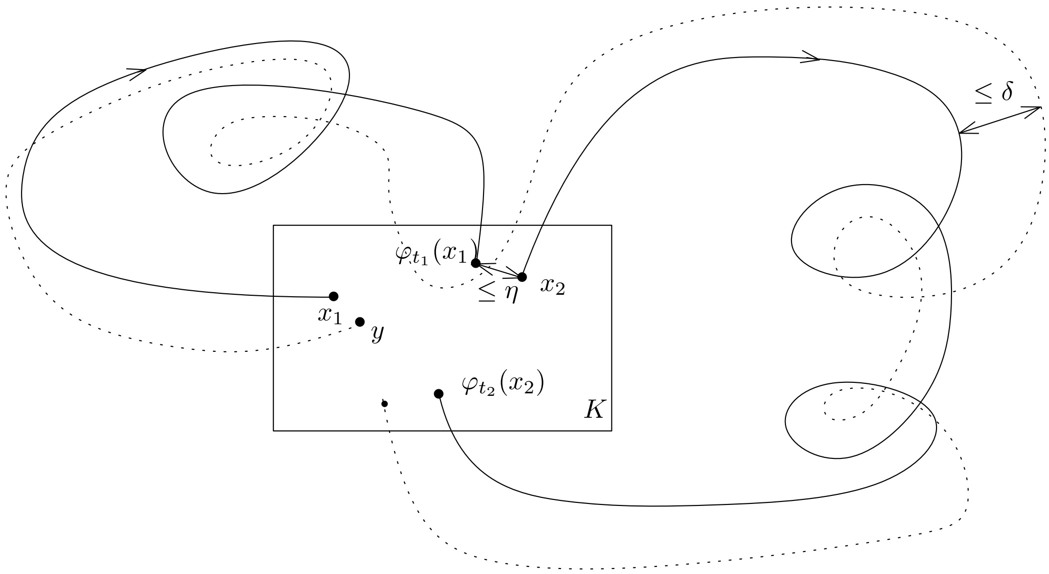

Figure 2.2: Finite exact shadowing for $N = 2$

Lemma 2.8. *Let $\varphi\colon M \to M$ be a C^1-flow satisfying* (2)*. If φ satisfies the local product structure (see Definition 2.7), then φ satisfies the finite exact shadowing property (see Definition 2.6).*

Proof. It is enough to prove it for $N = 2$, and the general case follows by successive uses of the $N = 2$ version. Let K be a compact subset and fix $\delta > 0$. Let $\delta' = \frac{\delta}{1+b}$ where b comes from the upper bound constant in (2). As $\overline{B(K,1)}$ is compact, using the local product structure, we obtain $\eta > 0$ such that for all $x, y \in \overline{B(K,1)}$ such that $d(x,y) \leq \eta$, there exists $z \in M$ and $|\tau| < \delta'$ such that $d(\varphi_t x, \varphi_t z) \leq \delta'$ for all $t \leq 0$ and $d(\varphi_{t+\tau} y, \varphi_t z) \leq \delta'$ for all $t \geq 0$.

Fix $x_1, x_2 \in K$ and $t_1, t_2 \geq 0$ such that $d(\varphi_{t_1}(x_1), x_2) < \eta$, that is, as in the definition of finite exact shadowing. Apply then the local product structure for $\varphi_{t_1}(x_1)$ and x_2: it gives us $z \in M$ and $|\tau| < \delta'$ such that $d(\varphi_{t+t_1}(x_1), \varphi_t(z)) \leq \delta'$ for all $t \leq 0$ and $d(\varphi_{t+\tau}(x_2), \varphi_t(z)) \leq \delta$ for all $t \geq 0$.

Set $y = \varphi_{-t_1}(z)$. It satisfies $d(\varphi_t(y), \varphi_t(x_1)) \leq \delta' < \delta$ for $0 \leq t \leq t_1$ and, by (2),

$$d(\varphi_{t+t_1}(y), \varphi_t(x_2)) = d(\varphi_t(z), \varphi_t(x_2)) \leq d(\varphi_t(z), \varphi_{t+\tau}(x_2)) + d(\varphi_{t+\tau}(x_2), \varphi_t(x_2)) \leq \delta' + b|\tau| < \delta$$

for $t \geq 0$. The result follows. □

2.7 H-flows

Definition 2.9. *A H-flow is a C^1-flow $(\varphi_t)_{t\in\mathbb{R}}$ on a complete Riemannian manifold satisfying the following properties :*

1. *for every $\tau \in [-1,1]$, the time τ map φ_τ is Lipschitz, as in (1);*
2. *the parametrization of the flow satisfies the bounds (2);*
3. *the flow is topologically transitive, see Definition 2.2;*
4. *the flow satisfies the finite exact shadowing property, see Definition 2.6;*
5. *the flow satisfies the closing lemma, see Definition 2.3;*
6. *the flow is expansive, see Definition 2.4.*

Remark 2.10. By Lemma 2.8, an expansive flow satisfying the assumptions (transitivity, local product and closing lemma) of [CS10] and the controls (1) and (2) is a H-flow.

Theorem 2.11. *Let N be a Riemannian manifold with pinched negative curvature, bounded from above and from below by uniform negative constants. Assume that the injectivity radius is bounded from below by a positive constant. Let $M = T^1N$ be its unit tangent bundle, endowed with the distance associated with the Sasaki metric. If the geodesic flow $(g_t)_{t\in\mathbb{R}}$ is topologically transitive, then it is a H-flow.*

Proof. The bound (1) saying that the geodesic flow on M at time τ, for $\tau \in [-1, 1]$, is Lispchitz continuous, with a uniform Lipschitz constant, will follow from a uniform bound on the differential of the geodesic flow on the unit tangent bundle M for the Sasaki metric.

First note that, as the sectional curvature is bounded, the Riemann curvature tensor, which is completely determined by the sectional curvature, is also bounded (see for instance [GHL90, Chapter III, Theorem 3.8] or [BK81, Section 6.1]). Now, the differential of the geodesic flow on the tangent bundle M is

$$D_{(p,v)}g_t(X, Y) = (J(t), J'(t)),$$

where J is the Jacobi field along the geodesic γ defined by $(\gamma(0), \gamma'(0)) = (p, v)$ with initial condition $J(0) = X$ and $J'(0) = Y$ (see for instance [Bal95, Lemma 1.13]). Therefore, in order to bound uniformly the differential of the geodesic flow, it is enough to bound uniformly $\|J(t)\|$ and $\|J'(t)\|$ for $t \in [-\tau, \tau]$. From a direct computation (see [Bal95, Proof of Proposition 1.19]), we obtain

$$\begin{aligned}\left(||J(t)||^2 + ||J'(t)||^2\right)' &= 2\left\langle J'(t), J(t) + J''(t)\right\rangle = 2\left\langle J'(t), J(t) - R(J(t), \gamma'(t))\gamma'(t)\right\rangle \\ &\leq 2C||J'(t)||||J(t)|| \leq C\left(||J(t)||^2 + ||J'(t)||^2\right)\end{aligned}$$

for some $C > 0$ where R is the curvature tensor. Using Grönwall's inequality, we obtained the desired bound.

The bounds (2) follow from an elementary computation : in the classical horizontal/vertical coordinates of TTN, the geodesic vector field generating the geodesic flow satisfies $X(v) = (v, 0)$ so that its Sasaki norm satisfies $\|X(v)\| = \|v\| = 1$. Topological transitivity is an assumption. Finite exact shadowing follows from the fact that geodesic flows satisfy a local product structure, and from Lemma 2.8. Closing lemma is due to Eberlein [Ebe96] in nonpositive curvature, and a short proof in the particular case of negative curvature is given in [CS10]. Expansivity follows from Proposition 2.5. □

2.8 Properties of H-flows

In the whole section, φ is a H-flow on a complete Riemannian manifold M, and d is a Riemannian distance.

Lemma 2.12. *Let φ be a H-flow. For every nonempty open set $U \subset M$, there exists a periodic orbit that intersects U.*

Proof. Choose $x \in U$ and $\varepsilon > 0$ such that $B(x, 2\varepsilon) \subset U$. The closing lemma (see Definition 2.3) associates to x and ε some $\delta > 0$ and $T_{\min} > 0$. Without loss of generality, $\delta \leq \varepsilon$. By transitivity (see Definition 2.2), there exists $t \geq T_{\min}$ such that $B(x, \delta/2) \cap \varphi_t(B(x, \delta/2)) \neq \emptyset$. Choose some $\varphi_t(y)$ in this intersection, and apply the closing lemma 2.3 to y. As $d(\varphi_t(y), y) < \delta \leq \varepsilon$, there exists a periodic point $z \in B(y, \varepsilon) \subset B(x, 2\varepsilon) \subset U$. This concludes the proof. □

Notation. For a compact set $K \subset M$ whose interior is nonempty, denote by τ_K the minimal period of a periodic orbit (possibly a multiply covered orbit) with period ≥ 1 intersecting K. Such an orbit always exists for a H-flow as the interior of K is nonempty.

Lemma 2.13 (Separation of orbits)**.** *Let φ be a H-flow on M. For every $\nu > 0$, there exists $\tau_0 > 0$ such that for every $\tau_1 \geq 1$, there exists $\varepsilon_1 > 0$ such that the following holds. For all periodic points x_0, x_1 with respective periods $T_0, T_1 \geq \tau_1$, satisfying $0 \leq T_1 - T_0 \leq \tau_0$, if, for all $s \in [0, T_0 - \tau_1]$ we have*

$$d(\varphi_s(x_0), \varphi_s(x_1)) < \varepsilon_1 \tag{4}$$

then $x_1 = \varphi_u(x_0)$ for some $u \in [-\nu, \nu]$.

Proof. Fix $\nu > 0$. Let $\varepsilon' > 0$ be given by Definition 2.4 of expansivity associated with ν. Let $\tau_0 = \varepsilon'/2b$, where $b > 0$ is the constant of (2). Fix $\tau_1 \geq 1$. Let $C(\tau_1) = C \geq 1$ be a uniform Lipschitz constant for every φ_t with $t \in [0, \tau_1]$ (such a constant exists by the Lipschitz condition in the definition of H-flows). Let $\varepsilon_1 = \varepsilon'/2C$. Note that $\varepsilon_1 \leq \varepsilon'/2$.

Consider x_0, x_1, T_0 and T_1 as in the statement of the lemma. Then, for every $t \in [T_0 - \tau_1, T_0]$,

$$d(\varphi_t(x_0), \varphi_t(x_1)) \leq C d(\varphi_{T_0-\tau_1}(x_0), \varphi_{T_0-\tau_1}(x_1)) < C\varepsilon_1 = \frac{\varepsilon'}{2},$$

where the last inequality holds because of (4). Therefore, for every $t \in [0, T_0]$,

$$d(\varphi_t(x_0), \varphi_t(x_1)) < \frac{\varepsilon'}{2}.$$

Thus, for every $t \in [0, T_0]$, because of this last inequality and because of (2),

$$d\left(\varphi_{t\frac{T_1}{T_0}}(x_1), \varphi_t(x_0)\right) \leq d\left(\varphi_{t\frac{T_1}{T_0}}(x_1), \varphi_t(x_1)\right) + d\left(\varphi_t(x_1), \varphi_t(x_0)\right) < bt\left|\frac{T_1}{T_0} - 1\right| + \frac{\varepsilon'}{2} \leq b\tau_0 + \frac{\varepsilon'}{2} \leq \varepsilon'.$$

As $\varphi_t(x_0)$ and $\varphi_{t\frac{T_1}{T_0}}(x_1)$ are T_0 periodic, the inequality

$$d\left(\varphi_{t\frac{T_1}{T_0}}(x_1), \varphi_t(x_0)\right) < \varepsilon'$$

holds for every $t \in \mathbb{R}$. From the expansivity property, see Definition 2.4, we get a parameter u such that $|u| \leq \nu$ and $x_1 = \varphi_u(x_0)$. □

Lemma 2.14 (Uniform transitivity)**.** *Let φ be a H-flow on M. Let $K' \subset K \subset M$ be two compact subset with nonempty interior. Let $\delta > 0$. There exists $\sigma > 0$ such that for every $x, y \in K$ and for every $S \geq \sigma$ there are $z \in M$ and $T \in [S - \tau_K, S + \tau_K]$ such that*

$$\varphi_{[0,T]}(z) \cap K' \neq \emptyset, \quad d(x, z) < \delta \quad \textit{and} \quad d(y, \varphi_T(z)) < \delta.$$

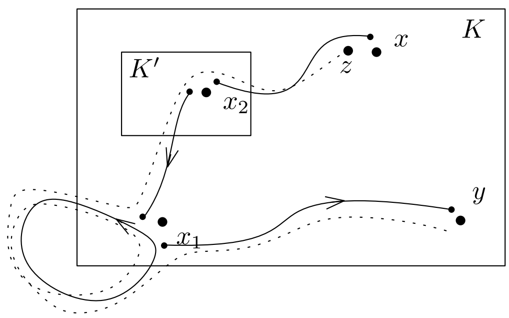

Figure 2.3: Proof of Uniform Transitivity

Proof. See Figure 2.3. Recall that τ_K is the length of the shortest periodic orbit with period ≥ 1 intersecting K. Let η be given by the finite exact shadowing property 2.6 with parameters K, δ and $N = 4$. Cover K with finitely many balls $B_i = B(x_i, \eta/2)$ of radius $\eta/2$. Without loss of generality, we may assume that the periodic orbit γ used to define τ_K intersects B_1 and that $B(x_2, \delta + \eta/2) \subset K'$.

For all i, j, there exists $T_{i,j} > 0$ such that $\varphi_{T_{i,j}}(B_i) \cap B_j \neq \emptyset$ (this is the transitivity property from Definition 2.2). Let $\sigma_0 = 3\max_{i,j}\{T_{i,j}\} + \tau_K$.

Let $x, y \in K$. In particular, there exists i, j such that $x \in B_i$, $y \in B_j$. Let $S \geq \sigma_0$. Fix $n \geq 2$ such that $T_{i,2} + T_{2,1} + n\tau_K + T_{1,j} \in [S - \tau_K, S + \tau_K]$. Use the finite exact shadowing (see Definition 2.6) to concatenate an orbit from B_i to B_2 (given by the definition of $T_{i,2}$), and orbit from B_2 to B_1 (given by the definition of $T_{2,1}$) $n\gamma$ and an orbit from B_1 to B_j (given by the definition of $T_{1,j}$). The initial point of the concatenated orbit is the desired point z. □

The following lemma allows us to build a periodic orbit from a pseudo-orbit, see Figure 2.4.

Lemma 2.15 (Multiple closing lemma)**.** *Let φ be a H-flow on M. Let $K \subset M$ be a compact subset. For all $\delta > 0$, $\nu > 0$ and $N \in \mathbb{N}^*$, there exist $T_{\min} > 0$ and $\eta > 0$ such that for all $x_1, \dots, x_N \in K$, and $T_1, \dots, T_N > 0$ with $\sum_{i=1}^N T_i \geq T_{\min}$ and $\varphi_{T_i}(x_i) \in B(x_{i+1}, \eta)$ for $i = 1, \dots, N$ (where $x_{N+1} = x_1$), the following property holds. There exists a periodic orbit γ with period*

$$\ell(\gamma) \in \left[\sum_{i=1}^N T_i - \nu, \sum_{i=1}^N T_i + \nu\right]$$

such that for every $0 \leq i \leq N$, and every $s \in [0, T_i]$, we have

$$d(\gamma(s + \sum_{j=1}^{i-1} T_j), \varphi_s(x_i)) < \delta\,.$$

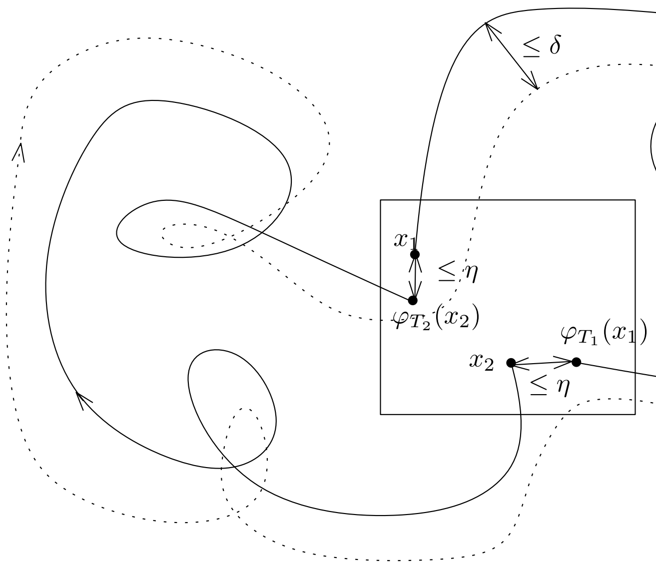

Figure 2.4: Multiple closing Lemma for $N = 2$

Proof. This follows immediately from the finite exact shadowing property 2.6 and the closing lemma 2.3. □

The following lemma allows us to build a periodic orbit from a finite number of orbits with endpoints in a compact set, see Figure 2.5.

Lemma 2.16 (Uniform multiple closing lemma)**.** *Let φ be a H-flow on M. Let $K' \subset K \subset M$ be two compact subsets with nonempty interior, let $\nu > 0$, $\delta > 0$ and $N \in \mathbb{N}^*$. Then, there exist $\sigma > 0$ and $T_{\min} > 0$ such that for every $S \geq \sigma$, for all $x_1, \dots, x_N \in K$, all $T_1, \dots, T_N > 0$ that satisfy $\sum_{i=1}^N T_i \geq T_{\min}$, and such that for every $1 \leq i \leq N$, $\varphi_{T_i}(x_i) \in K$, the following property holds. There exists a periodic orbit γ with period*

$$\ell(\gamma) \in \left[\sum_{i=1}^N T_i + NS - \tau_K - \nu, \sum_{i=1}^N T_i + NS + \tau_K + \nu\right]$$

that intersects $\overset{\circ}{K'}$ and there exist $\tau_1, \dots, \tau_{N-1}$, $\tau_i \in [S - \tau_K, S + \tau_K]$ such that that for every $0 \leq i \leq N$, and every $s \in [0, T_i]$, we have

$$d(\gamma(s + \sum_{j=1}^{i-1} (T_j + \tau_j)), \varphi_s(x_i)) < \delta\,.$$

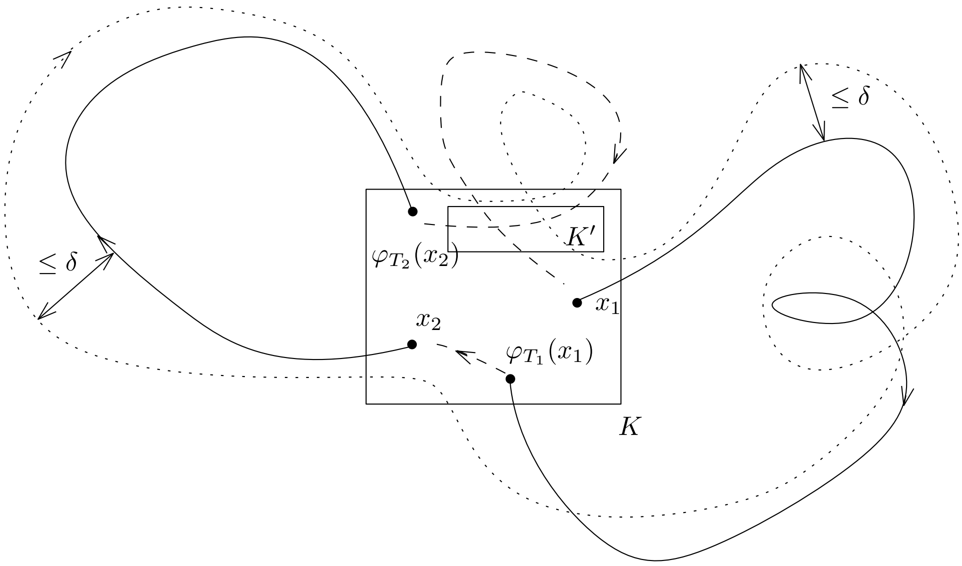

Figure 2.5: Uniform multiple closing Lemma for $N = 2$

Proof. We will use the transitivity property 2.2 to construct pieces of orbits from $\varphi_{T_i}(x_i)$ to x_{i+1}, then the multiple closing lemma 2.15 to obtain a periodic orbit. We will carefully choose the length of the last orbit to ensure $\ell(\gamma) \in \left[\sum_{i=1}^N T_i + NS - \tau_K - \nu, \sum_{i=1}^N T_i + NS + \tau_K + \nu\right]$ as opposed to $\ell(\gamma) \in \left[\sum_{i=1}^N T_i + NS - N\tau_K - \nu, \sum_{i=1}^N T_i + NS + N\tau_K + \nu\right]$ if one ins not careful.

More precisely, assume δ is small enough so that there exist a ball $B(x', 2\delta) \subset K' \subset K$. Lemma 2.15 with parameters K, N, δ and ν gives us $T_{\min}$ and η. Let σ_0 be given by the uniform transitivity lemma (Lemma 2.14) with parameters K, δ and $B(x', \delta)$. Let $\sigma = \sigma_0 + (N-1)\tau_K$.

Let $x_1, \ldots, x_N$ and $T_1, \ldots T_N$ and S be as in the statement of the Lemma. By successive uses of the uniform transitivity 2.14, we can build pieces of orbits of respective lengths $\tau_i \in [S - \tau_K, S + \tau_K]$ from $B(\varphi_{T_i}(x_i), \eta)$ to $B(x_{i+1}, \eta)$ for $i = 1, \ldots N-1$. The uniform transitivity 2.14, with constant $S_0 = NS - \sum_{i=1}^N \tau_i \geq NS - (N-1)S - (N-1)\tau_K \geq \sigma_0$, gives us a piece of orbit from $B(\varphi_{T_N}(x_N), \eta)$ to $B(x_1, \eta)$ intersecting $B(x', \delta)$ and of length $\tau_n \in [NS - \sum_{i=1}^N \tau_i - \tau_K, NS - \sum_{i=1}^N \tau_i + \tau_K]$. Therefore, we have $\sum_{i=1}^N \tau_i \in [NS - \tau_K, NS + \tau_K]$. Lemma 2.15 gives us a periodic orbit which satisfies all the desired conditions. □

With the same arguments, we get the following variant of lemmas 2.15 and 2.16. We let the proof to the reader.

Lemma 2.17 (Variation around the multiple closing lemma)**.** *Let φ be a H-flow on M. Let $K' \subset K \subset M$ be two compact subsets with nonempty interior, let $\nu > 0$, $\delta > 0$ and $N \in \mathbb{N}^*$. Then, there exist $\sigma > 0$ and $T_{\min} > 0$ and $\eta > 0$ such that for every $S \geq \sigma$, for all $x_1, \ldots, x_N \in K$, all $T_1, \ldots, T_N > 0$ that satisfy $\sum_{i=1}^N T_i \geq T_{\min}$, and such that for every $1 \leq i \leq N$, $\varphi_{T_i}(x_i) \in B(x_{i+1}, \eta)$ (with $x_{N+1} = x_1$), the following property holds. There exists a periodic orbit γ with period*

$$\ell(\gamma) \in \left[\sum_{i=1}^N T_i + S - \tau_K - \nu, \sum_{i=1}^N T_i + S + \tau_K + \nu\right]$$

that intersects $\overset{\circ}{K'}$ and such that for every $0 \leq i \leq N-1$, and every $s \in [0, T_i]$, we have

$$d(\gamma(s + \sum_{j=1}^{i-1} T_j), \varphi_s(x_i)) < \delta\,.$$

3 Entropies

This section is devoted to the introduction of different notions of entropy, both associated to a dynamics $(\varphi_t)_t$ and to a φ-invariant probability measure. Some subadditivity properties for periodic orbits are shown in the framework of H-flows, which imply some consequences for the Gurevich entropy. The definitions of entropies at infinity are also presented.

Remark 3.1. All notions of entropies, denoted here h, concern a single transformation, and satisfy the relation $h(\varphi^n) = |n|\, h(\varphi)$. Classically, the entropy of a flow $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ is defined as the entropy of its time-one map φ_1. In this section, we write φ instead of φ_1.

3.1 Kolmogorov-Sinai entropy

Let μ be a φ-invariant measure and ξ be a finite or countable μ-measurable partition. The entropy of the partition ξ with respect to μ is the quantity

$$H(\xi,\mu) = \sum_{\substack{C\in\xi \\ \mu(C)>0}} -\mu(C)\log(\mu(C)) \geq 0\,.$$

Given two measurable partitions ξ_1, ξ_2, we denote by $\xi_1 \vee \xi_2$ the refinement of the two partitions, that is the partition whose elements are the sets $A \cap B$ with positive measure, for $A \in \xi_1$ and $B \in \xi_2$. Given a finite or countable μ-measurable partition ξ, for every $n \geq 1$, set $\xi_n := \bigvee_{i=0}^{n-1} \varphi_{-i}(\xi)$.

Definition 3.2. *The Kolmogorov-Sinai entropy of φ with respect to μ is*

$$h_{\mathrm{KS}}(\mu) = \sup_{\xi} \lim_{n\to+\infty} \frac{1}{n} H(\xi_n,\mu)\,,$$

where the supremum is taken over all finite or countable μ-measurable partitions ξ with $H(\xi,\mu) < \infty$.

Denote by $\mathcal{M}_\varphi$ (resp. $\mathcal{M}_\varphi^{\mathrm{erg}}$) the set of φ-invariant (resp. φ-invariant ergodic) probability measures. The set $\mathcal{M}_\varphi$ is a convex set whose extremal points are exactly the ergodic invariant measures $\mu \in \mathcal{M}_\varphi^{\mathrm{erg}}$. Moreover, the entropy map $\mu \in \mathcal{M}_\varphi \mapsto h_{\mathrm{KS}}(\mu) \in \mathbb{R}^+$ is affine [Wal82, Theorem 8.1]. This justifies the following theorem-definition.

Definition 3.3. *The variational entropy of the flow φ is the supremum*

$$h_{\mathrm{var}}(\varphi) = \sup_{\mu\in\mathcal{M}_\varphi} h_{\mathrm{KS}}(\mu) = \sup_{\mu\in\mathcal{M}_\varphi^{\mathrm{erg}}} h_{\mathrm{KS}}(\mu)\,.$$

A measure realizing this supremum, when it exists, is called a measure of maximal entropy *of φ.*

3.2 Katok and Brin-Katok entropies

For $x \in M$, $\varepsilon, T > 0$, let $B(x,\varepsilon,T)$ be the associated *dynamical ball*, i.e.

$$B(x,\varepsilon,T) = \{y \in M :\ \forall t \in [0,T],\ d(\varphi_t(x),\varphi_t(y)) < \varepsilon\}.$$

Definition 3.4. *Let $\mu \in \mathcal{M}_\varphi$, $\alpha \in (0,1)$ and $T, \varepsilon > 0$. A finite set E is said to be $(T,\varepsilon,\alpha,\mu)$*-spanning *if*

$$\mu\left(\cup_{x\in E} B(x,\varepsilon,T)\right) \geq \alpha.$$

Fact 3.5. *Let M be a manifold and μ a probability measure on M. For all $T > 0$, $\varepsilon > 0$, $\alpha \in (0,1)$ there exists a finite $(T,\varepsilon,\alpha,\mu)$-spanning set.*

Proof. As M is exhaustible by compact subsets, there exists a compact subset $K \subset M$ such that $\mu(K) \geq \alpha$. Then $\cup_{x\in K} B(x,\varepsilon,T)$ is an open cover for K. Any finite subcover provides us with a finite $(T,\varepsilon,\alpha,\mu)$-spanning set. □

The following fact is immediate from the definition.

Fact 3.6. *Let $M(T,\varepsilon,\alpha,\mu)$ be the minimal cardinality of a $(T,\varepsilon,\alpha,\mu)$-spanning set. It is a non-decreasing quantity in $\alpha > 0$ and non-increasing in $\varepsilon > 0$.*

Definition 3.7 (Katok entropy [Kat80])**.** *The* Katok entropy *of μ is defined as*

$$h_{\mathrm{Kat}}(\mu) = \inf_{\alpha>0} \sup_{\varepsilon>0} \limsup_{T\to\infty} \frac{1}{T} \log M(T,\varepsilon,\alpha,\mu) \;=\; \lim_{\alpha\to 0} \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log M(T,\varepsilon,\alpha,\mu)\,.$$

Definition 3.8 (Brin-Katok local entropy [BK83])**.** *Let K be a compact subset of M. The* upper, resp. lower, local entropy *on K is defined as*

$$\overline{h}_{\mathrm{loc}}(\mu,K) = \sup_{\varepsilon>0} \operatorname*{ess\,sup}_{x\in K} \limsup_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T}\log\mu\left(B(x,T,\varepsilon)\right) = \lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{x\in K} \limsup_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T}\log\mu\left(B(x,T,\varepsilon)\right)\,,$$

resp.

$$\underline{h}_{\mathrm{loc}}(\mu,K) = \sup_{\varepsilon>0} \operatorname*{ess\,inf}_{x\in K} \liminf_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T}\log\mu\left(B(x,T,\varepsilon)\right) = \lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{x\in K} \liminf_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T}\log\mu\left(B(x,T,\varepsilon)\right)\,.$$

The upper (resp. lower) Brin-Katok *entropy of μ is defined as*

$$\overline{h}_{\mathrm{BK}}(\mu) = \sup_{K\subset M\, compact} \overline{h}_{\mathrm{loc}}(\mu,K)\,,$$

resp.

$$\underline{h}_{\mathrm{BK}}(\mu) = \inf_{K\subset M\, compact} \underline{h}_{\mathrm{loc}}(\mu,K)\,.$$

Remark 3.9. Poincaré recurrence theorem implies that μ-almost every point in K returns to K infinitely often and therefore

$$\limsup_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T}\log\mu\left(B(x,T,\varepsilon)\right)$$

is well defined μ-almost everywhere.

3.3 Gurevic entropy

3.3.1 Definition

A periodic point of φ is a couple (x,T), with $x\in M$ and $T>0$ such that $\varphi_T(x)=x$. A periodic orbit γ is a couple $(\{\varphi_t(x), t\in\mathbb{R}\}, T)$, where (x,T) is a periodic point. The period $T>0$ is denoted by $\ell(\gamma)$. Note that, with this definition, the orbits of (x,T) and (x,nT), for $n\geq 2$, are distinct. A parametrized periodic orbit is a periodic map from $\mathbb{R}$ to M, still denoted by γ, of the form

$$s\in\mathbb{R}\mapsto\gamma(s)=\varphi_s(\varphi_{t_0}(x))\,,\quad \text{for a fixed real number } t_0\in\mathbb{R}.$$

By abuse of notation, we speak of a point $z\in\gamma$ instead of a point in the image of the orbit associated with γ. We denote by $\mathcal{P}$ the set of periodic orbits of the flow φ.

Let $K\subset M$ be a compact set. Given any $x\in\gamma$, denote by $\ell(\gamma\cap K)=|\{t\in[0,\ell(\gamma)],\, \varphi_t(x)\in K\}|$ the length of the intersection of the periodic orbit γ with the set K. This quantity does not depend on the point x used in its definition.

Given any $0<T_0\leq T_1$, we define

$$\mathcal{P}_K(T_0,T_1)=\{\gamma\in\mathcal{P},\, \ell(\gamma)\in[T_0,T_1], \gamma\cap K\neq\emptyset\}\,.$$

Define also

$$\mathcal{P}_K(T_0)=\{\gamma\in\mathcal{P},\, \ell(\gamma)\leq T_0, \gamma\cap K\neq\emptyset\}\,.$$

Theorem 3.10. *Let φ be a H-flow on (M,d). Let $K\subset M$ be a compact subset with nonempty interior and $\sigma>0$. The quantity*

$$h_{\mathrm{Gur}}(\varphi,K,\sigma)=\limsup_{L\to\infty}\frac{1}{L}\log\#\mathcal{P}_K(L,L+\sigma) \tag{5}$$

does depend neither on $\sigma>0$ nor on K. It is called the Gurevic entropy of the flow φ *and denoted by $h_{\mathrm{Gur}}(\varphi)$.*

Moreover, for every compact subset $K\subset M$ with nonempty interior, and every $\sigma\geq 5\tau_K$, with τ_K being the period of the shortest periodic orbit with period at least 1 that intersects K, the Gurevic entropy is a true limit:

$$h_{\mathrm{Gur}}(\varphi)=\lim_{L\to+\infty}\frac{1}{L}\log\#\mathcal{P}_K(L,L+\sigma)\,. \tag{6}$$

The first part of the Theorem is relatively elementary and classical, and follows from Facts 3.12 and 3.13. The second part is more difficult and follows from subadditivity properties proved in Section 3.3.3.

Before the proof of the theorem, let us give an immediate corollary, that will be useful in Section 7.2.

Corollary 3.11. *Under the assumptions of Theorem 3.10, if $h_{\mathrm{Gur}}(\varphi)>0$, then*

$$\lim_{L\to\infty}\frac{\#\mathcal{P}_K\left(\frac{L+\sigma}{2}\right)}{\#\mathcal{P}_K(L,L+\sigma)}=0\,.$$

3.3.2 The Gurevic entropy does depend neither on K nor on σ

We prove here the first part of Theorem 3.10.

Fact 3.12. *Under the assumptions of Theorem 3.10, the Gurevic entropy satisfies*

$$h_{\mathrm{Gur}}(\varphi,K,\sigma)=\limsup_{L\to+\infty}\frac{1}{L}\log\#\mathcal{P}_K(L)\,.$$

In particular, $h_{\mathrm{Gur}}(\varphi,K,\sigma)=h_{\mathrm{Gur}}(\varphi,K)$ does not depend on the constant σ.

Proof. Fix some $\sigma>0$. As $\mathcal{P}_K(L,L+\sigma)\subset\mathcal{P}_K(L+\sigma)$, the inequality

$$h_{\mathrm{Gur}}(\varphi,K,\sigma)\leq\limsup_{L\to+\infty}\frac{1}{L}\log\#\mathcal{P}_K(L)$$

is immediate.

The proof of $h_{\mathrm{Gur}}(\varphi,K,\sigma)\geq\limsup_{L\to+\infty}\frac{1}{L}\log\#\mathcal{P}_K(L)$ is classical. Let $\delta>0$ and denote as h the entropy $h_{\mathrm{Gur}}(\varphi,K,\sigma)$. If $h=+\infty$, there is noting to prove. We now assume $h<\infty$. There exists $T_0>0$ such that for $T>T_0$, $\#\mathcal{P}_K(T,T+\sigma)\leq e^{T(h+\delta)}$. Let $N_T=\left\lfloor\frac{T-T_0}{\sigma}\right\rfloor-1$. We have

$$\mathcal{P}_K(T)\ \subset\ \mathcal{P}_K(T_0)\cup\bigcup_{n=0}^{N_T}\mathcal{P}_K(T_0+n\sigma,T_0+(n+1)\sigma)\,.$$

Therefore

$$\begin{aligned}\#P_K(T)&\leq P_K(T_0)+\sum_{n=0}^{N_T}\#\mathcal{P}_K(T_0+n\sigma,T_0+(n+1)\sigma)\\&\leq\mathcal{P}_K(T_0)+\sum_{n=0}^{N_T}e^{(T_0+n\sigma)(h+\delta)}\\&\leq\mathcal{P}_K(T_0)+\frac{T}{\sigma}e^{T(h+\delta)}.\end{aligned}$$

It follows that

$$\limsup_{T\to\infty}\frac{1}{T}\log\#P_K(T)\leq h_{\mathrm{Gur}}(\varphi,K,C)+\delta\,.$$

As the inequality holds for any $\delta>0$, the result follows. □

Fact 3.13. *If φ is a H-flow, and $K \subset M$ is a compact set with nonempty interior, then $h_{\mathrm{Gur}}(\varphi, K) = h_{\mathrm{Gur}}(\varphi)$. In particular, the Gurevic entropy $h_{\mathrm{Gur}}(\varphi)$ does not depend on K.*

Proof. If $K \subset K'$ are any two compact sets, we have $h_{\mathrm{Gur}}(\varphi, K) \leq h_{\mathrm{Gur}}(\varphi, K')$.

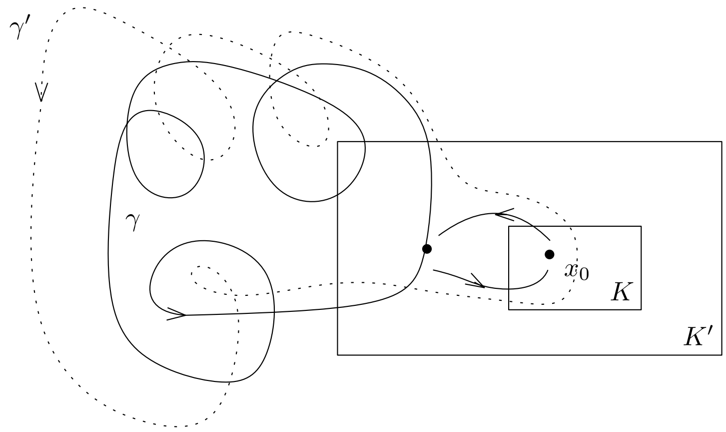

Figure 3.1: The Gurevic entropy does not depend on the compact

The reverse inequality can be obtained, as explained below, from the transitivity, finite exact shadowing, and closing lemma properties as gathered in Lemma 2.15.

Let $\sigma' > 0$. Choose $\varepsilon > 0$ and $x_0 \in K$ such that the open ball $B_0 = B(x_0, \varepsilon)$ is included in K. By selecting a smaller ε if necessary, we may also assume that 2ε is adapted to the separation of orbits (Lemma 2.13) for parameters $\nu = \tau_1 = 1$. This lemma also gives us τ_0.

Let $0 < \eta < \varepsilon/2$ and $T_{\min}$ be given by Lemma 2.15 with parameters K', $N = 3$, $\delta = \varepsilon/2$ and $\nu = 1$. Cover K' with finitely many open balls $B_i = B(x_i, \eta)$. Without loss of generality, assume that $x_0 \in B_0$. By transitivity (2.14), there exist pieces of orbits c_i with length $\ell_i \geq 1$ from B_i to B_0 and c_i' with length $\ell_i' \geq 1$ from B_0 to B_i. Set $L_0 = \max\{\ell_i, \ell_i'\}$. We may assume that ℓ_i and ℓ_i' are bounded below by $T_{\min}$.

Given a periodic orbit γ in $\mathcal{P}_{K'}(L, L+\sigma')$, we may assume, without loss of generality, that $\gamma(0) \in K$. By Lemma 2.15, one can concatenate the pieces of orbits γ (starting at $\gamma(0)$), c_i and c_i' to get an $\varepsilon/2$-close periodic orbit γ' that goes through $B(x_0, \varepsilon) \subset K$, and has length in $[\ell(\gamma) + \ell_i + \ell_i' - 1, \ell(\gamma) + \ell_i + \ell_i' + 1] \subset [L, L + \sigma' + 2L_0 + 1]$.

This construction gives us a map from $\mathcal{P}_{K'}(L, L + \sigma')$ to $\mathcal{P}_K(L, L + \sigma)$, for $\sigma = \sigma' + 2L_0 + 1$ depending on K and K'. Let us control the cardinal of its preimages. Let γ_0 and γ_1 be periodic orbits in $\mathcal{P}_{K'}(L, L + C')$ of period $\ell(\gamma_0)$ and $\ell(\gamma_1)$, lying in the preimage of $\gamma \in \mathcal{P}_K(L, L + \sigma)$. One may choose the orgins of γ_0, γ_1 and γ such that there exist t_1 satisfying the following

- for all $0 \leq t \leq \ell(\gamma_0)$, we have $d(\gamma_0(t), \gamma(t)) \leq \varepsilon/2$
- for all $0 \leq t \leq \ell(\gamma_1)$, we have $d(\gamma_1(t), \gamma(t + t_1)) \leq \varepsilon/2$.

Therefore, for all $0 \leq t \leq \min(\ell(\gamma_0), \ell(\gamma_1))$, by the previous inequalities and by (2),

$$d(\gamma_0(t), \gamma_1(t)) \leq \varepsilon + b|t_1|.$$

Thus if $|t_1| \leq \varepsilon/b$ and $|\ell(\gamma_1) - \ell(\gamma_0)| \leq \tau_0$, Lemma 2.13 proves that $\gamma_0 = \gamma_1$. As $|t_1| \leq L + \sigma$ and $|\ell(\gamma_1) - \ell(\gamma_0)| \leq \sigma'$, we get the inequality

$$\#\mathcal{P}_{K'}(L, L + \sigma') \leq \left\lceil \frac{(L + \sigma)b}{\varepsilon} \right\rceil \left\lceil \frac{\sigma'}{\tau_0} \right\rceil \#\mathcal{P}_K(L, L + \sigma)\,.$$

The equality $h_{\mathrm{Gur}}(\varphi, K) = h_{\mathrm{Gur}}(\varphi, K')$ follows immediately. This proves $h_{\mathrm{Gur}}(\varphi, K)$ does not depend on K. □

3.3.3 First subadditivity properties

We now prove a subadditivity property, that is a comparison between counts of orbits of periods L_1, L_2 and $L_1 + L_2$. This will be a key ingredient to prove the last part of Theorem 3.10 which says that the Gurevic entropy is a true limit.

Recall that τ_K is the period of the shortest periodic orbit with period ≥ 1 that intersects the interior of K.

Proposition 3.14. *Let φ be a H-flow on M. Let $K \subset M$ be a compact subset with nonempty interior. Let $\sigma_0 \geq 4\tau_K$. There exist constants D and σ such that for all $L_1, L_2 \gg 1$,*

$$\#\mathcal{P}_K(L_1, L_1 + \sigma_0)\,\#\mathcal{P}_K(L_2, L_2 + \sigma_0) \leq D\,(L_1 + L_2)\,\#\mathcal{P}_K(L_1 + L_2 + \sigma, L_1 + L_2 + \sigma + \sigma_0)\,. \qquad (7)$$

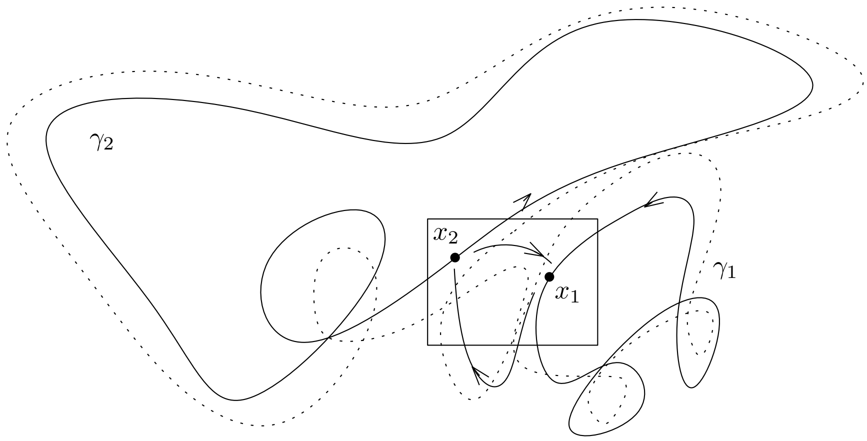

Figure 3.2: First subadditivity property

Proof. See Figure 3.2. Fix the compact set K. We will build a map

$$f\colon \mathcal{P}_K(L_1, L_1 + \sigma_0) \times \mathcal{P}_K(L_2, L_2 + \sigma_0) \to \mathcal{P}_K(L_1 + L_2 + \sigma, L_1 + L_2 + \sigma + \sigma_0)\,,$$

whose preimages, for L_1, L_2 large enough and a suitable choice of the constant σ, have a cardinality bounded by $D \times (L_1 + L_2)$, for a suitable constant $D > 0$. Inequality 7 will follow immediately.

Step 1. Construction of f.

Lemma 2.13 associates with $\nu = 1$ some $\tau_0 > 0$, and for $\tau_1 = 2\tau_0 + 2\tau_K$ some $\varepsilon > 0$. Lemma 2.16 applied to the compact set K associates to $\nu = 1$, $\delta = \frac{\varepsilon}{4}$ and $N = 2$ some numbers $T_{\min} > 0$ and $\sigma' > 0$.

Consider $L_1, L_2 \geq T_{\min}$, and a pair of periodic orbits $\gamma_1 \in \mathcal{P}_K(L_1, L_1{+}\sigma_0)$ and $\gamma_2 \in \mathcal{P}_K(L_2, L_2{+}\sigma_0)$, with respective lengths $\ell(\gamma_1)$ and $\ell(\gamma_2)$. The periodic orbit $f(\gamma_1, \gamma_2)$ is defined as follows. As γ_1 and γ_2 intersect K, we can reparametrize them so that their origins $\gamma_1(0), \gamma_2(0)$ belong to K. Apply Lemma 2.16 with $x_1 = \gamma_1(0), x_2 = \gamma_2(0), T_1 = \ell(\gamma_1), T_2 = \ell(\gamma_2)$ and

$$S = S(\gamma_1, \gamma_2) = \frac{L_1 + L_2 + 2\sigma_0 - \ell(\gamma_1) - \ell(\gamma_2)}{2} + \sigma' + \tau_K \geq \sigma'\,.$$

We get a periodic orbit $\gamma = f(\gamma_1, \gamma_2)$ that intersects K, with length

$$\ell(\gamma) \in [\ell(\gamma_1) + \ell(\gamma_2) + 2S - \tau_K - 1, \ell(\gamma_1) + \ell(\gamma_2) + 2S + \tau_K + 1] \subset [L_1 + L_2 + \sigma, L_1 + L_2 + \sigma + \sigma_0]\,,$$

where $\sigma = 2\sigma_0 + 2\sigma' + \tau_K - 1$. Moreover, there exists $\tau \in [S - \tau_K, S + \tau_K]$ such that

- for every $s \in [0, \ell(\gamma_1)]$, we have $d(\gamma_1(s), \gamma(s)) < \dfrac{\varepsilon}{4}$;
- for every $s \in [0, \ell(\gamma_2)]$, we have $d(\gamma_2(s), \gamma(\ell_1 + \tau + s)) < \dfrac{\varepsilon}{4}$.

The periodic orbit $f(\gamma_1,\gamma_2)=\gamma$ belongs therefore to $\mathcal{P}_K(L_1+L_2+\sigma, L_1+L_2+\sigma+\sigma_0)$. Moreover, our construction provides an origin of the orbit γ, i.e. a marked point on its image.

Step 2. Bound on the cardinality of each preimage.

Assume that $\gamma_1,\widetilde{\gamma}_1 \in \mathcal{P}_K(L_1,L_1+\sigma_0)$ and $\gamma_2,\widetilde{\gamma}_2 \in \mathcal{P}_K(L_2,L_2+\sigma_0)$ are such that $f(\gamma_1,\gamma_2) = f(\widetilde{\gamma}_1,\widetilde{\gamma}_2)=\gamma$. The constructions of $f(\gamma_1,\gamma_2)$ and $f(\widetilde{\gamma}_1,\widetilde{\gamma}_2)$ lead to the same orbit γ by assumption, but with maybe different origins. Without loss of generality, we can shift the parametrization of γ so that the origin given by the construction of $\gamma=f(\gamma_1,\gamma_2)$ starting from γ_1 and γ_2 is $\gamma(0)$. Let $\gamma(s_0)$ be the origin of γ given by the construction of $\gamma=f(\widetilde{\gamma}_1,\widetilde{\gamma}_2)$ starting from $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$.

Step 2.a Bound on the cardinality of each preimage, when the lengths are prescribed.

We prove the following statement. If the orbits satisfy $f(\gamma_1,\gamma_2)=f(\widetilde{\gamma}_1,\widetilde{\gamma}_2)$ and

$$|s_0| < \frac{\varepsilon}{4b}\,, \qquad |\ell(\gamma_1)-\ell(\widetilde{\gamma}_1)| < \tau_0\,, \qquad |\ell(\gamma_2)-\ell(\gamma_2)| < \tau_0\,, \tag{8}$$

where b is the constant of property (2) in definition 2.9, then the orbits coincide: $(\gamma_1,\gamma_2)=(\widetilde{\gamma}_1,\widetilde{\gamma}_2)$.

By the definition of the function f, the following holds:

- for every $s\in[0,\ell_1]$, we have $d(\gamma(s),\gamma_1(s)) < \dfrac{\varepsilon}{4}$;
- for every $s\in[0,\widetilde{\ell}_1]$, we have $d(\gamma(s_0+s),\widetilde{\gamma}_1(s)) < \dfrac{\varepsilon}{4}$.

Thus, for every $s\in[0,\min(\ell(\gamma_1),\ell(\widetilde{\gamma}_1)]$ we obtain

$$d(\gamma_1(s),\widetilde{\gamma}_1(s)) \le d(\gamma_1(s),\gamma(s)) + d(\gamma(s),\gamma(s_0+s)) + d(\gamma(s_0+s),\widetilde{\gamma}_1(s)) < \frac{\varepsilon}{4} + |s_0|b + \frac{\varepsilon}{4} < \varepsilon\,.$$

By Lemma 2.13, we deduce that $\gamma_1=\widetilde{\gamma}_1$.

Again by the construction of f, there exist $\tau\in[S(\gamma_1,\gamma_2)-\tau_K, S(\gamma_1,\gamma_2)+\tau_K]$ and $\widetilde{\tau}\in[S(\widetilde{\gamma}_1,\widetilde{\gamma}_2)-\tau_K, S(\widetilde{\gamma}_1,\widetilde{\gamma}_2)+\tau_K]$ such that

- for every $s\in[0,\ell(\gamma_2)]$, we have $d(\gamma(\ell(\gamma_1)+\tau+s),\gamma_2(s)) < \dfrac{\varepsilon}{4}$;
- for every $s\in[0,\ell(\widetilde{\gamma}_2)]$, we have $d(\gamma(s_0+\ell(\widetilde{\gamma}_1)+\widetilde{\tau}+s),\widetilde{\gamma}_2(s)) < \dfrac{\varepsilon}{4}$

Let $\bar{s}=\ell(\widetilde{\gamma}_1)-\ell(\gamma_1)+\widetilde{\tau}-\tau$. Up to swapping γ_1 and $\widetilde{\gamma}_1$, we may assume $\bar{s}\ge 0$. Moreover, we have $|\bar{s}|\le 2\sigma_0+2\tau_K$. Then, for every $s\in[0,\min(\ell(\gamma_2)-\bar{s},\ell(\widetilde{\gamma}_2))]$, we have

$$\begin{aligned} d(\gamma_2(\bar{s}+s),\widetilde{\gamma}_2(s)) \le & d(\gamma_2(\bar{s}+s),\gamma(\ell(\gamma_1)+\tau+\bar{s}+s)) + d(\gamma(\ell(\gamma_1)+\tau+\bar{s}+s),\gamma(s_0+\ell(\widetilde{\gamma}_1)+\widetilde{\tau}+s)) \\ & + d(\gamma(s_0+\ell(\widetilde{\gamma}_1)+\widetilde{\tau}+s),\widetilde{\gamma}_2(s)) \\ < & \frac{\varepsilon}{4} + |s_0|b + \frac{\varepsilon}{4} < \varepsilon\,. \end{aligned}$$

By Lemma 2.13, since $|\bar{s}|\le 2\sigma_0+2\tau_K=\tau_1$, we conclude that also $\gamma_2=\widetilde{\gamma}_2$.

Step 2.b. Conclusion.

So far, we have shown that, as soon as (γ_1,γ_2) and $(\widetilde{\gamma}_1,\widetilde{\gamma}_2)$ satisfy (8), if they have the same image under f, then they are the same periodic orbits. Since

$$|s_0| \le L_1+L_2+\sigma+\sigma_0\,, \qquad |\ell(\gamma_1)-\ell(\widetilde{\gamma}_1)| \le \sigma_0\,, \qquad |\ell(\gamma_2)-\ell(\gamma_2)| \le \sigma_0\,,$$

we deduce that the cardinality of any preimage through f of a periodic orbit in $\mathcal{P}_K(L_1+L_2+\sigma, L_1+L_2+\sigma+\sigma_0))$ is bounded by

$$\left\lceil\frac{\sigma_0}{\tau_0}\right\rceil^2 \left\lceil\frac{4b}{\varepsilon}(L_1+L_2+\sigma+\sigma_0)\right\rceil\,.$$

Choose the constant D so that

$$\left\lceil\frac{\sigma_0}{\tau_0}\right\rceil^2 \left\lceil\frac{4b}{\varepsilon}(L_1+L_2+\sigma+\sigma_0)\right\rceil \le D(L_1+L_2)\,.$$

The desired bound 7 follows immediately. □

We will use the subadditivity property shown in Proposition 3.14 to deduce that the exponential growth rate of the cardinality of the set of periodic orbits intersecting K has a limit. We need the following result, see [dBE52, Theorem 23].

Theorem 3.15 (de Bruijn–Erdös)**.** *Let $t > 0 \mapsto \psi(t)$ be a positive and increasing map. Assume that $\int_1^\infty \frac{\psi(t)}{t^2}\,dt < \infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$u_{n+m} \le u_n + u_m + \psi(n+m) \qquad \text{for } \frac{n}{2} \le m \le 2n\,.$$

Then $\lim_{n\to\infty} \frac{u_n}{n} = L$ for some $L \in \mathbb{R}\cup\{-\infty\}$.

We can now conclude the proof of Theorem 3.10, by showing that (6) holds.

Proof of (6) *in Theorem 3.10.* Fix a compact set K with nonempty interior. Let $\hat\sigma$ be the constant σ given by Proposition 3.14 for $\sigma_0 = 4\tau_K$ (with respect to the notation of Proposition 3.14).

Step 1. The sequence $\frac{1}{n}\log\#\mathcal{P}_K(n-\hat\sigma, n-\hat\sigma+4\tau_K)$ converges to $h_{\mathrm{Gur}}(\varphi)$.

Step 1.a The sequence $\frac{1}{n}\log\#\mathcal{P}_K(n-\hat\sigma, n-\hat\sigma+4\tau_K)$ has a limit.

Define a sequence $(u_n)_{n\in\mathbb{N}}$ as

$$u_n = -\log\#\mathcal{P}_K(n-\hat\sigma, n-\hat\sigma+4\tau_K)$$

for $n\in\mathbb{N}, n\gg 1$. By Proposition 3.14, for n,m large enough, the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies

$$u_{n+m} \le \log(D(n+m-2\hat\sigma)) + u_n + u_m \le \log(D(n+m)+1) + u_n + u_m\,.$$

Observe that the function $\psi : t>0 \mapsto \log(Dt+1)$ is positive and increasing for $t>0$ and satisfies $\int_1^\infty \frac{\log(Dt+1)}{t^2}\,dt < \infty$. Therefore, by Theorem 3.15, the sequence $(\frac{u_n}{n})_{n\in\mathbb{N}}$ converges and $\lim_{n\to\infty}\frac{-u_n}{n} \le h_{\mathrm{Gur}}(\varphi)$.

Step 1.b. The limit is $h_{\mathrm{Gur}}(\varphi)$.

We now prove $\lim_{n\to\infty}\frac{-u_n}{n} = h_{\mathrm{Gur}}(\varphi)$. It is enough to prove $\lim_{n\to\infty}\frac{-u_n}{n} \ge h_{\mathrm{Gur}}(\varphi)$. Let $(L_n)_{n\in\mathbb{N}}$ be such that

$$\lim_{n\to+\infty}\frac{\log\#\mathcal{P}_K(L_n, L_n+3\tau_K)}{L_n} = h_{\mathrm{Gur}}(\varphi)\,.$$

Let $(N_n)_{n\in\mathbb{N}}$ be the sequence of integers such that

$$N_n - \hat\sigma \le L_n < N_n - \hat\sigma + 1.$$

Then (as $\tau_K \ge 1$)

$$\mathcal{P}_K(L_n, L_n+3\tau_K) \subset \mathcal{P}_K(N_n-\hat\sigma, N_n-\hat\sigma+4\tau_K)\,.$$

Therefore

$$h_{\mathrm{Gur}}(\varphi) = \lim_{n\to\infty}\frac{\log\#\mathcal{P}_K(L_n, L_n+3\tau_K)}{L_n} \le \lim_{n\to\infty}\frac{\log\#\mathcal{P}_K(N_n-\hat\sigma, N_n-\hat\sigma+4\tau_K)}{N_n-\hat\sigma} = \lim_{n\to+\infty}\frac{-u_{N_n}}{N_n}\,.$$

Thus $\lim_{n\to\infty}\frac{-u_n}{n} = h_{\mathrm{Gur}}(\varphi)$.

Step 2. The sequence $\frac{1}{L_n}\log\#\mathcal{P}_K(L_n, L_n+5\tau_K)$ converges to $h_{\mathrm{Gur}}(\varphi)$.

Let $(L_n)_n$ be a sequence such that $L_n\to\infty$. We now prove that $\left(\frac{\log\#\mathcal{P}_K(L_n, L_n+5\tau_K)}{L_n}\right)_{n\in\mathbb{N}}$ can only have $h_{\mathrm{Gur}}(\varphi)$ as subsequential limit and therefore converges to $h_{\mathrm{Gur}}(\varphi)$. Recall that $\hat\sigma > 0$ is the constant given by Proposition 3.14 for $\sigma_0 = 4\tau_K$. Let $(N_n)_{n\in\mathbb{N}}$ be the sequence of integers such that

$$L_n \le N_n - \hat\sigma < L_n + 1.$$

From the previous step, we know that $\left(\frac{\log\#\mathcal{P}_K(N_n-\hat\sigma, N_n-\hat\sigma+4\tau_K)}{N_n}\right)_{n\in\mathbb{N}}$ converges to $h_{\mathrm{Gur}}(\varphi)$. Moreover, as $\tau_K \ge 1$,

$$\mathcal{P}_K(N_n-\hat\sigma, N_n-\hat\sigma+4\tau_K) \subset \mathcal{P}_K(L_n, L_n+5\tau_K)$$

and we obtain

$$h_{\mathrm{Gur}}(\varphi) = \lim_{n\to\infty} \frac{\log \#\mathcal{P}_K(N_n - \hat{\sigma}, N_n - \hat{\sigma} + 4\tau_K)}{N_n} \leq \limsup_{n\to\infty} \frac{\log \#\mathcal{P}_K(L_n, L_n + 5\tau_K)}{L_n} \leq h_{\mathrm{Gur}}(\varphi)\,.$$

Thus the sequence $\left(\frac{\log \#\mathcal{P}_K(L_n, L_n + 5\tau_K)}{L_n}\right)_{n\in\mathbb{N}}$ converges to $h_{\mathrm{Gur}}(\varphi)$ and

$$\lim_{L\to+\infty} \frac{\log \#\mathcal{P}_K(L, L + 5\tau_K)}{L} = h_{\mathrm{Gur}}(\varphi)\,.$$

Step 3. General case.
Let $\sigma \geq 5\tau_K$. As

$$\frac{\log \#\mathcal{P}_K(L, L + 5\tau_K)}{L} \leq \frac{\log \#\mathcal{P}_K(L, L + \sigma)}{L}$$

and

$$\limsup_{L\to\infty} \frac{\log \#\mathcal{P}_K(L, L + \sigma)}{L} = h_{\mathrm{Gur}}(\varphi),$$

we obtain

$$\lim_{L\to+\infty} \frac{\log \#\mathcal{P}_K(L, L + \sigma)}{L} = h_{\mathrm{Gur}}(\varphi)\,.$$

□

3.4 Entropies at infinity

Definition 4.11 of *Strong positive recurrence* involves a notion of *entropy at infinity*. We introduce here different notions of entropy at infinity and compare them in section 4. The rough idea is to measure the exponential growth rate of the dynamics outside a large compact set K and then let K grow to exhaust M.

More precisely, for defining Gurevic entropy at infinity, we consider periodic orbits that intersect K but spend only a small proportion of time in K. For the variational entropy at infinity, we shall consider the supremum of measured entropies of probability measures that give a small measure to a large compact set K .

3.4.1 Gurevic entropy at infinity

Definition 3.16. *Let $K \subset M$ be a compact subset with nonempty interior. Let $\alpha > 0$, $L > 0$ and $\sigma > 0$. Define*

$$\mathcal{P}_K^{\alpha}(L, L+\sigma) = \{\gamma \in \mathcal{P}_K(L, L+\sigma) : \ \ell(\gamma \cap K) < \alpha \ell(\gamma)\}\,,$$

and

$$h_{\mathrm{Gur}}^{K,\alpha}(\varphi) = \limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_K^{\alpha}(L, L+\sigma)\,.$$

The Gurevic entropy at infinity *of the flow φ is defined by*

$$h_{\mathrm{Gur}}^{\infty}(\varphi) := \inf_K \lim_{\alpha\to 0} h_{\mathrm{Gur}}^{K,\alpha}(\varphi)\,,$$

where the infimum is taken over all compact subsets $K \subset M$ with nonempty interior.

Fact 3.17. *Under the hypotheses of the definition,* $\limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_K^{\alpha}(L, L+\sigma)$ *does not depend on σ and therefore $h_{\mathrm{Gur}}^{K,\alpha}(\varphi)$ is well-defined.*

Proof. By definition, if $\sigma \leq \sigma'$ then

$$\limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_K^{\alpha}(L, L+\sigma) \leq \limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_K^{\alpha}(L, L+\sigma').$$

We now prove

$$\limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_K^{\alpha}(L, L+2\sigma) \leq \limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_K^{\alpha}(L, L+\sigma).$$

This is enough to conclude the proof of the fact. We have

$$\mathcal{P}_K(L, L+2\sigma) \subset \mathcal{P}_K(L, L+\sigma) \cup \mathcal{P}_K(L+\sigma, L+2\sigma)$$

therefore

$$\#\mathcal{P}_K(L, L+2\sigma) \leq 2\max\left(\#\mathcal{P}_K(L, L+\sigma), \#\mathcal{P}_K(L+\sigma, L+2\sigma)\right).$$

As

$$\limsup_{L\to\infty} \frac{1}{L}\log \#\mathcal{P}_K(L, L+\sigma) = \limsup_{L\to\infty} \frac{1}{L}\log\left(\max\left(\#\mathcal{P}_K(L, L+\sigma), \#\mathcal{P}_K(L+\sigma, L+2\sigma)\right)\right)$$

we have

$$\limsup_{L\to\infty} \frac{1}{L}\log \#\mathcal{P}^\alpha_K(L, L+2\sigma) \leq \limsup_{L\to\infty} \frac{1}{L}\log \#\mathcal{P}^\alpha_K(L, L+\sigma)$$

as required. □

Fact 3.18. *Let K be a compact subset with nonempty interior. The map $\alpha > 0 \mapsto h^{K,\alpha}_{\mathrm{Gur}}(\varphi)$ is non-decreasing. Moreover, let K' be a compact subset such that $K \subset \overset{\circ}{K'}$. We then have*

$$h^{K',\alpha}_{\mathrm{Gur}}(\varphi) \leq h^{K,2\alpha}_{\mathrm{Gur}}(\varphi).$$

Proof. The first assertion is a direct consequence of the definition.

For the second assertion, we follow the arguments of the proof of Fact 3.13. Let $\sigma > 0$. We import the notation from the proof of Fact 3.13. Let us assume additionally that $\varepsilon \leq d(\partial K, \partial K')$. We associate to any periodic orbit of $\mathcal{P}^\alpha_{K'}(L, L+\sigma)$ a periodic orbit of $\mathcal{P}_K(L, L+\sigma')$, for some $\sigma' \geq \sigma$, which spends a time at most $\alpha L + \sigma'$ in K. For L large enough, $\alpha L + \sigma' \leq 2\alpha L$ and we obtain a map from $\mathcal{P}^{2\alpha}_{K'}(L, L+\sigma')$ to $\mathcal{P}^\alpha_K(L, L+\sigma)$. The bound

$$\left\lceil \frac{\sigma_0}{\tau_0} \right\rceil^2 \left\lceil \frac{4b}{\varepsilon}(L_1 + L_2 + \sigma + \sigma_0) \right\rceil.$$

on the number of preimages given in proof of Fact 3.13 remain valid. Therefore, there exists some $D > 0$ such that, for $L \gg 1$

$$\#\mathcal{P}^\alpha_{K'}(L, L+\sigma) \leq L\#\mathcal{P}^{2\alpha}_K(L, L+\sigma').$$

Since $h^{K,\alpha}_{\mathrm{Gur}}(\varphi)$ does not depend on the constant σ, the result follows. □

3.4.2 Variational entropy at infinity

As in [GST23], we introduce the variational entropy at infinity.

Definition 3.19. *The variational entropy at infinity of the flow φ is*

$$\begin{aligned} h^\infty_{\mathrm{var}}(\varphi) &:= \lim_{\varepsilon\to 0}\inf_K \sup\{h_{\mathrm{KS}}(\mu) :\ \mu \in \mathcal{M}_\varphi, \mu(K) \leq \varepsilon\} \\ &= \lim_{\varepsilon\to 0}\inf_K \sup\{h_{\mathrm{KS}}(\mu) :\ \mu \in \mathcal{M}^{\mathrm{erg}}_\varphi, \mu(K) \leq \varepsilon\}, \\ &= \lim_{\varepsilon\to 0}\inf_K \sup\{h_{\mathrm{Kat}}(\mu) :\ \mu \in \mathcal{M}^{\mathrm{erg}}_\varphi, \mu(K) \leq \varepsilon\}, \end{aligned}$$

where the infimum is taken over all compact subsets $K \subset M$.

The equality between the two first quantities on the right follows from the fact that the entropy map $\mu \in \mathcal{M}_\varphi \mapsto h_{\mathrm{KS}}(\mu)$ is convex and ergodic measures are the extremal points of $\mathcal{M}_\varphi$. The last equality follows from Theorem 4.2.

Remark 3.20. Observe that, in Definition 3.19, the quantity $\sup\{h_{\mathrm{Kat}}(\mu) :\ \mu \in \mathcal{M}^{\mathrm{erg}}_\varphi, \mu(K) \leq \varepsilon\}$ (as well as the others appearing in the equalities) is non-decreasing in ε and non-increasing in K. Therefore, it is possible to invert the order of $\lim_{\varepsilon\to 0}$ and $\inf_K$, i.e.,

$$\lim_{\varepsilon\to 0}\inf_K \sup\{h_{\mathrm{Kat}}(\mu) :\ \mu \in \mathcal{M}^{\mathrm{erg}}_\varphi, \mu(K) \leq \varepsilon\} = \inf_K \lim_{\varepsilon\to 0} \sup\{h_{\mathrm{Kat}}(\mu) :\ \mu \in \mathcal{M}^{\mathrm{erg}}_\varphi, \mu(K) \leq \varepsilon\}.$$

4 Comparison of entropies

4.1 Comparison of measure-theoretic entropies

Our main theorem (Theorem 1.1) establishes the existence of a measure that maximizes all notions of measured entropy. This measure will be obtained as a limit of averages of periodic measures. As a consequence, on the one hand, it is a priori not known to be ergodic, and on the other hand, its Katok and Brin-Katok entropies are the only ones that are computable. Therefore, we will need general statements to be able to compare all kinds of entropies.

Proposition 4.1 (Riquelme [Riq18]). *Let φ be a Lipschitz flow on a manifold M and $\mu \in \mathcal{M}_\varphi$ an invariant probability measure. If $K \subset M$ is a compact subset, then, for $\varepsilon > 0$ small enough,*

$$\int_K \limsup_{\substack{n\to+\infty \\ \varphi^n(x)\in K}} -\frac{1}{n} \log \mu(B(x,\varepsilon,n))\, d\mu \leq h_{\mathrm{KS}}(\mu)\,.$$

This proposition is proven in [Riq18] (see also [GST23, Appendix A]) when μ is ergodic, and the non ergodic case, very similar, is only briefly mentioned. As it is crucial for us, we give a proof of this statement.

Proof. In [Riq18, Theorem 2.10], Riquelme uses a proposition due to Ledrappier, see [Led13, Proposition 6.3], to build a partition $\mathcal{P}$ such that (without ergodicity), for μ-almost every x, for every n such that $\varphi^n(x) \in K$, we have $\mathcal{P}^n(x) \subset B(x,n,\varepsilon)$, where for a partition $\mathcal{P}$ we denote by $\mathcal{P}(x)$ the element of the partition containing x, and where $\mathcal{P}^n$ is the measurable partition consisting of all possible intersections of elements of $\varphi^{-i}\mathcal{P}$, for $i = 0, \dots, n-1$. It follows that

$$\int_K \limsup_{\substack{n\to+\infty \\ \varphi^n(x)\in K}} -\frac{1}{n} \log\left(\mu(B(x,\varepsilon,n)\right)\, d\mu \leq \int_M \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(\mathcal{P}^n(x)\right)\, d\mu\,.$$

The non-ergodic version of Shannon-McMillan-Breiman theorem ensures that $-\frac{1}{n}\log\mu\left(\mathcal{P}^n(x)\right)$ converges almost surely, so that the right hand side is in fact a true (almost sure) limit. This theorem is stated without proof in [Mn87, Theorem 1.2, Chapter IV]. It is stated and proven in [Kre85, Theorem 2.5] in a more general framework, and the proof of [Pet83, Theorem 2.3, p. 261] in the ergodic case adapts almost *verbatim* to the non-ergodic case.

By Fatou's Lemma, we get

$$\int_M \lim_{n\to\infty} -\frac{1}{n} \log \mu\left(\mathcal{P}^n(x)\right)\, d\mu \leq \liminf_{n\to\infty} \int_M -\frac{1}{n} \log \mu(\mathcal{P}^n(x))\, d\mu\,.$$

By definition of the entropy of a partition, we have

$$\int_M -\frac{1}{n} \log \mu\left(\mathcal{P}^n(x)\right)\, d\mu = \frac{1}{n} H(\mathcal{P}^n,\mu)\,,$$

and this quantity converges to $h(\mu,\mathcal{P}) \leq h_{\mathrm{KS}}(\mu)$. □

Theorem 4.2 (Brin-Katok [BK83], Katok [Kat80], Riquelme [Riq18]). *Let φ be a Lipschitz flow on a complete Riemannian manifold M. Let $\mu \in \mathcal{M}_\varphi^{\mathrm{erg}}$. Then*

$$h_{\mathrm{Kat}}(\mu) = \underline{h}_{\mathrm{BK}}(\mu) = \overline{h}_{\mathrm{BK}}(\mu) = h_{\mathrm{KS}}(\mu)\,.$$

Proof. The inequality $h_{\mathrm{KS}}(\mu) \leq h_{\mathrm{Kat}}(\mu)$ is stated in [Kat80] in the compact case, but the proof does not use compactness. The inequality $h_{\mathrm{KS}}(\mu) \leq \underline{h}_{\mathrm{BK}}(\mu)$ is stated in [BK83] in the compact case but the proof does not use compactness either.

Riquelme [Riq18, Theorems 2.8, 2.9, 2.10, 2.13] establishes the other (in)equalities. □

The following intermediate result, of independent interest, is proven in section 4.2.

Theorem 4.3. *Let φ be a H-flow on a manifold M. For every invariant probability measure $\mu \in \mathcal{M}_\varphi$, one has*

$$h_{\text{Kat}}(\mu) \leq h_{\text{Gur}}(\varphi)\,.$$

Our construction of a measure of maximal entropy in this paper will produce a measure that is *a priori* not necessarily ergodic, so that it is worth noting the following corollary. Recall that we use the notation $\mathcal{M}_\varphi$ for the set of φ-invariant probability measures, and $\mathcal{M}_\varphi^{\text{erg}}$ for the set of φ-invariant, ergodic, probability measures.

Corollary 4.4. *Let φ be a H-flow on a manifold M. We have*

$$\begin{aligned} h_{\text{var}}(\varphi) &= \sup_{\mu\in\mathcal{M}_\varphi} h_{\text{KS}}(\mu) = \sup_{\mu\in\mathcal{M}_\varphi^{\text{erg}}} h_{\text{KS}}(\mu) = \sup_{\mu\in\mathcal{M}_\varphi^{\text{erg}}} h_{\text{Kat}}(\mu) = \sup_{\mu\in\mathcal{M}_\varphi^{\text{erg}}} \underline{h}_{\text{BK}}(\mu) \\ &\leq \sup_{\mu\in\mathcal{M}_\varphi} h_{\text{Kat}}(\mu) \\ &\leq h_{\text{Gur}}(\varphi)\,. \end{aligned}$$

4.2 Katok entropy is smaller than Gurevic entropy - Proof of Theorem 4.3

In this subsection, we are going to prove Theorem 4.3. The rough idea goes as follows. Given a φ-invariant probability measure μ and compact set K, we can assume that each point of a spanning set for μ (whose cardinality is used to calculate its Katok entropy) lies in K and comes back to K after a time T. Thanks to the transitivity property and the finite exact shadowing property, we can close up the piece of orbit of each point of the spanning set to obtain a periodic orbit intersecting K. By controlling the default of injectivity of such a procedure, we will conclude that the Gurevich entropy is larger than the Katok entropy of μ. We start now with the details of the proof.

Firstly, we introduce the notion of separating spanning sets and prove that they can be equivalently used to define the Katok entropy of a measure.

Definition 4.5. *Let $T > 0$ and $\varepsilon > 0$. A set $E \subset X$ is (ε, T)-separating if for all $x, y \in E, x \neq y$ there exists $t \in [0, T]$ such that*

$$d(\varphi_t(x), \varphi_t(y)) \geq \varepsilon\,.$$

Let $\mu \in \mathcal{M}_\varphi$.

Definition 4.6. *Let $T > 0$, $\varepsilon > 0$ and $\alpha \in (0, 1)$. A set E is a separating $(T, \varepsilon, \alpha, \mu)$-spanning set if it is a $(T, \varepsilon, \alpha, \mu)$-spanning set (see Definition 3.4) and it is $(\frac{\varepsilon}{2}, T)$-separating.*

Recall that $M(T, \varepsilon, \alpha, \mu)$ denotes the minimal cardinality of a $(T, \varepsilon, \alpha, \mu)$-spanning set. Similarly, denote by $M'(T, \varepsilon, \alpha, \mu)$ the minimal cardinality of a separating $(T, \varepsilon, \alpha, \mu)$-spanning set. We then have $M(T, \varepsilon, \alpha, \mu) \leq M'(T, \varepsilon, \alpha, \mu)$, since any separating $(T, \varepsilon, \alpha, \mu)$-spanning set is also a $(T, \varepsilon, \alpha, \mu)$-spanning set.

Lemma 4.7. *Let $T > 0$, $\varepsilon > 0$ and $\alpha \in (0, 1)$. Then $M'(T, 2\varepsilon, \alpha, \mu) \leq M(T, \varepsilon, \alpha, \mu)$.*

Proof. Let E be a $(T, \varepsilon, \alpha, \mu)$-spanning set of minimal cardinality $M = M(T, \varepsilon, \alpha, \mu)$. Enumerate the elements of E as $\{x_1, \ldots, x_M\}$. We select a subset E' of E as follows.

1. The point $x_1 \in E'$.
2. Consider the dynamical ball centered at x_1 of radius 2ε. For $i > 1$, we erase the point x_i, i.e., $x_i \notin E'$, if and only if the dynamical ball $B(x_i, \varepsilon, T) \subset B(x_1, 2\varepsilon, T)$.
3. We consider the next point x_j among the remaining ones. We have not erased it at the previous step; we then keep it and say that it belongs to E'.
4. We iterate now the erasing procedure, starting with x_j. Consider the dynamical ball $B(x_j, 2\varepsilon, T)$. For $i > j$, we erase the point x_i, i.e., $x_i \notin E'$, if and only if $B(x_i, \varepsilon, T) \subset B(x_j, 2\varepsilon, T)$.

We have then $\#E' \leq \#E = M(T,\varepsilon,\alpha,\mu)$. We are now going to show that E' is a separating $(T,2\varepsilon,\alpha,\mu)$-spanning set. This will imply then $M'(T,2\varepsilon,\alpha,\mu) \leq \#E'$, concluding our proof. For every $x \in E$

1. either $x \in E'$. In this case observe that $B(x,\varepsilon,T) \subset B(x,2\varepsilon,T)$;
2. or $x \notin E'$. In this case, it means that there exists another $\bar{x} \in E'$ such that $B(x,\varepsilon,T) \subset B(\bar{x},2\varepsilon,T)$.

Thus

$$\bigcup_{x\in E} B(x,\varepsilon,T) \subset \bigcup_{x\in E'} B(x,2\varepsilon,T)\,,$$

and so

$$\mu\left(\bigcup_{x\in E'} B(x,2\varepsilon,T)\right) \geq \mu\left(\bigcup_{x\in E} B(x,\varepsilon,T)\right) \geq \alpha\,,$$

where the last inequality comes from the fact that E is a $(T,\varepsilon,\alpha,\mu)$-spanning set. So, the set E' is a $(T,2\varepsilon,\alpha,\mu)$-spanning set.

Moreover, let $x_i, x_j \in E'$ with $i > j$. There exists $t \in [0,T]$ such that $d(\varphi_t(x_i),\varphi_t(x_j)) \geq \varepsilon$. Indeed, if not, it would imply that

$$B(x_i,\varepsilon,T) \subset B(x_j,2\varepsilon,T)\,,$$

which is in contradiction with the construction of E'. That is, E' is also (ε,T)-separating, which concludes the proof. □

We then deduce the following corollary.

Corollary 4.8. *Let $\mu \in \mathcal{M}_\varphi$. Then*

$$h_{\mathrm{Kat}}(\mu) = \inf_{\alpha>0}\sup_{\varepsilon>0}\limsup_{T\to+\infty}\frac{1}{T}\log\left(M'(T,\varepsilon,\alpha,\mu)\right)\,.$$

Notice that $\varepsilon \mapsto M(T,\varepsilon,\alpha,\mu)$ and $\alpha \mapsto M(T,\varepsilon,\alpha,\mu)$ are non increasing. Yet, while $\alpha \mapsto M'(T,\varepsilon,\alpha,\mu)$ is also non-decreasing, we have a prioi no control on $\varepsilon \mapsto M'(T,\varepsilon,\alpha,\mu)$.

We now prove a lemma analogous to Lemma 4.7 with the extra condition that the points in the separating-spanning set should belong to a fixed compact set K. This is the first step to prove Theorem 4.3.

Lemma 4.9. *Let E be a $(T,\varepsilon,\alpha,\mu)$-spanning set and let $K \subset M$ be a subset such that $\mu(K) = \beta$. Then there exists a separating $(T,2\varepsilon,\alpha+\beta-\mu(M),\mu)$-spanning set $E' \subset K$ such that $\#E' \leq \#E$.*

Proof. We construct the set E' inductively as follows. Let us enumerate E as $\{x_1,\dots,x_N\}$.

1. If $B(x_1,\varepsilon,T) \cap K \neq \emptyset$, then we choose $y_1 \in B(x_1,\varepsilon,T)\cap K$ and set $E'_1 = \{y_1\}$. Otherwise $E'_1 = \emptyset$.
2. For $i \geq 2$, if
$$B(x_i,\varepsilon,T) \cap K \cap (M \setminus \bigcup_{y_j\in E'_{i-1}} B(y_j,\varepsilon,T)) \neq \emptyset\,,$$
then we choose $y_i \in B(x_i,\varepsilon,T)\cap K \cap (M\setminus \bigcup_{y_j\in E'_{i-1}} B(y_j,\varepsilon,T))$ and add y_i to E'_{i-1} to obtain E'_i. Otherwise let $E'_i = E'_{i-1}$.

Let $E' = E'_N$. Observe that $\#E' \leq \#E$ and that $E' \subset K$. Moreover, we have

$$\bigcup_{x\in E} B(x,\varepsilon,T)\cap K \subset \bigcup_{y\in E'} B(y,2\varepsilon,T)\,. \tag{9}$$

Indeed, if $y_i \in E'$, then in particular $y_i \in B(x_i,\varepsilon,T)\cap K$ and so

$$B(x_i,\varepsilon,T)\cap K \subset B(x_i,\varepsilon,T) \subset B(y_i,2\varepsilon,T)\,.$$

If x_i is such that there is no corresponding $y_i \in E'$, then $B(x_i, \varepsilon, T) \cap K \cap (M \setminus \bigcup_{y_j \in E'_{i-1}} B(y_j, \varepsilon, T)) = \emptyset$. If $B(x_i, \varepsilon, T) \cap K = \emptyset$, then the empty set is clearly contained in $\bigcup_{y \in E'} B(y, 2\varepsilon, T)$. If not, the only possibility is that $B(x_i, \varepsilon, T) \cap K \subset \bigcup_{y_j \in E'_{i-1}} B(y_j, \varepsilon, T)$; then

$$B(x_i, \varepsilon, T) \cap K \subset \bigcup_{y \in E'} B(y, \varepsilon, T) \subset \bigcup_{y \in E'} B(y, 2\varepsilon, T) .$$

We now argue that E' is a $(T, 2\varepsilon, \alpha + \beta - \mu(M), \mu)$-separating set. Indeed, from (9),

$$\begin{aligned}
\mu\left(\bigcup_{y \in E'} B(y, 2\varepsilon, T)\right) &\geq \mu\left(\bigcup_{x \in E} B(x, \varepsilon, T) \cap K\right) \\
&\geq \mu\left(\bigcup_{x \in E} B(x, \varepsilon, T)\right) + \mu(K) - \mu(M) \\
&\geq \alpha + \beta - \mu(M) ,
\end{aligned}$$

where the last inequality comes from E being a $(T, \varepsilon, \alpha, \mu)$-spanning set. Moreover the set $E' \subset K$ is also separating. Indeed, let $y, y' \in E'$, $y \neq y'$. In particular, by the construction of E', it means that $y' \notin B(y, \varepsilon, T)$ (or viceversa): there exists $\tau \in [0, T]$ such that $d(\varphi_\tau(y), \varphi_\tau(y')) \geq \varepsilon$. ☐

Proof of Theorem 4.3. Let $\mu \in \mathcal{M}_\varphi$. We are now going to prove that the Katok entropy of μ is smaller or equal to the Gurevic entropy. Let $\varepsilon > 0$ and $\alpha_1 \in (0, 1)$. Pick $\alpha' \in (0, 1)$ such that $\alpha_1 + \alpha' \in (0, 1)$. Let $\alpha_0 = \alpha_1 + \alpha'$.

Let $K \subset M$ be a compact subset with nonempty interior such that $\mu(K) \geq 1 - \alpha'/2$ (such a K exists as M is exhaustible by compact sets). Let us apply Lemma 2.16 at K, $\delta = \frac{\varepsilon}{3}$, $\nu = \tau_K$ and $N = 1$. The lemma gives us constants $\sigma > 0, T_{\min} > 0$. Let $T \geq T_{\min}$ and $K' = K \cap \varphi_{-T}(K)$. Observe that $\mu(K') \geq 1 - \alpha'$ as μ is φ-invariant. Let E be a $(T, \varepsilon, \alpha_0, \mu)$ spanning set. By Lemma 4.9, there exists a separating $(T, 2\varepsilon, \alpha_1, \mu)$ spanning set E' such that $\#E' \leq \#E$ and $E' \subset K'$.

We can associate to every $x \in E'$ a periodic point y, thanks to Lemma 2.16. Indeed, since $x \in E' \subset K'$, it holds that both x and $\varphi_T(x)$ belongs to K, and by construction $T \geq T_{\min}$. Thus, there exists $y \in M$ and $L \in [T + \sigma - 2\tau_K, T + \sigma + 2\tau_K]$ so that $\varphi_L(y) = y$ and $d(\varphi_t(y), \varphi_t(x)) < \frac{\varepsilon}{3}$ for every $t \in [0, T]$.

We can give an upper bound on the number of points in E' that could be associated to the same periodic point. Fix $x, x' \in E'$ with $x \neq x'$. Let γ and γ' be the associated periodic orbits. Assume $\gamma = \gamma'$. Then, there exists u such that, for all $s \in [0, T]$,

$$d(\varphi_s(x), \gamma(s)) < \frac{\varepsilon}{3} \qquad \text{and} \qquad d(\varphi_s(x'), \gamma(s + u)) < \frac{\varepsilon}{3} .$$

Moreover, since E' is (ε, T) separating, then there exists $\tau \in [0, T]$ such that

$$d(\varphi_\tau(x), \varphi_\tau(x')) \geq \varepsilon .$$

Therefore,

$$d(\varphi_\tau(y), \varphi_\tau(y')) = d(\gamma(\tau), \gamma(\tau + u)) \geq \frac{\varepsilon}{3} .$$

From the right inequality of (2), we deduce that $|u| \geq \varepsilon/3b$. Consequently, there are at most $\left\lceil (T + \sigma + 2\tau_K) \frac{3b}{\varepsilon} \right\rceil$ points of E' that could correspond to points on the same periodic orbit. This implies that

$$\#E' \leq \left\lceil \frac{3b}{\varepsilon}(T + \sigma + 2\tau_K) \right\rceil \#\mathcal{P}_K(T + \sigma - 2\tau_K, T + \sigma + 2\tau_K) .$$

By the definition of Katok entropy for μ, by Proposition 3.12 and by Corollary 4.8, we deduce that

$$\limsup_{T \to +\infty} \frac{1}{T} \log \left(M'(T, 2\varepsilon, \alpha_1, \mu) \right) \leq h_{\mathrm{Gur}}(\varphi)$$

and therefore

$$h_{\mathrm{Kat}}(\mu) \leq h_{\mathrm{Gur}}(\varphi) .$$

☐

4.3 Comparison of entropies at infinity

With similar ideas, we can also compare Gurevic and variational entropies at infinity.

Theorem 4.10 (Comparison of entropies at infinity)**.** *Let φ be a H-flow on (M,d). The entropies at infinity satisfy*

$$h_{\mathrm{var}}^{\infty}(\varphi) \leq h_{\mathrm{Gur}}^{\infty}(\varphi)\,.$$

Proof. If $h_{\mathrm{Gur}}^{\infty}(\varphi) = +\infty$ there is nothing to prove. We now assume $h_{\mathrm{Gur}}^{\infty}(\varphi) < \infty$.

Assume $h_{\mathrm{var}}^{\infty}(\varphi) < \infty$. Fix some small $\alpha > 0$. Thanks to Remark 3.20, we choose a large compact set K_0, a small $0 < \eta < \frac{1}{2}$, and an ergodic probability measure $\mu \in \mathcal{M}_{\varphi}^{\mathrm{erg}}$ with $\mu(K_0) \leq \eta$ so that

$$\left|h_{\mathrm{Gur}}^{\infty}(\varphi) - h_{\mathrm{Gur}}^{K_0,4\eta}(\varphi)\right| \leq \alpha \quad \text{and} \quad |h_{\mathrm{var}}^{\infty}(\varphi) - h_{\mathrm{Kat}}(\mu)| \leq \alpha\,.$$

We follow very closely the proof of Theorem 4.3, with a few modifications.

Let $\varepsilon > 0$ and $\alpha_1 \in (0,1)$. Pick α' such that $\alpha_0 = \alpha_1 + \alpha' \in (0,1)$. Fix $w \in \mathrm{int}(K_0)$. Choose $\delta \leq \frac{\varepsilon}{3}$ small enough so that $\mu(B(K_0,\delta)) \leq 2\eta$ and $B(w,\delta) \subset K_0$. Fix K_2 a compact set such that $K_2 \supset B(K_0,\delta)$ and $\mu(K_2) \geq 1 - \frac{\alpha'}{4}$. Let $\Omega = B(K_0,\delta)$.

By Birkhoff ergodic theorem, for μ-almost every $x \in M$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{\Omega}(\varphi_t(x))\,dt = \mu(\Omega)\,,$$

where 1_Ω is the indicator function of Ω. As $\mu(\Omega) \leq 2\eta$ and $\mu(K_2) \geq 1 - \frac{\alpha'}{4}$, there exists a subset $A \subset K_2$ with $\mu(A) \geq 1 - \frac{\alpha'}{2}$ and $S_{\min} > 0$, such that for all $x \in A$ and all $T \geq S_{\min}$,

$$\frac{1}{T}\int_0^T 1_{\Omega}(\varphi_t(x))\,dt \leq 3\eta\,. \tag{10}$$

Note that, elements of the closure of A, denoted as $\overline{A}$, also satisfies condition (10). Indeed, let $(x_n)_n$ be a sequence of elements of A converging to x. Then, for all $T \geq S_{\min}$, as Ω is an open set and by Fatou's lemma

$$\frac{1}{T}\int_0^T 1_{\Omega}(\varphi_t(x))\,dt \leq \frac{1}{T}\int_0^T \liminf_{n\to\infty} 1_{\Omega}(\varphi_t(x_n))\,dt \leq \liminf_{n\to\infty} \frac{1}{T}\int_0^T 1_{\Omega}(\varphi_t(x_n))\,dt \leq 3\eta.$$

We will use Lemma 2.16 with parameters $\overline{B(w,\delta)} \subset K_2$, δ as above, $\nu = \tau_{K_2}$ and $N = 1$. This lemma gives us constants $T_{\min}$ and σ. Let $T \geq \max(T_{\min}, S_{\min})$. Set $K_1 = \overline{A} \cap \varphi_{-T}(\overline{A})$, and observe that $\mu(K_1) \geq 1 - \alpha'$.

Let E be a separating $(T,\varepsilon,\alpha_0,\mu)$-spanning set. Without loss of generality, we can assume that for every $x \in E$, $\mu(B(x,\varepsilon,T)) > 0$.

By Lemma 4.9, there exists a separating $(T,2\varepsilon,\alpha_1,\mu)$ spanning set E' such that $\#E' \leq \#E$ and $E' \subset K_1$. Without loss of generality, we can assume that for every $x \in E'$, $\mu(B(x,2\varepsilon,T)) > 0$.

Observe that, since $E' \subset \overline{A}$, every point $x \in E'$ satisfies inequality (10). Fix now a point $x \in E'$. Note that $x \in K_2$ and $\varphi_T(x) \in K_2$.

By Lemma 2.16, we obtain a periodic orbit γ such that

- for all $t \in [0,T]$, we have $d(\varphi_t(x), \gamma(t)) \leq \delta$;
- $\ell(\gamma) \in [T + \sigma - 2\tau_{K_2}, T + \sigma + 2\tau_{K_2}]$;
- γ intersects $B(w,\delta)$.

Therefore, since by (10) the piece of orbit $\varphi_{[0,T]}(x)$ spends at most a total amount of time of $3\eta T$ in $B(K,\delta) = \Omega$, we have

$$\ell(\gamma \cap K_0) = \ell(\gamma_{[0,T]} \cap K_0) + \ell(\gamma_{[T,\ell(\gamma)]} \cap K_0) \leq 3\eta T + \sigma + 2\tau_{K_2}.$$

Thus $\ell(\gamma \cap K_0) \leq 4\eta\ell(\gamma)$ for $T \gg 1$ and therefore $\gamma \in \mathcal{P}_{K_0}^{4\eta}(T + \sigma - 2\tau_{K_2}, T + \sigma + 2\tau_{K_2})$.

The end of the proof is the same as the one of Theorem 4.3. Since E' is (ε, T)-separating, we deduce that

$$\#E' \leq \left\lceil \frac{3b}{\varepsilon}(T+\sigma+2\tau_{K_2}) \right\rceil \#\mathcal{P}^{4\eta}_{K_0}(T+\sigma-2\tau_{K_2}, T+\sigma+2\tau_{K_2}) .$$

By the definition of Katok entropy for μ, by Proposition 3.12 and by Corollary 4.8, we deduce that

$$\limsup_{T\to+\infty} \frac{1}{T} \log M'(T, 2\varepsilon, \alpha_1, \mu) \leq h^{K_0,4\eta}_{\mathrm{Gur}}(\varphi)$$

and therefore

$$h_{\mathrm{Kat}}(\mu) \leq h^{K_0,4\eta}_{\mathrm{Gur}}(\varphi) ,$$

and at the end

$$h^{\infty}_{\mathrm{var}}(\varphi) \leq h^{\infty}_{\mathrm{Gur}}(\varphi) + 2\alpha .$$

As α was arbitrary, the result follows.

If $h^{\infty}_{\mathrm{var}}(\varphi) = +\infty$, we choose μ ergodic such that $h_{\mathrm{Kat}}(\mu) \geq N$ and proceed as in the previous case. We obtain $N \leq h^{K_0,4\eta}_{\mathrm{Gur}}(\varphi)$ for arbitrarily big N and therefore $h^{\infty}_{\mathrm{Gur}}(\varphi) = \infty$. □

4.4 Strong positive recurrence

In [ST21, GST23], for geodesic flows in negative curvature, the geodesic flow is said *strongly positively recurrent* if its entropy at infinity is strictly smaller than its topological entropy. In this context, all notions of entropy (resp. entropy at infinity) coincide, as proven in [GST23]. Here, it is not the case. That motivates the following terminology.

Definition 4.11. *The flow $(\varphi_t)_{t\in\mathbb{R}}$ is h_{Gur}-*strongly positively recurrent *if $h^{\infty}_{\mathrm{Gur}}(\varphi) < h_{\mathrm{Gur}}(\varphi)$. It is h_{var}-*strongly positively recurrent *if $h^{\infty}_{\mathrm{var}}(\varphi) < h_{\mathrm{var}}(\varphi)$.*

Remark 4.12. Theorems 4.10 and 1.1 show that if φ is a h_{Gur}-strongly positively recurrent H-flow, then it is also h_{var}*-strongly positively recurrent*.

4.5 Gurevic entropies

The end of this section is devoted to the proof of the following proposition.

Proposition 4.13. *Let φ be a H-flow. Then $h^{\infty}_{\mathrm{Gur}}(\varphi) < \infty$ if and only if $h_{\mathrm{Gur}}(\varphi) < \infty$.*

We will start with two preliminary results.

The following lemma is a rephrasing of the finiteness of entropy on compact sets. It will be useful to control the entropy using the entropy at infinity in the proof of the proposition and also in the proof of Theorem 5.26.

Lemma 4.14. *Let $K_0 \subset K_1$ be two compact subsets of M. Let φ be an expansive flow on M. Let $C > 0$. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{P}_{K_0}(T, T+C), \gamma \subset K_1\} < \infty .$$

Proof. Consider the closure of the set of all periodic orbits of φ that are contained in the compact set K_1. Denote such a closed set by X. Then, endowing it with the distance inherited from that of the whole M, the set X is a metric space. The flow φ restricted to X is still expansive. Since we can find a bigger compact set that contains it, the set X is also compact. Applying then [BW72, Theorem 5], the result follows immediately. □

The following proposition is an adaptation of Proposition 3.14.

Proposition 4.15. *Let $\varphi\colon M \to M$ be a H-flow. Let K_0, K_1 be two compact subsets of M with nonempty interior and such that $K_0 \subset \mathring{K_1}$. Let $0 < \alpha < 1$ and $\sigma_0 \geq 4\tau_{K_1}$. There exist constants $D > 0$ and $\sigma > 0$ such that for all $L_1, L_2 \gg 1$ with $\frac{\alpha}{3}L_2 \geq L_1$, one has*

$$\#\mathcal{P}_{K_0}(L_1, L_1+\sigma_0)\, \#\mathcal{P}^{\alpha/3}_{K_1}(L_2, L_2+\sigma_0) \leq D\,(L_1+L_2)\, \#\mathcal{P}^{\alpha}_{K_0}(L_1+L_2+\sigma, L_1+L_2+\sigma+\sigma_0) .$$

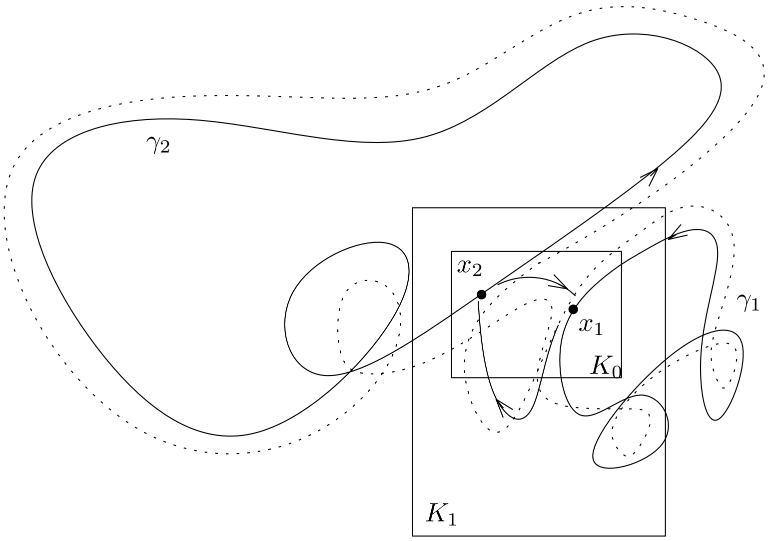

Figure 4.1: Modified subadditivity property

Proof of Proposition 4.15. The proof is a direct adaptation of the proof of Proposition 3.14. See Figure 4.1. We import the notation from this proof. Choose $\varepsilon > 0$ as in the proof of Proposition 3.14. Without loss of generality, we can assume that $\varepsilon/4 < d(\partial K_0, \partial K_1)$. The only new property to check is that if $\gamma_1 \in \mathcal{P}_{K_0}(L_1, L_1 + \sigma_0)$, $\gamma_2 \in \mathcal{P}_{K_1}^{\alpha/3}(L_2, L_2 + \sigma_0)$ and $\frac{\alpha}{3} L_2 \geq L_1$ then $\gamma = f(\gamma_1, \gamma_2)$ satisfies $\ell(\gamma \cap K_0) < \alpha \ell(\gamma)$. The very same construction in the proof of Proposition 3.14 gives us the constants $D > 0$ and $\sigma > 0$. Recall that γ consists in four pieces

- a piece $\varepsilon/4$-close to γ_1 on an interval of length $\ell(\gamma_1)$;
- a piece $\varepsilon/4$-close to γ_2 on an interval of length $\ell(\gamma_2)$;
- two transition pieces of total length $\leq \sigma + \sigma_0$.

Therefore

$$\begin{aligned}
\ell(\gamma \cap K_0) &\leq \ell(\gamma_1) + \ell(\gamma_2 \cap K_1) + \sigma + \sigma_0 \\
&< \ell(\gamma_1) + \frac{\alpha}{3} \ell(\gamma_2) + \sigma + \sigma_0 \\
&\leq L_1 + \sigma_0 + \frac{\alpha}{3}(L_2 + \sigma_0) + \sigma + \sigma_0 \\
&\leq \frac{2\alpha}{3} L_2 + \sigma + 3\sigma_0 \\
&\leq \alpha L_2 \\
&\leq \alpha \ell(\gamma)
\end{aligned}$$

for $L_2 \gg 1$. Thus $\gamma \in \mathcal{P}_{K_0}^{\alpha}(L_1 + L_2 + \sigma, L_1 + L_2 + \sigma + \sigma_0)$. The rest of the proof is unchanged. □

Proof of Proposition 4.13. If $h_{\mathrm{Gur}}(\varphi) < \infty$ then $h_{\mathrm{Gur}}^{\infty}(\varphi) < \infty$.

We now assume $h_{\mathrm{Gur}}^{\infty}(\varphi) < \infty$. There exists K_0 compact and $\alpha_0 > 0$ such that for all $\alpha \leq \alpha_0$

$$h_{\mathrm{Gur}}^{K_0,\alpha}(\varphi) \leq h_{\mathrm{Gur}}^{\infty}(\varphi) + 1.$$

Therefore, for $L \gg 1$,

$$\#\mathcal{P}_{K_0}^{\alpha_0}(L, L + \sigma_0) \leq e^{\left(h_{\mathrm{Gur}}^{\infty}(\varphi) + 2\right) L}.$$

Let K_1 be a compact subset of M such that $K_0 \subset \overset{\circ}{K_1}$. We now use Proposition 4.15 with parameters K_0, K_1, α_0 and $\sigma_0 = 5\tau_{K_0}$. For all $L_1, L_2 \gg 1$ with $\frac{\alpha_0}{3} L_2 \geq L_1$

$$\#\mathcal{P}_{K_0}(L_1, L_1 + \sigma_0) \, \#\mathcal{P}_{K_1}^{\alpha_0/3}(L_2, L_2 + \sigma_0) \leq D\,(L_1 + L_2)\, \#\mathcal{P}_{K_0}^{\alpha_0}(L_1 + L_2 + \sigma, L_1 + L_2 + \sigma + \sigma_0)\,.$$

Let us first assume there exist an increasing sequence $(L_2^n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} L_2^n = \infty$ and, for all $n \geq 0$,

$$\#\mathcal{P}_{K_1}^{\alpha/3}(L_2^n, L_2^n + \sigma_0) \geq 1.$$

Let $L_1^n = \frac{\alpha_0}{3} L_2^n$. Then, for all $n \gg 1$

$$\begin{aligned}\#\mathcal{P}_{K_0}(L_1^n, L_1^n + \sigma_0) &\leq D\left(1 + \frac{3}{\alpha_0}\right) L_1^n \#\mathcal{P}_{K_0}^{\alpha_0}\left(\left(1 + \frac{3}{\alpha_0}\right) L_1^n + \sigma, \left(1 + \frac{3}{\alpha_0}\right) L_1^n + \sigma + \sigma_0\right) \\ &\leq D\left(1 + \frac{3}{\alpha_0}\right) L_1^n e^{\left(h_{\mathrm{Gur}}^{\infty}(\varphi)+2\right)\left(\left(1+\frac{3}{\alpha_0}\right)L_1^n + \sigma\right)}.\end{aligned}$$

As the Gurevic entropy is a true limit, not only a superior limit when φ is a H-flow (see Theorem 3.10), we then have

$$h_{\mathrm{Gur}}(\varphi) \leq (h_{\mathrm{Gur}}^{\infty}(\varphi) + 2)\left(1 + \frac{3}{\alpha_0}\right) < \infty.$$

We now assume

$$\#\mathcal{P}_{K_1}^{\alpha_0/3}(L_2, L_2 + \sigma_0) = 0$$

for all $L_2 \gg 1$. In particular, there do not exist arbitrarily long chords of ∂K_1 outside K_1 (otherwise, by transitivity and the uniform multiple closing lemma 2.16, one may construct a periodic orbit in some $\mathcal{P}_{K_1}^{\alpha_0/3}(L_2, L_2 + \sigma_0)$). Therefore all the periodic orbits intersecting K_0 are contained in some compact K_2. From Lemma 4.14, we obtain that the Gurevic entropy is finite. □

5 Gurevic entropy versus chord entropy

It will be useful in the sequel to count chords, i.e., pieces of orbits from the neighbourhood of a point to the neighbourhood of another point, and try to compare their number with the number of periodic orbits. In Section 5.1 we explain how to count chords and state some elementary properties of chords counts.

5.1 Chords

The aim of this section is to introduce these chords and prove some counting properties.

Let $K \subset M$ be a compact set. Let $x, y \in K$ and $\eta > 0$. Let $0 < T^- < T^+$. The set of chords from $B(x, \eta)$ to $B(y, \eta)$ with lengths in $[T^-, T^+]$ is

$$\mathcal{C}(x, y, \eta, T^-, T^+) = \{z \in B(x, \eta) : \ \varphi_{[T^-, T^+]}(z) \cap B(y, \eta) \neq \emptyset\}.$$

Definition 5.1. *Let $\delta > 0$. A set E is a $E(x, y, \eta, T^-, T^+, \delta)$-set if:*

1. *$E \subset \mathcal{C}(x, y, \eta, T^-, T^+)$;*
2. *E is a (δ, T^-)-separating set;*
3. *the set of chords $\mathcal{C}(x, y, \eta, T^-, T^+)$ is contained in the union of dynamical balls $\bigcup_{z\in E} B(z, \delta, T^-)$.*

Denote by $\mathcal{N}_{\mathcal{C}}(x, y, \eta, T^-, T^+, \delta)$ the maximal cardinality of a $E(x, y, \eta, T^-, T^+, \delta)$-set.

Fact 5.2. *Let E be a set satisfying points 1 and 2 of Definition 5.1. Then, there exists a set E' that is a $E(x, y, \eta, T^-, T^+, \delta)$-set and contains E.*

Proof. If the union of dynamical balls $\cup_{z\in E} B(z, \delta, T^-)$ does not contain $\mathcal{C}(x, y, \eta, T^-, T^+)$, then we pick a point

$$z' \in \mathcal{C}(x, y, \eta, T^-, T^+) \setminus \bigcup_{z\in E} B(z, \delta, T^-).$$

By construction, the set $\{z'\} \cup E$ is (δ, T^-)-separating and contained in $\mathcal{C}(x, y, \eta, T^-, T^+)$, i.e., it satisfies points 1 and 2 of Definition 5.1. If it is not a $E(x, y, \eta, T^-, T^+, \delta)$-set, we iterate the procedure. By compactness of the closure of $\mathcal{C}(x, y, \eta, T^-, T^+)$, this procedure will stop after a finite number of iterations. At the end, we obtain a set E' that contains E and which is a $E(x, y, \eta, T^-, T^+, \delta)$-set. □

Fact 5.3. *Let $x, y \in K$, $0 < \delta$ and $0 < T^- < T^+$. The map*

$$\eta \in (0, +\infty) \mapsto \mathcal{C}(x, y, \eta, T^-, T^+)$$

is non-decreasing for the inclusion. Moreover, the map

$$\eta \in (0, +\infty) \mapsto \mathcal{N}_{\mathcal{C}}(x, y, \eta, T^-, T^+, \delta) \in \mathbb{N}$$

is non-decreasing.

Proof. The first assertion is a direct consequence of the definition of chords. For the second assertion, fix $\eta_1 < \eta_2$, use the first assertion and apply Fact 5.2 to a $E(x, y, \eta_1, T^-, T^+, \delta)$-set of maximal cardinality E, to build a $E(x, y, \eta_2, T^-, T^+, \delta)$-set containing E. We deduce that

$$\#E = \mathcal{N}_{\mathcal{C}}(x, y, \eta_1, T^-, T^+, \delta) \leq \mathcal{N}_{\mathcal{C}}(x, y, \eta_2, T^-, T^+, \delta)\,.$$

□

Fact 5.4. *Let $x, y \in K$, $\eta > 0$ and $0 < T^- < T^+$. Then, the map*

$$\delta \in (0, +\infty) \mapsto \mathcal{N}_{\mathcal{C}}(x, y, \eta, T^-, T^+, \delta) \in \mathbb{N}$$

is non-increasing.

Proof. Let $\delta_1 \leq \delta_2$ First note that if a set is (δ_2, T^-)-separating then it is also (δ_1, T^-)-separating. Let E_2 be an $E(x, y, \eta, T^-, T^+, \delta_2)$-set of maximal cardinality. Then E_2 satisfies the first two points in the definition of $E(x, y, \eta, T^-, T^+, \delta_1)$-set. Using Fact 5.2, we obtain

$$\#E_2 = \mathcal{N}_{\mathcal{C}}(x, y, \eta, T^-, T^+, \delta_2) \leq \mathcal{N}_{\mathcal{C}}(x, y, \eta, T^-, T^+, \delta_1)$$

as required. □

Up to changing the parameter involved in the definition of the counting of chords, we show in Proposition 5.5 that such a number is uniform with respect to the points $x, y \in K$.

Proposition 5.5 (Chord counting does not depend on the endpoints of the chords)**.** *Let $K \subset M$ be a compact subset. For any η_1 and δ such that $0 < \eta_1 < \frac{\delta}{2}$, there exist $\sigma > 0$ and $0 < \widetilde{\eta}_0 < \eta_1/2$ such that for all x_0, y_0, x_1, y_1 in K, $0 < T_0^- < T_0^+$, $0 < \eta_0 \leq \widetilde{\eta}_0$ and, $S \geq \sigma$, we have*

$$\mathcal{N}_{\mathcal{C}}(x_0, y_0, \eta_0, T_0^-, T_0^+, \delta) \leq \mathcal{N}_{\mathcal{C}}(x_1, y_1, \eta_1, T_0^+ + S, T_0^+ + S + 2\tau_K, \delta/2)\,.$$

In particular, if $T_0^+ + \sigma \leq T_1^- < T_1^+$ and $T_1^+ - T_1^- \geq 2\tau_K$, we have

$$\mathcal{N}_{\mathcal{C}}(x_0, y_0, \eta_0, T_0^-, T_0^+, \delta) \leq \mathcal{N}_{\mathcal{C}}(x_1, y_1, \eta_1, T_1^-, T_1^+, \delta/2).$$

Proof. Figure 5.1 summarizes the proof. The first naive idea to prove the proposition is the following. By transitivity property 2.14, we find arcs of length $S/2$ respectively from $B(x_1, \eta_0)$ to $B(x_0, \eta_0)$ and from $B(y_0, \eta_0)$ to $B(y_1, \eta_0)$. The finite exact shadowing property 2.6 allows to concatenate every chord of length in $[T_0^-, T_0^+]$ from $B(x_0, \eta_0)$ to $B(y_0, \eta_0)$ with these arcs before and after it, to obtain a chord from $B(x_1, \eta_1)$ to $B(y_1, \eta_1)$. The resulting chord has length in $[T_0^- + S - 2\tau_K, T_0^+ + S + 2\tau_K]$ and the uncertainty on its length is higher than desired.

The proof is close to this naive idea, but we choose first an arc from $B(x_1, \eta_0)$ to $B(x_0, \eta_0)$, with length $S_1 \simeq S/2 \pm \tau_K$. Second, we consider an arbitrary chord of length $\ell \in [T_0^-, T_0^+]$. Third, by uniform transitivity, we choose an arc from $B(y_0, \eta_0)$ to $B(y_1, \eta_0)$ with length $S_2 \pm \tau_K$, where S_2 is chosen so that

$$S_1 + \ell + S_2 = T_0^+ + S + \tau_K$$

so that, after concatenation, the resulting chord has length in $[T_0^+ + S, T_0^+ + S + 2\tau_K]$.

Step 1. Choice of parameters. Let $\widetilde{\eta}_0 = \eta/2$, with η the constant given by the finite exact shadowing property 2.6 applied with K, $N = 3$ and $\delta = \eta_1/2$. Let σ_0 be the constant given by the

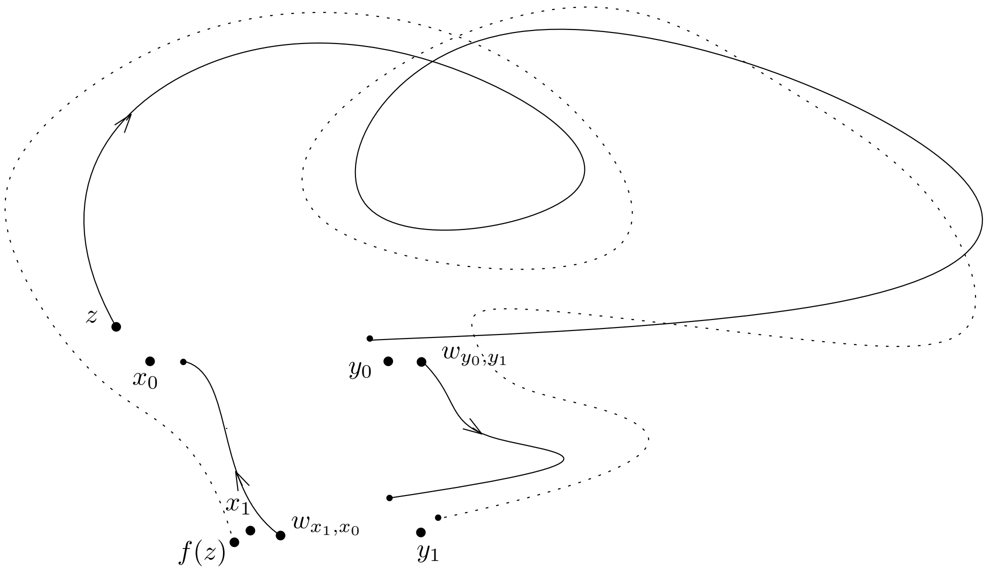

Figure 5.1: Chord counting does not depend on the endpoints of the chords

uniform transitivity property applied with parameters K and $\widetilde{\eta}_0$. Let $\sigma = 2\max(\sigma_0, \tau_K)$ and $S \geq \sigma$. Let $0 < \eta_0 \leq \widetilde{\eta}_0$.

Step 2. First use of transitivity. Transitivity property 2.14 gives us a point $w_{x_1,x_0} \in B(x_1, \widetilde{\eta}_0)$ and $\ell(w_{x_1,x_0}) \in [S/2 - \tau_K, S/2 + \tau_K]$ such that $\varphi_{\ell(w_{x_1,x_0})}(w_{x_1,x_0}) \in B(x_0, \widetilde{\eta}_0)$.

Step 3. Choice of a chord. Let E be a $E(x_0, y_0, \eta_0, T_0^-, T_0^+, \delta)$-set of maximal cardinality. For every $z \in E$, denote by $\ell(z) \in [T_0^-, T_0^+]$ a time such that $\varphi_{\ell(z)}(z) \in B(y_0, \eta_0)$.

Step 4. Second use of transitivity. Lemma 2.14 applied to K, $\widetilde{\eta}_0$, y_0, y_1 and $S_2 = S/2 + (T_0^+ - \ell(z)) + (S/2 + \tau_K - \ell(w_{x_1,x_0})) \geq S/2 \geq \sigma_0$ gives a point w_{y_0,y_1} and a length $\ell(w_{y_0,y_1}) \in [S_2 - \tau_K, S_2 + \tau_K]$ such that $w_{y_0,y_1} \in B(y_0, \widetilde{\eta}_0)$ and $\varphi_{\ell(w_{y_0,y_1})}(w_{y_0,y_1}) \in B(y_1, \widetilde{\eta}_0)$. By construction, we have

$$T_0^+ + S \leq \ell(z) + \ell(w_{x_1,x_0}) + \ell(w_{y_0,y_1}) \leq T_0^+ + S + 2\tau_K.$$

Step 5. Concatenation. By the choice of parameters in Step 1, we have $d(\varphi_{\ell(w_{x_1,x_0})}(w_{x_1,x_0}), z) < 2\widetilde{\eta}_0$ and $d(\varphi_{\ell(z)}(z), w_{y_0,y_1}) < 2\widetilde{\eta}_0$, By the finite exact shadowing property 2.6 there exists a point $f(z)$ such that

- for every $s \in [0, \ell(w_{x_1,x_0})]$, $d(\varphi_s(f(z)), \varphi_s(w_{x_1,x_0})) < \eta_1/2$;
- for every $s \in [0, \ell(z)]$, $d(\varphi_{\ell(w_{x_1,x_0})+s}(f(z)), \varphi_s(z)) < \eta_1/2$;
- for every $s \in [0, \ell(w_{y_0,y_1})]$, $d(\varphi_{\ell(w_{x_1,x_0})+\ell(z)+s}(f(z)), \varphi_s(w_{y_0,y_1})) < \eta_1/2$.

By construction,

$$d(f(z), x_1) \leq d(f(z), w_{x_1,x_0}) + d(w_{x_1,x_0}, x_1) < \eta_1/2 + \widetilde{\eta}_0 \leq \eta_1$$

and

$$\begin{aligned} d(\varphi_{\ell(w_{x_1,x_0})+\ell(z)+\ell(w_{y_0,y_1})}(f(z)), y_1) \leq & d(\varphi_{\ell(w_{x_1,x_0})+\ell(z)+\ell(w_{y_0,y_1})}(f(z)), \varphi_{\ell(w_{y_0,y_1})}(w_{y_0,y_1})) \\ & + d(\varphi_{\ell(w_{y_0,y_1})}(w_{y_0,y_1}), y_1) \\ < & \eta_1/2 + \widetilde{\eta}_0 \leq \eta_1. \end{aligned}$$

Thus, the point $f(z)$ belongs to $\mathcal{C}(x_1, y_1, \eta_1, T_0^+ + S, T_0^+ + S + 2\tau_K)$. Therefore, we have just defined a map $f : E \to \mathcal{C}(x_1, y_1, \eta_1, T_0^+ + S, T_0^+ + S + 2\tau_K)$

Step 6. Separation. Recall that E is a $E(x_0, y_0, \eta_0, T_0^-, T_0^+, \delta)$-set of maximal cardinality. For every $z \in E$, we built in the preceding steps a point $f(z) \in \mathcal{C}(x_1, y_1, \eta_1, T_0^+ + S, T_0^+ + S + 2\tau_K)$. We prove now that $E' = f(E)$ is $(\frac{\delta}{2}, T_0^+ + S)$-separating and that f is injective.

Let $z_1, z_2 \in E$ be such that $z_1 \neq z_2$. As E is (δ, T_0^-)-separating, there exists $u \in [0, T_0^-]$ such that $d(\varphi_u(z_1), \varphi_u(z_2)) \geq \delta$. Therefore, we have

$$\begin{aligned} d(\varphi_{u+\ell(z_{x_1,x_0})}(f(z_1)), \varphi_{u+\ell(z_{x_1,x_0})}(f(z_2))) &\geq d(\varphi_u(z_1), \varphi_u(z_2)) - d(\varphi_{u+\ell(z_{x_1,x_0})}(f(z_1)), \varphi_u(z_1)) \\ &\quad - d(\varphi_{u+\ell(z_{x_1,x_0})}(f(z_2)), \varphi_u(z_2)) \\ &> \delta - \eta_1 \geq \frac{\delta}{2} \end{aligned}$$

that is, since $T_0^- + \frac{S}{2} + \tau_K \leq T_0^+ + S$, E' is a $(\frac{\delta}{2}, T_0^+ + S)$-separating set and f is injective. Thus $\#E \leq \#E'$.

Conclusion. By Lemma 5.2, we obtain

$$\mathcal{N}_{\mathcal{C}}(x_0, y_0, \eta_0, T_0^-, T_0^+, \delta) \leq \mathcal{N}_{\mathcal{C}}\left(x_1, y_1, \eta_1, T_0^+ + S, T_0^+ + S + 2\tau_{\min}(K), \frac{\delta}{2}\right),$$

as required. ☐

5.2 Comparing chords and periodic orbits

We can now compare the number of chords with the number of periodic orbits of approximately the same length. Note that the admissible lengths of the chords/periodic orbits are intervals of the same length τ but shifted by σ. This is not critical to compare chord entropy and Gurevic entropy but will be crucial for upcoming statements.

Proposition 5.6. *Let $K \subset M$ be a compact subset of M with nonempty interior. Let $\tau > 2\tau_K$. Fix $\delta > 0$. There exist constants $D = D(\delta) > 0$, $T_{\min} > 0$ and $\sigma > 0$ such that for all $x, y \in K$, $T \geq T_{\min}$ and $0 < \eta < 1$, we have*

$$\mathcal{N}_{\mathcal{C}}(x, y, \eta, T, T+\tau, \delta) \;\leq\; D \times (T + \sigma + \tau) \times \#\mathcal{P}_K(T+\sigma, T+\sigma+\tau).$$

Proof. The idea of the proof is quite simple. We start with a chord from x to y whose length is in $[T, T+\tau]$, we use transitivity to build an almost-closed pseudo orbit following the chord from x to y and coming back to x. The closing lemma 2.16 allows us to close it into a closed orbit intersecting K. Then, we control the default of injectivity of the construction.

Step 1. Choice of parameters.
Let $T_{\min} > 0$ and $\sigma' > 0$ be the constants given by Lemma 2.16 applied to $K \subset \overline{B(K,1)}$, $\delta/3$, $\nu = \min(\tau/2 - \tau_K, 1)$ and $N = 1$. Without loss of generality, one may assume $T_{\min} \geq 1$. Let $\sigma = \sigma' - \tau_K - \nu$. One may assume $\sigma > 0$. Let $T \geq T_{\min}$. Let E be a $E(x, y, \eta, T, T+\tau, \delta)$-set of maximal cardinality.

Step 2. Construction of a map from chords to periodic orbits.
We now define a map

$$f : E \to \mathcal{P}_K(T+\sigma, T+\sigma+\tau).$$

Let $w \in E$. Lemma 2.16 gives us a periodic orbit $\gamma = f(w)$ with period in

$$[T + \sigma' - \tau_K - \nu, T + \sigma' + \tau_K + \nu] \subset [T+\sigma, T+\sigma+\tau]$$

that intersects K and $\frac{\delta}{3}$-shadows the orbit of w from $B(x,\eta)$ to $B(y,\eta)$. More precisely, for $w \in E$, let $\ell(w) \in [T, T+\tau]$ a time such that $\varphi_{\ell(w)}(w) \in B(y,\eta)$. Then, by construction, there exists an origin s_0 for the periodic orbit γ such that for all $s \in [0, \ell(w)]$ we have

$$d(\gamma(s+s_0), \varphi_s(w)) < \frac{\delta}{3}. \tag{11}$$

Step 3. Control of the cardinality of the preimages of f.
Let $\gamma \in \mathcal{P}_K(T+\sigma, T+\sigma+\tau)$ and let $w_1, w_2 \in E$ be such that $f(w_1) = f(w_2) = \gamma$. The construction $\gamma = f(w_1)$ gives us an origin s_1 and the construction $\gamma = f(w_2)$ an origin s_2.

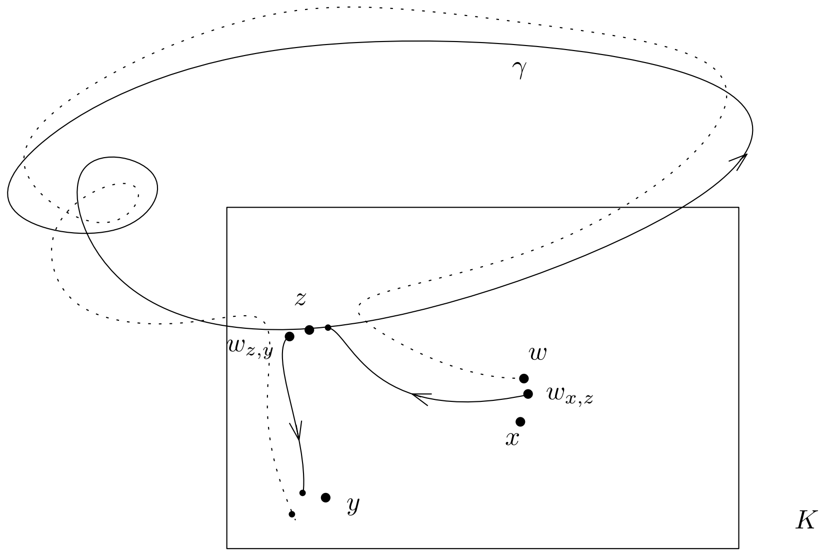

Figure 5.2: How to create a chord from a periodic orbit

For every $s \in [0,T]$ by (11) and by property (2) we obtain

$$\begin{aligned} d(\varphi_s(w_1), \varphi_s(w_2)) &\leq d(\varphi_s(w_1), \gamma(s+s_1)) + d(\gamma(s+s_1), \gamma(s+s_2)) + d(\varphi_s(w_2), \gamma(s+s_2)) \\ &\leq \frac{\delta}{3} + \frac{\delta}{3} + b|s_2 - s_1| \,. \end{aligned}$$

If $b|s_2 - s_1| < \frac{\delta}{3}$, then $w_2 \in B(w_1, \delta, T)$ and, as E is (δ, T)-separating, we obtain $w_1 = w_2$. Therefore, if $w_1 \neq w_2$ then $|s_2 - s_1| \geq \delta/3b$ and

$$\# f^{-1}(\gamma) \leq \left\lceil \frac{3b}{\delta}(T + \sigma + \tau) \right\rceil .$$

As E is a $E(x, y, \eta, T, T+\tau, \delta)$-set of maximal cardinality, we conclude that

$$\mathcal{N}_{\mathcal{C}}(x, y, \eta, T, T+\tau, \delta) \leq \left\lceil \frac{3b}{\delta} \times (T + \sigma + \tau) \right\rceil \times \#\mathcal{P}_K(T+\sigma, T+\sigma+\tau).$$

This is the desired result, with $D = \left\lceil \frac{3b}{\delta} \right\rceil + 1$. ☐

Proposition 5.7. *Let $K \subset M$ be a compact subset. Fix some $\tau > 2\tau_K$. There exists $\varepsilon_0 > 0$ and D such that for all $0 < \delta < \varepsilon_0/4$ and $0 < \eta < \varepsilon_0/2$, there exists $\sigma > 0$ such that for all $x, y \in K$ and $T > 0$, we have*

$$\#\mathcal{P}_K(T, T+\tau) \leq D\, \mathcal{N}_{\mathcal{C}}(x, y, \eta, T+\sigma, T+\sigma+\tau, \delta)\,.$$

Proof. The strategy of the proof is, once again, to use transitivity property 2.14 from x to a periodic orbit γ and from γ to y, and then the finite exact shadowing property 2.6 to get a chord from x to y that starts close from x, goes to γ, follows it, and then finishes close to y. One difficulty, as usual, is to control the (lack of) injectivity of the construction. See Figure 5.2.

Step 1. Choice of appropriate parameters. At the end of the construction, we will use the separation property 2.13 with $\nu = 1$ and $\tau_1 = \tau + 2\tau_K$. This lemma gives us some constants τ_0 and ε_0. Fix $0 < \delta < \varepsilon_0/4$ and $0 < \eta < \varepsilon_0/2$. We will apply the finite exact shadowing property 2.6 on the 1-neighbourhood $K' = \overline{B(K,1)}$ of K, with $N = 3$ pieces of orbits that we want to glue to get a shadowing orbit at distance at most $\eta/2$ of the initial pieces. Property 2.6 gives a constant $0 < \zeta < \eta/2$ associated with K', $N = 3$ and $\eta/2$. Now, transitivity property 2.14 on K' with precision ζ gives us a constant $\sigma_0 > 0$. Observe that by definition, $\tau_{K'} \leq \tau_K$. Set $\sigma = 2\sigma_0 + \tau$.

Step 2. A map from periodic orbits to chords. Let E be a $E(x, y, \eta, T+\sigma, T+\sigma+\tau, \delta)$-set of maximal cardinality. We define a map

$$f : \mathcal{P}_K(T, T+\tau) \to E\,,$$

with controlled lack of injectivity as follows.

Step 2a. Transitivity. Start with a periodic orbit $\gamma \in \mathcal{P}_K(T, T+\tau)$. Choose a starting point $z \in \gamma \cap K$ and a parametrization of γ such that $z = \gamma(0)$. By transitivity property 2.14, we find $w_{x,z} \in B(x,\zeta)$ and $\ell(w_{x,z}) \in [\sigma_0 - \tau_{K'}, \sigma_0 + \tau_{K'}] \subset [\sigma_0 - \tau_K, \sigma_0 + \tau_K]$ such that $\varphi_{\ell(w_{x,z})}(w_{x,z}) \in B(z,\zeta)$. In particular,

$$\ell(w_{x,z}) + \ell(\gamma) \in [\sigma_0 + T - \tau_K, \sigma_0 + T + \tau + \tau_K].$$

Step 2b. Transitivity with well chosen length. For every $S \geq \sigma_0$, transitivity allows to find a chord from $B(z,\zeta)$ to $B(y,\zeta)$ with length in $[S - \tau_K, S + \tau_K]$. Choose

$$S = \sigma_0 + (\sigma_0 + \tau_K - \ell(w_{x,z})) + (T + \tau - \ell(\gamma)) \geq \sigma_0$$

and let $w_{z,y}$ denote the initial point of the associated chord and $\ell(w_{z,y})$ its length. By construction,

$$\ell(z_{x,z}) + \ell(\gamma) + \ell(z_{z,y}) \in [\sigma_0 + (\sigma_0 + \tau_K) + (T+\tau) - \tau_K, \sigma_0 + (\sigma_0 + \tau_K) + (T+\tau) + \tau_K].$$

Therefore (recall that $\sigma = 2\sigma_0 + \tau$ and $\tau > 2\tau_K$),

$$\ell(z_{x,z}) + \ell(\gamma) + \ell(z_{z,y}) \in [T+\sigma, T+\sigma+2\tau_K] \subset [T+\sigma, T+\sigma+\tau].$$

Step 2c. Finite exact shadowing. Recall that $w_{x,z} \in B(x,\zeta)$, $\varphi_{\ell(w_{z,y})}(w_{z,y}) \in B(y,\zeta)$ and $\zeta \leq \eta/2$. As $\varphi_{\ell(w_{x,z})}(w_{x,z}) \in B(\gamma(0),\zeta) = B(z,\zeta)$ and $w_{z,y} \in B(z,\zeta) = B(\gamma(\ell(\gamma)),\zeta)$, the finite exact shadowing property 2.6 gives a point $w \in B(w_{x,z}, \eta/2) \subset B(x,\eta)$ such that

- for $0 \leq s \leq \ell(w_{x,z})$, $d(\varphi_s(w), \varphi_s(w_{x,z})) < \frac{\eta}{2}$;
- for $0 \leq s \leq \ell(\gamma)$, $d(\varphi_{\ell(w_{x,z})+s}(w), \gamma(s)) < \frac{\eta}{2}$;
- for $0 \leq s \leq \ell(w_{z,y})$, $d(\varphi_{\ell(w_{x,z})+\ell(\gamma)+s}(w), \varphi_s(w_{z,y})) < \frac{\eta}{2}$.

In particular, $\varphi_{\ell(w_{x,z})+\ell(\gamma)+\ell(w_{z,y})}(w) \in B(y,\eta)$ and $w \in \mathcal{C}(x,y,\eta,T+\sigma,T+\sigma+\tau)$.

Step 2d. Construction of f. Since E is a $E(x,y,\eta,T+\sigma,T+\sigma+\tau,\delta)$-set, there exists a point $p \in E$ such that $w \in B(p,\delta,T+\sigma)$. Set $f(\gamma) = p$. If w belongs to more than one dynamical ball, just enumerate all the points in E and choose the first one.

Step 3. Bound the cardinality of the preimages of f. Consider $\gamma_1, \gamma_2 \in \mathcal{P}_K(T, T+\tau)$ such that $f(\gamma_1) = f(\gamma_2) = p \in E$. Divide the interval $[T, T+\tau]$ into intervals of length τ_0, where τ_0 is given by Lemma 2.13 as explained at the beginning of the proof. We now prove that if γ_1 and γ_2 satisfy $f(\gamma_1) = f(\gamma_2)$ and $|\ell(\gamma_1) - \ell(\gamma_2)| \leq \tau_0$, then $\gamma_1 = \gamma_2$. This will imply the desired result with $D = \lceil \tau/\tau_0 \rceil$.

Assume from now that $0 \leq |\ell(\gamma_2) - \ell(\gamma_1)| \leq \tau_0$, We want to show that for $s \in [0, T - 2\tau_K]$, we have $d(\gamma_1(s), \gamma_2(s)) \leq \varepsilon_0$, and then use the separation property 2.13.

The construction of $f(\gamma_1)$ (resp. $f(\gamma_2)$) involves chords with initial points w_{x,z_1} and $w_{z_1,y}$ (resp. w_{x,z_2} and $w_{z_2,y}$), and produces a point w_1 (resp. w_2) in $\mathcal{C}(x,y,\eta,T+\sigma,T+\sigma+\tau)$ such that $w_1 \in B(p,\delta,T+\sigma)$ (resp. $w_2 \in B(p,\delta,T+\sigma)$). This proves $w_1 \in B(w_2, 2\delta, T+\sigma)$. Without loss of generality, we may assume that $\ell(w_{x,z_1}) \leq \ell(w_{x,z_2})$.

By construction, for every $\ell(w_{x,z_1}) \leq s \leq \ell(\gamma_1) + \ell(w_{x,z_1})$, $\varphi_s(w_1)$ is $\eta/2$-close to γ_1. Similarly, for every $\ell(w_{x,z_2}) \leq s \leq \ell(\gamma_2) + \ell(w_{x,z_2})$, $\varphi_s(w_2)$ is $\eta/2$-close to γ_2. More precisely, for all $s \in [0, \ell(\gamma_1)]$,

$$d(\gamma_1(s), \varphi_{s+\ell(w_{x,z_1})}(w_1)) \leq \frac{\eta}{2}.$$

Therefore, for all $s \in [-(\ell(w_{x,z_2}) - \ell(w_{x,z_1})), \ell(\gamma_1) - (\ell(w_{x,z_2}) - \ell(w_{x,z_1}))]$

$$d(\gamma_1(s + (\ell(w_{x,z_2}) - \ell(w_{x,z_1})), \varphi_{s+\ell(w_{x,z_2})}(w_1)) \leq \frac{\eta}{2}.$$

Symmetrically, for all $s \in [0, \ell(\gamma_2)]$,

$$d(\gamma_2(s), \varphi_{s+\ell(w_{x,z_2})}(w_2)) \leq \frac{\eta}{2}.$$

Recall that $0 \le \ell(w_{x,z_2}) - \ell(w_{x,z_1}) \le 2\tau_K$, $T \le \ell(\gamma_1) \le T+\tau$ and $T \le \ell(\gamma_2) \le T+\tau$. Therefore, for all $s \in [0, T-2\tau_K]$, we have

$$\begin{aligned} d(\gamma_1(s+\ell(w_{x,z_2})-\ell(w_{x,z_1})), \gamma_2(s)) \le & d(\gamma_1(s+\ell(w_{x,z_2})-\ell(w_{x,z_1})), \varphi_{s+\ell(w_{x,z_2})}(w_1)) \\ & + d(\varphi_{s+\ell(w_{x,z_2})}(w_1), \varphi_{s+\ell(w_{x,z_2})}(w_2)) \\ & + d(\varphi_{s+\ell(w_{x,z_2})}(w_2), \gamma_2(s)) \\ < & \frac{\eta}{2} + 2\delta + \frac{\eta}{2} < \varepsilon_0. \end{aligned}$$

By Lemma 2.13, we deduce that $\gamma_1 = \gamma_2$. Thus, the cardinality of the preimage of any point by f is bounded by $D = \lceil \tau/\tau_0 \rceil$ and

$$\#\mathcal{P}_K(T, T+\tau) \le D\,\#E = D\,\mathcal{N}_{\mathcal{C}}(x, y, \eta, T+\sigma, T+\sigma+\tau, \delta).$$

as required. □

We end this section with a technical adaptation of Proposition 5.6 which will be useful to compare entropies at infinity in the proof of Theorem 5.26. The idea is to compare chords and orbits contained in specified compact subsets of M. This may be skipped on first reading.

Let K be a compact subset of M. Let $x, y \in K$ and let K' be a compact set containing K. Analogously to Definition 5.1, we consider $E_{K'}(x, y, \eta, T^-, T^+, \delta)$-sets which are $E(x, y, \eta, T^-, T^+, \delta)$-sets made up of chords from $B(x, \eta)$ to $B(y, \eta)$ *contained in* K'. We will denote by $\mathcal{N}_{\mathcal{C},K'}(x, y, \eta, T^-, T^+, \delta)$ the maximal cardinality of such sets.

Proposition 5.8. *Let $K \subset M$ be a compact subset of M with nonempty interior. Let $\tau > 2\tau_K$. Fix $0 < \delta < 3$. There exist constants $R_{\min} > 0$, $D = D(\delta) > 0$, $T_{\min} > 0$ and $\sigma > 0$ such that for all $x, y \in K$, $R \ge R_{\min}$, $T \ge T_{\min}$ and all $0 < \eta < 1$, we have*

$$\mathcal{N}_{\mathcal{C},K_R}(x, y, \eta, T, T+\tau, \delta) \;\le\; D \times (T+\sigma+\tau) \times \#\{\gamma \in \mathcal{P}_K(T+\sigma, T+\sigma+\tau), \gamma \subset K_{R+1}\}$$

where $K_R = \overline{B(K, R)}$.

Proof. The proof goes exactly as the proof of Proposition 5.6. One just have to choose $R_{\min}$ big enough so that $K_{R_{\min}}$ contains all chords connecting, by uniform transitivity, any couple of balls, among a finite family covering K, whose radius depends only on δ and K. As $\delta/3 < 1$, the periodic orbit γ is then contained in K_{R+1} if the original chord is contained in K_R. □

5.3 Chord entropy

In this section, we define a notion of entropy that counts chords with increasing length. We prove that for H-flows, it coincides with Gurevic entropy. This chord entropy will be easier to use than the standard Gurevic entropy.

Fix $x, y \in M$ and $\sigma > 0$, $\eta > 0$, $\delta > 0$. Let

$$h_{\mathcal{C}}(x, y, \eta, \delta) = \limsup_{T\to\infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}(x, y, \eta, T, T+\sigma, \delta). \tag{12}$$

Recall that $\mathcal{N}_{\mathcal{C}}(x, y, \eta, T, T+\sigma, \delta)$ counts chords and is defined in Definition 5.1. This does not depend on σ, as proved in the following lemma.

Lemma 5.9. *The quantity $h_{\mathcal{C}}(x, y, \eta, \delta)$ is non-decreasing in $\eta > 0$, non-increasing in $\delta > 0$ and does not depend on C.*

Proof. The first assertion comes from Fact 5.3 and the second assertion from Fact 5.4.

We now prove the last assertion. Choose two constants $0 < \sigma_1 < \sigma_2 < \infty$. Let n be the smallest integer such that $\sigma_2 \le n\sigma_1$. For all choices of x, y, δ, η and $T > 0$, we have

$$\mathcal{N}_{\mathcal{C}}(x, y, \eta, T, T+\sigma_1, \delta) \le \mathcal{N}_{\mathcal{C}}(x, y, \eta, T, T+\sigma_2, \delta) \le \sum_{j=0}^{n-1} \mathcal{N}_{\mathcal{C}}(x, y, \eta, T+j\sigma_1, T+(j+1)\sigma_1, \delta).$$

Indeed,

$$\mathcal{C}(x,y,\eta,T,T+\sigma_2) \subset \bigcup_{j=0}^{n-1} \mathcal{C}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1)$$

and if E is a $E(x,y,\eta,T,T+\sigma_2,\delta)$-set of maximal cardinality, then $E_j = E \cap \mathcal{C}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1)$ is (δ,T)-separating and therefore $(\delta,T+j\sigma_1)$-separating. By Fact 5.2, we deduce that

$$\#E_j \leq \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1,\delta)\,,$$

and thus, since $E \subset \bigcup_{j=0}^{n-1} E_j$,

$$\mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma_2,\delta) = \#E \leq \sum_{j=0}^{n-1} \#E_j \leq \sum_{j=0}^{n-1} \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1,\delta)\,.$$

Therefore,

$$\mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma_1,\delta) \leq \mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma_2,\delta) \leq n \max_{j=0\ldots,n-1} \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1,\delta)\,.$$

Yet

$$\begin{aligned}
\limsup_{T\to\infty} \frac{\log \mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma_1,\delta)}{T} &= \limsup_{T\to\infty} \max_{j=0\ldots,n-1} \frac{\log \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1,\delta)}{T+j\sigma_1} \\
&= \limsup_{T\to\infty} \frac{1}{T} \log \max_{j=0\ldots,n-1} \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1,\delta) \\
&= \limsup_{T\to\infty} \frac{1}{T} \log n \max_{j=0\ldots,n-1} \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+j\sigma_1,T+(j+1)\sigma_1,\delta).
\end{aligned}$$

Therefore

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma_1,\delta) = \limsup_{T\to\infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma_2,\delta).$$

□

Lemma 5.10. *Let $K \subset M$ be a compact subset. Let $\delta > 0$ and $\eta_1 > 0$ such that $\eta_1 < \delta/2$. Then there exists $\tilde{\eta}_0$ such that for all x_0, y_0, x_1, y_1 in K and for all $0 < \eta_0 \leq \tilde{\eta}_0$,*

$$h_{\mathcal{C}}(x_0,y_0,\eta_0,\delta) \leq h_{\mathcal{C}}(x_1,y_1,\eta_1,\frac{\delta}{2}).$$

In particular

$$\lim_{\delta\to 0}\lim_{\eta\to 0} h_{\mathcal{C}}(x_0,y_0,\eta,\delta) = \lim_{\delta\to 0}\lim_{\eta\to 0} h_{\mathcal{C}}(x_1,y_1,\eta,\delta).$$

Proof. By Proposition 5.5, there exists σ and $\tilde{\eta}_0$ such that, for all $0 < \eta_0 \leq \tilde{\eta}_0$ and all $T > 0$

$$\mathcal{N}_{\mathcal{C}}(x_0,y_0,\eta_0,T,T+2\tau_K,\delta) \leq \mathcal{N}_{\mathcal{C}}(x_1,y_1,\eta_1,T+\sigma,T+\sigma+2\tau_K,\delta/2).$$

Therefore

$$h_{\mathcal{C}}(x_0,y_0,\eta_0,\delta) \leq h_{\mathcal{C}}(x_1,y_1,\eta_1,\frac{\delta}{2}).$$

Considering the limit when $\eta_0 \to 0$ and then when $\eta_1 \to 0$, we obtain

$$\lim_{\eta\to 0} h_{\mathcal{C}}(x_0,y_0,\eta,\delta) \leq \lim_{\eta\to 0} h_{\mathcal{C}}(x_1,y_1,\eta,\frac{\delta}{2}).$$

We now consider the limit when $\delta \to 0$ to obtain

$$\lim_{\delta\to 0}\lim_{\eta\to 0} h_{\mathcal{C}}(x_0,y_0,\eta,\delta) \leq \lim_{\delta\to 0}\lim_{\eta\to 0} h_{\mathcal{C}}(x_1,y_1,\eta,\delta).$$

Therefore, by inverting the roles,

$$\lim_{\delta\to 0}\lim_{\eta\to 0} h_{\mathcal{C}}(x_0,y_0,\eta,\delta) = \lim_{\delta\to 0}\lim_{\eta\to 0} h_{\mathcal{C}}(x_1,y_1,\eta,\delta).$$

□

Definition 5.11. *We define the* chord entropy *as*

$$h_{\mathcal{C}}(\varphi) = \lim_{\delta\to 0}\lim_{\eta\to 0}\limsup_{T\to\infty}\frac{1}{T}\log\mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\sigma,\delta)\,.$$

Observe that the chord entropy does not depend on the choice of x, y and σ.

Theorem 5.12. *Let φ be a H-flow on M. The Gurevic entropy coincides with the chord entropy:*

$$h_{\text{Gur}}(\varphi) = h_{\mathcal{C}}(\varphi).$$

Moreover, for every fixed compact set K with nonempty interior, there exists $\alpha_0 > 0$ such that for all $x, y \in K$, $\tau \geq 5\tau_K$, $0 < \delta < \alpha_0/2$ and $0 < \eta < \alpha_0/2$, the quantity

$$\frac{1}{T}\log\mathcal{N}_C(x,y,\eta,T,T+\tau,\delta)$$

converges towards $h_{\text{Gur}}(\varphi)$ when $T \to +\infty$. Thus

$$h_{\mathcal{C}}(\varphi) = h_{\text{Gur}}(\varphi) = \lim_{T\to\infty}\frac{1}{T}\log\mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\tau,\delta).$$

Proof. It follows immediately from Propositions 5.6 and 5.7, as we will see now.

Fix $\tau = 5\tau_K$. Fix ε_0 as in Proposition 5.7. Let $\alpha_0 = \min(\varepsilon_0/2, 1)$. Choose $K \subset M$ compact and $x, y \in K$. Fix $0 < \delta < \alpha_0/2$ and $0 < \eta < \alpha_0/2$. From Propositions 5.6 and 5.7, there exists σ, σ', D and D' such that for all $T \gg 1$

$$\frac{1}{D}\#\mathcal{P}_K(T,T+\tau) \leq \mathcal{N}_{\mathcal{C}}(x,y,\eta,T+\sigma,T+\sigma+\tau,\delta) \leq D'(T+\sigma+\sigma'+\tau)\#\mathcal{P}_K(T+\sigma+\sigma',T+\sigma+\sigma'+\tau).$$

As h_{Gur} is a true limit (Theorem 3.10), we obtain that the following limits exist and

$$\lim_{T\to\infty}\frac{1}{T}\log\mathcal{P}_K(T,T+\tau) = \lim_{T\to\infty}\frac{1}{T}\log\mathcal{N}_{\mathcal{C}}(x,y,\eta,T,T+\tau,\delta)$$

Thus

$$h_{\text{Gur}}(\varphi) = h_{\mathcal{C}}(\varphi).$$

□

5.4 Entropy at infinity through chords

This section is devoted to the notion of chord entropy at infinity, that we will later compare with $h^{\infty}_{\text{Gur}}(\varphi)$. Fix a compact subset $K \subset M$, and two points $x, y \in \partial K$. For $\eta > 0$, we define the η-interior neighbourhood of K as

$$K_{-\eta} = K \setminus \cup_{x\in\partial K} B(x,\eta)\,.$$

We define a *chord outside $K_{-\eta}$ from $B(x,\eta)$ to $B(y,\eta)$* as a path from a point of $B(x,\eta)$ to a point of $B(y,\eta)$ that does not intersect $K_{-\eta}$. We now consider the set of chords outside $K_{-\eta}$ with controlled length. Define $\mathcal{C}^{K^c}(x,y,\eta,T^-,T^+)$ as

$$\{z \in B(x,\eta), \exists\tau \in [T^-,T^+] \text{ such that } \varphi_\tau(z) \in B(y,\eta) \text{ and } \varphi_{[0,\tau]}(z) \cap K_{-\eta} = \emptyset\}.$$

Observe that some sets $\mathcal{C}^{K^c}(x,y,\eta,T^-,T^+)$ could be empty for all T^- and T^+. Following Definition 5.1, we introduce the following notations.

Definition 5.13. *Let $\delta > 0$. A set E is a $E^{K^c}(x,y,\eta,T^-,T^+,\delta)$-set if:*

1. *$E \subset \mathcal{C}^{K^c}(x,y,\eta,T^-,T^+)$;*
2. *E is a (δ,T^-)-separating set;*
3. *the set $\mathcal{C}^{K^c}(x,y,\eta,T^-,T^+)$ is contained in the union of dynamical balls $\bigcup_{z\in E} B(z,\delta,T^-)$.*

We define the number $\mathcal{N}_{\mathcal{C}}^{K^c}(x,y,\eta,T^-,T^+,\delta)$ *of chords from x to y outside $K_{-\eta}$ with length in $[T^-,T^+]$ as the maximal cardinality of a $E^{K^c}(x,y,\eta,T^-,T^+,\delta)$-set.*

The proof of the following fact is similar to the proof of Fact 5.2.

Fact 5.14. *Let E be a set satisfying points 1 and 2 of Definition 5.13. Then there exists a set E' containing E which is a $E^{K^c}(x,y,\eta,T^-,T^+,\delta)$-set.*

The following fact is analogous to Fact 5.3 and we omit its proof.

Fact 5.15. *Let K be a compact set with nonempty interior and $x,y \in \partial K$. Let $\delta > 0$ and $0 < T^- < T^+$. The map*

$$\eta \in (0,+\infty) \mapsto \mathcal{C}^{K^c}(x,y,\eta,T^-,T^+,\delta)$$

is non-decreasing for the inclusion. The map

$$\eta \in (0,+\infty) \mapsto \mathcal{N}_{\mathcal{C}}^{K^c}(x,y,\eta,T^-,T^+,\delta)$$

is non-decreasing.

The following fact is similar to Fact 5.4.

Fact 5.16. *Let K be a compact set with nonempty interior and $x,y \in \partial K$. Let $\eta > 0$ and $0 < T^- < T^+$. The map*

$$\delta \mapsto \mathcal{N}_{\mathcal{C}}^{K^c}(x,y,\eta,T^-,T^+,\delta)$$

is non-increasing.

Moreover, a proof similar to the proof of Lemma 5.9 gives the following result.

Lemma 5.17. *The exponential growth rate*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}^{K^c}(x,y,\eta,T,T+C,\delta)$$

does not depend on $C > 0$.

We can now start defining the chord entropy outside a compact set. Set

$$h_{\mathcal{C}}^{K^c}(x,y,\eta,\delta) = \limsup_{T\to\infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}^{K^c}(x,y,\eta,T,T+C,\delta)$$

and

$$h_{\mathcal{C}}^{K^c}(\eta,\delta) = \sup_{x,y\in\partial K} \limsup_{T\to\infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}^{K^c}(x,y,\eta,T,T+C,\delta) = \sup_{x,y\in\partial K} h_{\mathcal{C}}^{K^c}(x,y,\eta,\delta)\,.$$

The *chord entropy outside K* is defined as

$$h_{\mathcal{C}}^{K^c}(\varphi) = \lim_{\delta\to 0} \lim_{\eta\to 0} h_{\mathcal{C}}^{K^c}(\eta,\delta)\,. \tag{13}$$

This definition makes sense because the function $\eta \to h_{\mathcal{C}}^{K^c}(\eta,\delta)$ is non-decreasing, and the function $\delta \to h_{\mathcal{C}}^{K^c}(\eta,\delta)$ is non-increasing.

In the following proposition we prove that the above quantity is essentially non-increasing when K grows.

Proposition 5.18. *If $K_1 \subset \overset{\circ}{K}_2$ and K_1 has nonempty interior, then*

$$h_{\mathcal{C}}^{K_2^c}(\varphi) \leq h_{\mathcal{C}}^{K_1^c}(\varphi)\,.$$

This proposition is proved below. It motivates the following definition.

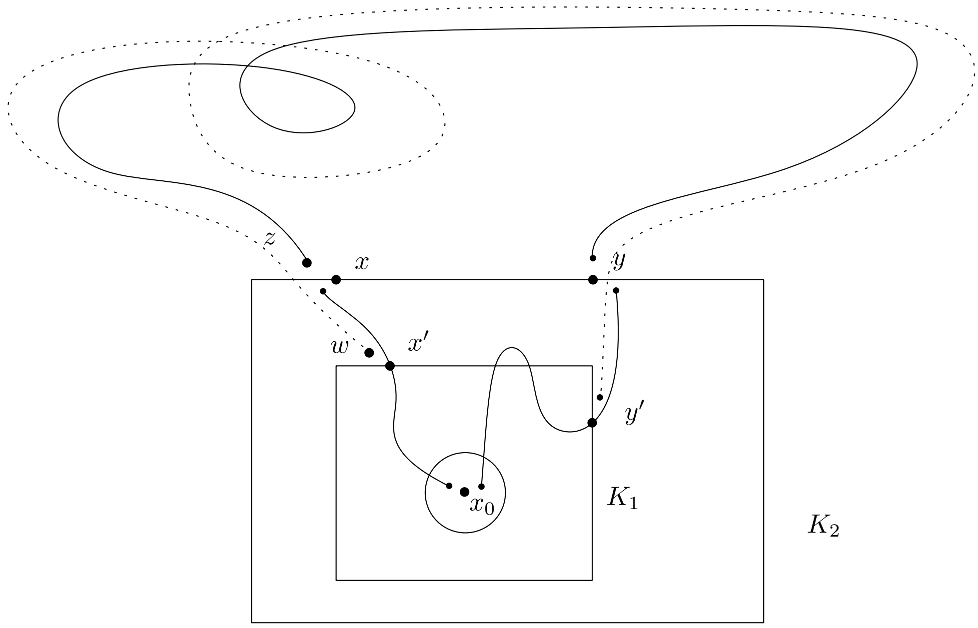

Figure 5.3: How to construct a chord connecting x' to y' from a chord connecting x to y

Definition 5.19. *The* chord entropy at infinity *is*

$$h_{\mathcal{C}}^{\infty}(\varphi) = \inf_{K} h_{\mathcal{C}}^{K^c}(\varphi) = \inf_{K} \lim_{\delta \to 0} \lim_{\eta \to 0} \sup_{x,y \in \partial K} \limsup_{T \to \infty} \frac{1}{T} \log \mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \eta, T, T + C, \delta) , \tag{14}$$

the infimum being taken over all compact sets K with nonempty interior.

The following corollary is an immediate consequence of Proposition 5.18 and of Definition 5.19.

Corollary 5.20 (Chord entropy at infinity is invariant under compact perturbations)**.** *Let $\varphi\colon M_1 \to M_1$ and $\psi\colon M_2 \to M_2$ be two H-flows such that there exist two compact sets $K_1 \subset M_1$ and $K_2 \subset M_2$ with $M_1 \setminus K_1 = M_2 \setminus K_2$, $\varphi(M_1 \setminus K_1) = \psi(M_2 \setminus K_2)$ and $\varphi_{|M_1\setminus K_1} = \psi_{|M_2\setminus K_2}$. Then*

$$h_{\mathcal{C}}^{\infty}(\varphi) = h_{\mathcal{C}}^{\infty}(\psi).$$

Proof of Proposition 5.18. The main idea of the proof is the following. For $x, y \in \partial K_2$, find some points $x', y' \in \partial K_1$ such that the number of chords from x to y outside K_2 is bounded by the number of chords from x' to y' outside K_1. As the chord entropy outside a compact set is defined by counting chords, the theorem is proved. More precisely, the points x' and y' have the following property: there exist a chord from x' to $B(x, \eta/2)$ contained in $M \setminus \mathring{K}_1$ and a chord from $B(y, \eta/2)$ to y' contained in $M \setminus \mathring{K}_1$. We can now concatenate these two chords and a chord from x to y outside K_2 to obtain a chord from x' to y' outside K_1. As this process is injective, we obtain the desired inequality between the number of chords and therefore a proof of the theorem. See Figure 5.3.

Step 1. Setting the parameters. Fix $0 < \delta < \min(d(\partial K_1, \partial K_2), 1)$ and $C > 0$. Since the interior of K_1 is nonempty, we can fix x_0 and $\delta_0 > 0$ such that $B(x_0, \delta_0) \subset \mathring{K}_1$. Fix $0 < \alpha < \min(\delta/4, \delta_0)$. By the finite exact shadowing property, see Definition 2.6, applied at the compact set $\overline{B(K_2, 1)}$, $\delta = \alpha$ and $N = 3$, we get some $0 < \eta < \alpha$.

Step 2. Comparing chords outside K_1 and K_2. We now show that for every $x, y \in \partial K_2$ there exists $x', y' \in \partial K_1$ and $\ell(x'), \ell(y') > 0$ such that

$$\mathcal{N}_{\mathcal{C}}^{K_2^c}\left(x, y, \frac{\eta}{2}, T, T + C, \delta\right) \leq \mathcal{N}_{\mathcal{C}}^{K_1^c}\left(x', y', \alpha, T + \ell(x') + \ell(y'), T + \ell(x') + \ell(y') + C, \frac{\delta}{2}\right)$$

for any $T > 0$.

Step 2a. How to find x' and y'? Let $x, y \in \partial K_2$. By Lemma 2.14 applied at points $x, x_0 \in K_2$ with $\delta = \frac{\eta}{2} > 0$, there exists $t > 0$ such that $\varphi_t(B(x_0, \frac{\eta}{2})) \cap B(x, \frac{\eta}{2}) \neq \emptyset$. Since $\frac{\eta}{2} < \delta_0$, we have $B\left(x_0, \frac{\eta}{2}\right) \subset \mathring{K}_1$. Therefore, there exist a point $x' \in \partial K_1$ and a time $0 < \ell(x') \leq t$ such that

$$\varphi_{\ell(x')}(x') \in B\left(x, \frac{\eta}{2}\right) \quad \text{and} \quad \varphi_{[0,\ell(x')]}(x') \cap \mathring{K}_1 = \emptyset\,.$$

By a similar argument, there exists a point $y' \in \partial K_1$ and a time $\ell(y') > 0$ such that

$$\varphi_{-\ell(y')}(y') \in B\left(y, \frac{\eta}{2}\right) \quad \text{and} \quad \varphi_{[-\ell(y'),0]}(y') \cap \mathring{K}_1 = \emptyset\,.$$

Step 2b Constructing a map f from chords connecting x to y, to chords connecting x' to y'. Let E be a $E^{K_2^c}(x, y, \frac{\eta}{2}, T, T + C, \delta)$-set of maximal cardinality. We define a map

$$f\colon E \to \mathcal{C}^{K_1^c}(x', y', \alpha, T + \ell(x') + \ell(y'), T + \ell(x') + \ell(y') + C)$$

as follows. Let $z \in E$. In particular, there exists $\ell(z) \in [T, T + C]$ such that $z \in B(x, \frac{\eta}{2}), \varphi_{\ell(z)}(z) \in B(y, \frac{\eta}{2})$ and $\varphi_{[0,\ell(z)]}(z) \cap (K_2)_{-\frac{\eta}{2}} = \emptyset$. Observe that

$$d(\varphi_{\ell(x')}(x'), z) < \eta \quad \text{and} \quad d(\varphi_{\ell(z)}(z), \varphi_{-\ell(y')}(y')) < \eta\,.$$

By the finite exact shadowing property, see Definition 2.6, applied at $x', z, \varphi_{-\ell(y')}(y') \in \overline{B(K_2, 1)}$, we obtain a point w such that

- for all $s \in [0, \ell(x')]$, $d(\varphi_s(w), \varphi_s(x')) < \alpha$;
- for all $s \in [0, \ell(z)]$, $d(\varphi_{\ell(x')+s}(w), \varphi_s(z)) < \alpha$;
- for all $s \in [0, \ell(y')]$, $d(\varphi_{\ell(x')+\ell(z)+s}(w), \varphi_{-\ell(y')+s}(y')) < \alpha$.

In particular, we have $w \in B(x', \alpha)$ and $\varphi_{\ell(x')+\ell(z)+\ell(y')}(w) \in B(y', \alpha)$ with $\ell(x') + \ell(z) + \ell(y') \in [T + \ell(x') + \ell(y'), T + \ell(x') + \ell(y') + C]$. Moreover, since $\varphi_{[0,\ell(x')]}(x')$ and $\varphi_{[-\ell(y'),0]}(y')$ do not intersect $\mathring{K}_1$, both $\varphi_{[0,\ell(x')]}(w)$ and $\varphi_{[\ell(x')+\ell(z),\ell(x')+\ell(z)+\ell(y')]}(w)$ do not intersect $(K_1)_{-\alpha}$. Additionally, since $\varphi_{[0,\ell(z)]}(z)$ does not intersect $(K_2)_{-\frac{\eta}{2}}$ and $\eta/2 < \delta < d(\partial K_1, \partial K_2)$, the arc $\varphi_{[\ell(x'),\ell(x')+\ell(z)]}(w)$ does not intersect $(K_1)_{-\alpha}$. This proves

$$w \in \mathcal{C}^{K_1^c}(x', y', \alpha, T + \ell(x') + \ell(y'), T + \ell(x') + \ell(y') + C).$$

Set $f(z) = w$.

Step 2c. The map f is injective. Let now $z_1, z_2 \in E$ and assume that $f(z_1) = f(z_2) = w$. By the construction of f and since $\ell(z_1) \geq T$ and $\ell(z_2) \geq T$, for every $s \in [0, T]$, we have

$$d(\varphi_s(z_1), \varphi_s(z_2)) \leq d(\varphi_s(z_1), \varphi_{\ell(x')+s}(w)) + d(\varphi_s(z_2), \varphi_{\ell(x')+s}(w)) < 2\alpha < \frac{\delta}{2}.$$

Since the set E is (δ, T)-separating, we conclude that $z_1 = z_2$, i.e., f is injective.

Step 2d. Consequences on the numbers of chords. Observe that the set $f(E)$ is contained in $\mathcal{C}^{K_1^c}(x', y', \alpha, T + \ell(x') + \ell(y'), T + \ell(x') + \ell(y') + C)$. Moreover, since E is (δ, T)-separating and $2\alpha < \frac{\delta}{2}$, the set $f(E)$ is $(\frac{\delta}{2}, T + \ell(x') + \ell(y'))$-separating. Since $\#E = \#f(E)$ (as f is injective) and by Fact 5.14, we then conclude that

$$\mathcal{N}_{\mathcal{C}}^{K_2^c}(x, y, \frac{\eta}{2}, T, T + C, \delta) \leq \mathcal{N}_{\mathcal{C}}^{K_1^c}(x', y', \alpha, T + \ell(x') + \ell(y'), T + \ell(x') + \ell(y') + C, \frac{\delta}{2}).$$

Step 3. Conclusion. The previous inequality implies that, for every $x, y \in \partial K_2$, there exists $x', y' \in \partial K_1$ such that

$$h_{\mathcal{C}}^{K_2^c}(x, y, \frac{\eta}{2}, \delta) \leq h_{\mathcal{C}}^{K_1^c}(x', y', \alpha, \frac{\delta}{2}) \leq h_{\mathcal{C}}^{K_1^c}(\alpha, \frac{\delta}{2})\,,$$

where $\eta < \alpha < \delta$. Considering then the supremum over $x, y \in \partial K_2$, we obtain

$$h_{\mathcal{C}}^{K_2^c}(\frac{\eta}{2}, \delta) \leq h_{\mathcal{C}}^{K_1^c}(\alpha, \frac{\delta}{2})\,;$$

taking the limit as $\eta \to 0$ and then the limit as $\alpha \to 0$, we have

$$\lim_{\eta \to 0} h_{\mathcal{C}}^{K_2^c}(\frac{\eta}{2}, \delta) \leq \lim_{\alpha \to 0} h_{\mathcal{C}}^{K_1^c}(\alpha, \frac{\delta}{2})\,.$$

We now let $\delta \to 0$ and obtain $h_{\mathcal{C}}^{K_2^c}(\varphi) \leq h_{\mathcal{C}}^{K_1^c}(\varphi)$, as required. ☐

Remark 5.21. The proof of Proposition 5.18 relies only on the finite exact shadowing property and the transitivity of the flow.

5.5 Gurevic entropy at infinity and chords entropy at infinity coincide

Our goal is now to show that counting the chords at infinity is the same as counting the periodic orbits at infinity, as presented in the following statement.

Theorem 5.22. *Let $\varphi : M \to M$ be a H-flow. Then*

$$h_{\mathrm{Gur}}^{\infty}(\varphi) = h_{\mathcal{C}}^{\infty}(\varphi)\,.$$

The following corollary is immediate from the above theorem and corollary 5.20.

Corollary 5.23 (Gurevic entropy at infinity is invariant under compact perturbations)**.** *Let $\varphi\colon M_1 \to M_1$ and $\psi\colon M_2 \to M_2$ be two H-flows such that there exist two compact sets $K_1 \subset M_1$ and $K_2 \subset M_2$ with $M_1 \setminus K_1 = M_2 \setminus K_2$, $\varphi(M_1 \setminus K_1) = \psi(M_2 \setminus K_2)$ and $\varphi_{|M_1 \setminus K_1} = \psi_{|M_2 \setminus K_2}$. Then*

$$h_{\mathrm{Gur}}^{\infty}(\varphi) = h_{\mathrm{Gur}}^{\infty}(\psi)\,.$$

The proof of the inequality $h_{\mathcal{C}}^{\infty}(\varphi) \leq h_{\mathrm{Gur}}^{\infty}(\varphi)$ is easier and done in Proposition 5.24 below. The hard inequality is $h_{\mathrm{Gur}}^{\infty}(\varphi) \leq h_{\mathcal{C}}^{\infty}(\varphi)$. Indeed, we saw in section 5.2 that chords and periodic orbits have the same exponential growth rate. However, the Gurevic entropy at infinity counts periodic orbits that spend a small proportion of time in K, but an unbounded amount of time, whereas the chord entropy at infinity counts chords outside K, that can be closed into periodic orbits that spend a bounded amount of time in K. We follow the strategy developed in [GST23] and cut a periodic orbit that spends most of its time outside K into successive excursions outside K. This is expressed in the technical Theorem 5.26 whose Corollary 5.27 gives the desired inequality.

Proposition 5.24. *Let $\varphi : M \to M$ be a H-flow. Then*

$$h_{\mathcal{C}}^{\infty}(\varphi) \leq h_{\mathrm{Gur}}^{\infty}(\varphi).$$

Proof. The main idea of the proof is the following: the chord entropy at infinity can be approximated by counting separated chords outside a big compact K which start in a neighborhood of $x \in \partial K$ and end in neighborhood of $y \in \partial K$. These orbits can be closed to obtain different periodic orbits which intersect K but stay a finite amount of time in K. As these orbits contribute to the Gurevic entropy at infinity we obtain that the chord entropy at infinity is smaller than the Gurevic entropy at infinity. We now give a detailed proof of the proposition.

First note that if $h_{\mathcal{C}}^{\infty}(\varphi) = \infty$, then $h_{\mathcal{C}}(\varphi) = h_{\mathrm{Gur}}(\varphi) = \infty$ and, from Lemma 4.13, $h_{\mathrm{Gur}}^{\infty}(\varphi) = \infty$: the proposition is proved. Therefore, in the remaining part of the proof, we may assume $h_{\mathcal{C}}^{\infty}(\varphi) < \infty$.

Step 1. We quantitatively approximate the chord entropy at infinity by counting chords from x to y outside a compact set. Fix $\varepsilon > 0$ and $\sigma_0 > 0$. By the definition of $h_{\mathcal{C}}^{\infty}(\varphi)$ (Definition 5.19) and Proposition 5.18, there exists a compact set K_0 with nonempty interior such that, for every compact set K for which $K_0 \subset \mathring{K}$, we have

$$h_{\mathcal{C}}^{K^c}(\varphi) \in \left[h_{\mathcal{C}}^{\infty}(\varphi), h_{\mathcal{C}}^{\infty}(\varphi) + \frac{\varepsilon}{8}\right]\,.$$

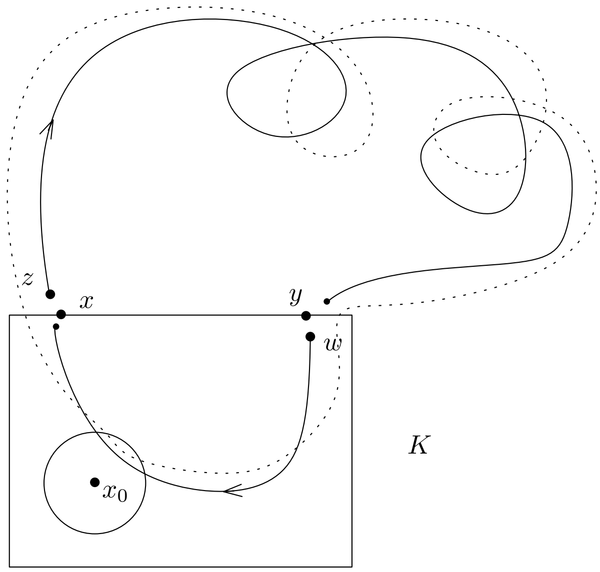

Figure 5.4: Construction of f

Fix a compact set K such that $K_0 \subset \mathring{K}$. Without loss of generality, we may assume there exists $x_0 \in \mathring{K}$ such that $B(x_0, 2\varepsilon) \subset K$. There exists $0 < \delta < \varepsilon$ so that

$$\lim_{\eta \to 0} \sup_{x,y \in \partial K} \limsup_{T \to \infty} \frac{1}{T} \log \left(\mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \eta, T, T+\sigma_0, \delta) \right) \in [h_{\mathcal{C}}^{\infty}(\varphi) - \varepsilon/4, h_{\mathcal{C}}^{\infty}(\varphi) + \varepsilon/4] \,.$$

Fix $0 < \alpha < \frac{\delta}{3}$. Apply the multiple closing lemma (Lemma 2.15) to the compact $\overline{B(K, \varepsilon)}$, $\delta = \alpha$, $\nu = 1$ and $N = 2$ to obtain a time $T_{\min} > 0$ and a parameter $0 < \eta < \alpha$. Coming back to the chord entropy, we can assume that $\eta < \delta$ is small enough such that

$$\sup_{x,y \in \partial K} \limsup_{T \to \infty} \frac{1}{T} \log \left(\mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \frac{\eta}{2}, T, T+\sigma_0, \delta) \right) \in [h_{\mathcal{C}}^{\infty}(\varphi) - \varepsilon/2, h_{\mathcal{C}}^{\infty}(\varphi) + \varepsilon/2] \,.$$

Consider then $x, y \in \partial K$ such that

$$\limsup_{T \to \infty} \frac{1}{T} \log \left(\mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \frac{\eta}{2}, T, T+\sigma_0, \delta) \right) \in [h_{\mathcal{C}}^{\infty}(\varphi) - \varepsilon, h_{\mathcal{C}}^{\infty}(\varphi) + \varepsilon] \,. \tag{15}$$

Apply the uniform transitivity (Lemma 2.14) to the compact set K, the compact set $K' = \overline{B(x_0, \varepsilon)}$ and $\delta = \frac{\eta}{2}$ and obtain a time $\sigma > 0$. Without loss of generality, we may assume $\sigma - \tau_K \geq 1$.

Step 2. We construct a map f from chords to periodic orbits (see Figure 5.4) Fix $\rho > 0$. Let

$$T > \max \left(T_{\min}, \frac{\sigma + \tau_K + 1}{\rho}, \sigma_0 + \sigma + \tau_K + 1 \right) \,. \tag{16}$$

Consider a set E that is a $E^{K^c}(x, y, \frac{\eta}{2}, T, T+\sigma_0, \delta)$-set of maximal cardinality, i.e.,

$$\#E = \mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \frac{\eta}{2}, T, T+\sigma_0, \delta).$$

We now build a map

$$f \colon E \to \mathcal{P}_{K_{-2\alpha}}^{\rho}(T, T+\sigma_0')$$

where $\sigma_0' = \sigma_0 + \sigma + \tau_K + 1$ and $\mathcal{P}_{K_{-2\alpha}}^{\rho}(T, T+\sigma_0')$ is defined in Definition 3.16. We fix once for all a point $w \in B(y, \frac{\eta}{2})$ such that $\varphi_{\ell(w)}(w) \in B(x, \frac{\eta}{2})$ for some $\ell(w) \in [\sigma - \tau_K, \sigma + \tau_K]$ and $\varphi_{[0,\ell(w)]}(w) \cap \overline{B(x_0, \varepsilon)} \neq \emptyset$. Such a point exists thanks to uniform transitivity (Lemma 2.14) applied at the points $x, y \in \partial K \subset K$.

Let $z \in E$. In particular, $z \in B(x, \frac{\eta}{2})$ and $\varphi_{\ell(z)}(z) \in B(y, \frac{\eta}{2})$ for some $\ell(z) \in [T, T+\sigma_0]$. Moreover, $\varphi_{[0,\ell(z)]}(z) \cap K_{-\frac{\eta}{2}} = \emptyset$ (this also comes from the definition of E, see Definition 5.13). By the multiple closing lemma (Lemma 2.15) applied at the points z and w and times $\ell(z)$ and $\ell(w)$, we obtain a periodic orbit γ of period $l(\gamma) \in [\ell(z) + \ell(w) - 1, \ell(z) + \ell(w) + 1] \subset [T, T+\sigma_0']$ such that

- for all $s \in [0, \ell(z)]$, we have $d(\gamma(s), \varphi_s(z)) < \alpha$;
- for all $s \in [0, \ell(w)]$, we have $d(\gamma(\ell(z)+s), \varphi_s(w)) < \alpha$.

In particular, as $\varphi_{[0,\ell(z)]}(z) \cap K_{-\frac{\eta}{2}} = \emptyset$, we have $\gamma([0,\ell(z)]) \cap K_{-\frac{\eta}{2}-\alpha} = \emptyset$. As $\eta < \alpha$, we obtain $\gamma([0,\ell(z)]) \cap K_{-2\alpha} = \emptyset$. Therefore

$$\frac{\ell(\gamma \cap K_{-2\alpha})}{\ell(\gamma)} \leq \frac{\ell(\gamma)-\ell(z)}{T} \leq \frac{\ell(w)+1}{T} \leq \frac{\sigma+\tau_K+1}{T} < \rho$$

(where the last inequality is satisfied as T has been chosen big enough, according to (16)).

We now prove that γ intersects $K_{-2\alpha}$. As $\varphi_{[\ell(z),\ell(z)+\ell(w)]}(w) \cap \overline{B(x_0,\varepsilon)} \neq \emptyset$, we have that $\gamma([\ell(z), \ell(z)+\ell(w)]) \cap B(x_0, \varepsilon+\alpha) \neq \emptyset$. As $\alpha < \frac{\delta}{3} < \frac{\varepsilon}{3}$ and $B(x_0, 2\varepsilon) \subset K$, we obtain $B(x_0, \varepsilon+\alpha) \subset K_{-2\alpha}$ and $\gamma \cap K_{-2\alpha} \neq \emptyset$. This proves $\gamma \in \mathcal{P}^{\rho}_{K_{-2\alpha}}(T, T+\sigma_0')$. Let $f(z) = \gamma$.

Step 3. The map f is almost injective. We now control the cardinality of the preimage by f of every periodic orbit. Let $z_1, z_2 \in E$ be such that $f(z_1) = f(z_2) = \gamma$. Let s_1, resp. s_2, be the origin of γ that comes with the construction of $f(z_1)$, resp. $f(z_2)$. Without loss of generality me may assume $0 \leq s_1 \leq s_2 < l(\gamma)/2$.

We first prove $|s_2 - s_1| \leq \sigma_0'$. By construction, $\gamma([s_1, s_1+T]) \cap K_{-2\alpha} = \emptyset$ and $\gamma([s_2, s_2+T]) \cap K_{-2\alpha} = \emptyset$. Moreover $\gamma([s_1+T, s_1+\ell(\gamma)]) \cap K_{-2\alpha} \neq \emptyset$ and $\gamma([s_2+T, s_2+l(\gamma)]) \cap K_{-2\alpha} \neq \emptyset$. As $T > \sigma_0'$, again by (16), we have $T > \ell(\gamma)/2$. Therefore, $[s_2+T, s_2+\ell(\gamma)] \subset [s_1, s_1+\ell(\gamma)+T]$ and, to ensure $\gamma([s_1+T, s_1+\ell(\gamma)]) \cap K_{-2\alpha} \neq \emptyset$ and $\gamma([s_2+T, s_2+\ell(\gamma)]) \cap K_{-2\alpha} \neq \emptyset$, we must have $[s_1+T, s_1+\ell(\gamma)] \cap [s_2+T, s_2+\ell(\gamma)] \neq \emptyset$, since $\gamma([s_1, s_1+T]) = \gamma([s_1+\ell(\gamma), s_1+\ell(\gamma)+T])$ cannot intersect $K_{-2\alpha}$. Therefore $s_2 - s_1 \leq \ell(\gamma) - T$. Thus $0 \leq s_2 - s_1 \leq \sigma_0'$.

Then, for all $\tau \in [0, T]$ we have, using (2),

$$\begin{aligned} d(\varphi_\tau(z_1), \varphi_\tau(z_2)) &\leq d(\varphi_\tau(z_1), \gamma(s_1+\tau)) + d(\gamma(s_1+\tau), \gamma(s_2+\tau)) + d(\varphi_\tau(z_2), \gamma(s_2+\tau)) \\ &< \alpha + b|s_2 - s_1| + \alpha\,. \end{aligned}$$

If $|s_2 - s_2| \leq \alpha/b$, then

$$d(\varphi_\tau(z_1), \varphi_\tau(z_2)) \leq 3\alpha < \delta\,.$$

Since E is a (δ, T)-separating set, we conclude that $z_1 = z_2$. Therefore

$$\# f^{-1}(\gamma) \leq \left\lceil \frac{b\sigma_0'}{\alpha} \right\rceil .$$

Step 4. Conclusion. We just proved that

$$\mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \frac{\eta}{2}, T, T+\sigma_0, \delta) \leq \left\lceil \frac{b\sigma_0'}{\alpha} \right\rceil \# \mathcal{P}^{\rho}_{K_{-2\alpha}}(T, T+\sigma_0')\,.$$

By taking the exponential growth rate in the previous inequality when $T \to +\infty$ we get

$$\limsup_{T\to\infty} \frac{1}{T} \log \left(\mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \frac{\eta}{2}, T, T+\sigma_0, \delta) \right) \leq \limsup_{T\to\infty} \frac{1}{T} \log \left(\# \mathcal{P}^{\rho}_{K_{-2\alpha}}(T, T+\sigma_0') \right) = h_{\mathrm{Gur}}^{K_{-2\alpha},\rho}(\varphi)$$

and therefore, by (15),

$$h_{\mathcal{C}}^{\infty}(\varphi) \leq \varepsilon + h_{\mathrm{Gur}}^{K_{-2\alpha},\rho}(\varphi).$$

We now consider the limit $\rho \to 0$ to obtain

$$h_{\mathcal{C}}^{\infty}(\varphi) \leq \varepsilon + \lim_{\rho\to 0} h_{\mathrm{Gur}}^{K_{-2\alpha},\rho}(\varphi).$$

We can now take the infimum over K (as $K \mapsto \lim_{\rho\to 0} h_{\mathrm{Gur}}^{K,\rho}(\varphi)$ is non-increasing with respect to inclusion (Fact 3.18), taking the infimum means taking big compact sets K and therefore is compatible with the conditions on K). By the arbitrariness of ε, we conclude that $h_{\mathcal{C}}^{\infty}(\varphi) \leq h_{\mathrm{Gur}}^{\infty}(\varphi)$. □

As said above, the other inequality is more difficult, but the proof follows closely the proof of [GST23, Theorem 5.1].

Given two compacts sets $K_1 \subset K_2$, $\sigma_0 > 0$ and $0 < \alpha \leq 1$, we introduce for all $T > 0$ the set of periodic orbits with length roughly T that intersect K_1 and spend a small amount of time in K_2:

$$\mathcal{P}(K_1, K_2, \alpha, T, T+\sigma_0) = \{\gamma \in \mathcal{P}_{K_1}(T, T+\sigma_0),\, \ell(\gamma \cap K_2) \leq \alpha \ell(\gamma)\}\,.$$

As in [GST23], we need the following lemma.

Lemma 5.25. *Let $\varphi\colon M \to M$ be a H-flow. Given any compact sets $K_1 \subset K_2$ with $\mathring{K}_1 \neq \emptyset$, $\sigma_0 > 0$ and $\alpha > 0$, there exists $0 < \delta < 1$ such that*

$$\begin{aligned}\limsup_{T\to+\infty} \frac{1}{T} \log \#\mathcal{P}^{\alpha}_{K_2}(T, T+\sigma_0) &\leq \limsup_{T\to+\infty} \frac{1}{T} \log \#\mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0)\\ &\leq \limsup_{T\to+\infty} \frac{1}{T} \log \#\mathcal{P}^{2\alpha}_{(K_2)_{-\delta}}(T, T+\sigma_0)\,,\end{aligned}$$

where $(K_2)_{-\delta} = K_2 \setminus \bigcup_{x\in\partial K_2} B(x,\delta)$.

Proof. The second inequality is easily obtained, thanks to the inclusion

$$\mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0) \subset \mathcal{P}^{2\alpha}_{(K_2)_{-\delta}}(T, T+\sigma_0).$$

We now prove the first inequality. The proof goes as follows. If γ is a periodic orbit intersecting K_2 then, using transitivity and the multiple closing lemma, we can make it do a small detour to intersect K_1. As the detour is controlled, we still control the time spent in K_2 (in fact $(K_2)_{-\delta}$) as well as the period of the new periodic orbit. This leads to an inequality between the number of periodic orbits in $\mathcal{P}^{\alpha}_{K_2}(T, T+\sigma_0)$ and $\mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0+\sigma_0')$, for some suitable $\sigma_0' > 0$. This proves

$$\limsup_{T\to+\infty} \frac{1}{T} \log \#\mathcal{P}^{\alpha}_{K_2}(T, T+\sigma_0) \leq \limsup_{T\to+\infty} \frac{1}{T} \log \#\mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0+\sigma_0')\,.$$

We then prove that the limit superior does not depend on σ_0'. See Figure 5.5. We now give a detailed proof.

Step 1. Setting the parameters. Since $\mathring{K}_1 \neq \emptyset$, we can fix a point $x_0 \in \mathring{K}_1$ such that $B(x_0, \delta_0) \subset \mathring{K}_1$ for some $\delta_0 > 0$. Lemma 2.13 (i.e. the separation of orbits) applied with $\nu = 1$ and $\tau_1 = 1$ gives us $\tau_0 > 0$ and $\varepsilon_1 > 0$. Fix $0 < \delta < \frac{\varepsilon_1}{3}$ small enough such that $d(\partial K_1, \partial K_2) > 2\delta$ and $B(x_0, \delta_0 + \delta) \subset \mathring{K}_1$. From Lemma 2.15 (i.e. the multiple closing lemma) applied at the compact set $\overline{B(K_2, 1)}$, with $\delta > 0$, $\nu = 1$ and $N = 2$, we obtain a time $T_{\min} > 0$ and $\eta > 0$. From unifom transitivity, i.e., Lemma 2.14, applied at the compact sets $\overline{B(x_0, \delta_0)} \subset \overline{B(K_2, 1)}$ with $\frac{\eta}{2} > 0$, we obtain a time $\sigma > 0$.

We now cover the compact set $\overline{B(K_2, 1)}$ with N balls $B(x_i, \frac{\eta}{2})$. For all $1 \leq i \leq N$, by transitivity applied at the center of the ball $B(x_i, \frac{\eta}{2})$, there exists a point $z_i \in B(x_i, \frac{\eta}{2})$ and a time $\ell(z_i) \in [\sigma + 1, \sigma + 2\tau_{K_2} + 1]$ such that $\varphi_{\ell(z_i)}(z_i) \in B(x_i, \frac{\eta}{2})$ and $\varphi_{[0,\ell(z_i)]}(z_i) \cap \overline{B(x_0, \delta_0)} \neq \emptyset$. Let $\sigma_0' = \sigma + 2(\tau_{K_2} + 1)$.

Step 2. We construct a map between our two sets of periodic orbits. Fix

$$T \geq \max\left(T_{\min}, \frac{\sigma_0' + \alpha\sigma_0}{\alpha}\right)\,; \tag{17}$$

we now construct a map

$$f\colon \mathcal{P}^{\alpha}_{K_2}(T, T+\sigma_0) \to \mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0+\sigma_0')\,.$$

Let $\gamma \in \mathcal{P}^{\alpha}_{K_2}(T, T+\sigma_0)$ and assume, without loss of generality, that $\gamma(0) \in K_2$. Let $1 \leq i \leq N$ be such that $\gamma(0) \in B(x_i, \frac{\eta}{2})$. Notice that

$$d(\varphi_{\ell(z_i)}(z_i), \gamma(0)) < \eta \qquad \text{and} \qquad d(\gamma(0), z_i) = d(\varphi_{\ell(\gamma)}(\gamma(0)), z_i) < \eta.$$

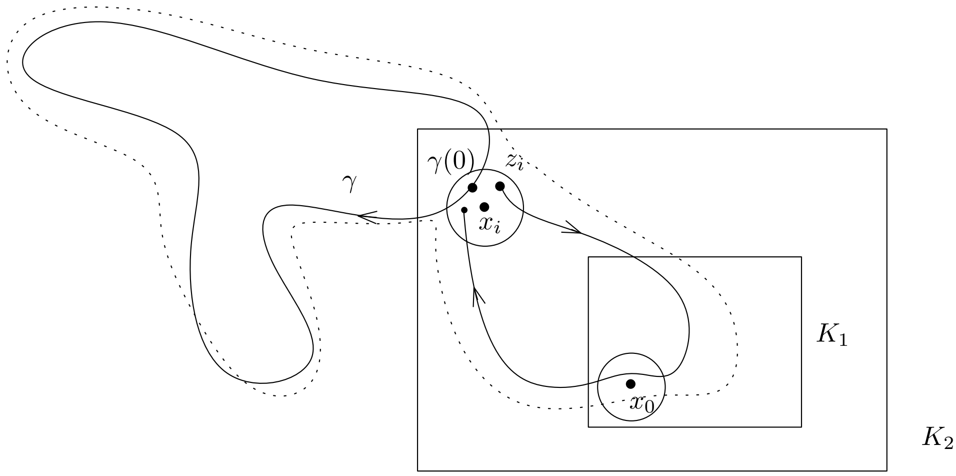

Figure 5.5: Construction of a periodic orbit intersecting K_1.

Therefore, also by (17), we can use Lemma 2.15 with points $x_1 = z_i$, $x_2 = \gamma(0)$, and times $\ell(z_i) \in [\sigma+1, \sigma+2\tau_{K_2}+1]$, $\ell(\gamma) \in [T, T+\sigma_0]$ to obtain a periodic orbit β of period $\ell(\beta) \in [\ell(\gamma)+\ell(z_i)-1, \ell(\gamma)+\ell(z_i)+1] \subset [T, T+\sigma_0+\sigma_0']$ which is δ-close first to $\varphi_{[0,\ell(z_i)]}(z_i)$ and then to γ.

The orbit β intersects K_1 because $\varphi_{[0,\ell(z_i)]}(z_i)$ meets $\overline{B(x_0,\delta_0)}$ and $B(x_0, \delta_0+\delta) \subset \mathring{K}_1$. Moreover, since $\ell(\gamma \cap K_2) \le \alpha \ell(\gamma)$ and β is δ-close to γ on $[\ell(z_i), \ell(z_i)+\ell(\gamma)]$, we have

$$\ell(\beta([\ell(z_i), \ell(z_i)+\ell(\gamma)])) \cap (K_2)_{-\delta}) \le \alpha(T+\sigma_0)$$

and therefore

$$\ell(\beta \cap (K_2)_{-\delta}) \le \alpha(T+\sigma_0) + \ell(\beta) - \ell(\gamma) \le \alpha(T+\sigma_0) + \sigma_0' \,.$$

So

$$\frac{\ell(\beta \cap (K_2)_{-\delta})}{\ell(\beta)} \le \frac{\alpha(T+\sigma_0)+\sigma_0'}{T} \le 2\alpha \,,$$

as T has been chosen big enough, according to (17). Thus, $\beta \in \mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0+\sigma_0')$.

Set $f(\gamma) = \beta$.

Step 3. The map f is almost injective. We want now to control the loss of injectivity of the function f defined above. Let $\gamma_1, \gamma_2 \in \mathcal{P}^{\alpha}_{K_2}(T, T+C)$ be such that $f(\gamma_1) = f(\gamma_2) = \beta$. Assume that $0 \le \ell(\gamma_1) - \ell(\gamma_2) \le \tau_0$ (where τ_0 is given by Lemma 2.13 as explained above). The point $\gamma_1(0)$ belongs to $B(x_i, \frac{\eta}{2})$, for some $1 \le i \le N$; additionally, let us assume that also $\gamma_2(0)$ belongs to the same ball $B(x_i, \frac{\eta}{2})$. Actually, for this second choice we have N possibilities, and we will count them when considering the cardinality of the preimage of a periodic orbit.

The construction of $f(\gamma_i)$ provides an origin of the orbit β, that we denote s_j, for $j = 1, 2$. We can assume without loss of generality that $s_1 = 0$. For any $s \in [0, \ell(\gamma_2)]$, one has

$$\begin{aligned} d(\gamma_1(s), \gamma_2(s)) &\le d(\gamma_1(s), \beta(\ell(z_i)+s)) + d(\beta(\ell(z_i)+s), \beta(\ell(z_i)+s_2+s)) + d(\beta(\ell(z_i)+s_2+s), \gamma_2(s)) \\ &\le \delta + b|s_2| + \delta \,, \end{aligned}$$

where b comes from (2). If $|s_2| \le \frac{\delta}{b}$, then

$$d(\gamma_1(s), \gamma_2(s)) \le 3\delta < \varepsilon_1 \,.$$

We can then conclude, by Lemma 2.13, that $\gamma_1 = \gamma_2$. It follows that the cardinal of the preimage of β by f is bounded by $N \left\lceil \frac{\sigma_0}{\tau_0} \right\rceil \left\lceil \frac{(T+\sigma_0+\sigma_0')b}{\delta} \right\rceil$. Consequently

$$\#\mathcal{P}^{\alpha}_{K_2}(T, T+\sigma_0) \le N \left\lceil \frac{\sigma_0}{\tau_0} \right\rceil \left\lceil \frac{(T+\sigma_0+\sigma_0')b}{\delta} \right\rceil \#\mathcal{P}(K_1, (K_2)_{-\delta}, 2\alpha, T, T+\sigma_0+\sigma_0') \,.$$

This proves

$$\limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}^{\alpha}_{K_2}(T,T+\sigma_0) \quad\leq\quad \limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}(K_1,(K_2)_{-\delta},2\alpha,T,T+\sigma_0+\sigma_0')\,. \tag{18}$$

Step 4. The limit superior $\limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}(K_1,K_2,\alpha,T,T+\sigma_0)$ does not depend on σ_0. Let $0<\sigma_1<\sigma_2$. Let $n=\lfloor\sigma_2/\sigma_1\rfloor$. Then

$$\mathcal{P}(K_1,K_2,\alpha,T,T+\sigma_2)\subset\cup_{i=0}^{n}\mathcal{P}(K_1,K_2,\alpha,T+i\sigma_1,T+(i+1)\sigma_1)$$

and

$$\begin{aligned}\#\mathcal{P}(K_1,K_2,\alpha,T,T+\sigma_1)\leq&\#\mathcal{P}(K_1,K_2,\alpha,T,T+\sigma_2)\\ \leq&(n+1)\max_{i=0\dots n}\#\mathcal{P}(K_1,K_2,\alpha,T+i\sigma_1,T+(i+1)\sigma_1)\,.\end{aligned}$$

Therefore,

$$\limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}(K_1,K_2,\alpha,T,T+\sigma_1)=\limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}(K_1,K_2,\alpha,T,T+\sigma_2)\,.$$

Thus, from this equality and from (18), we conclude

$$\limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}^{\alpha}_{K_2}(T,T+\sigma_0) \quad\leq\quad \limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}(K_1,(K_2)_{-\delta},2\alpha,T,T+\sigma_0)$$

as required. □

We are now able to state and prove the analogue of [GST23, thm 5.1] in our context.

Theorem 5.26. *Let $\varphi\colon M\to M$ be a H-flow. Let $K\subset M$ be a compact set with $\overset{\circ}{K}\neq\emptyset$. Let $\varepsilon>0$. There exist a map $\alpha\in(0,1)\to\psi(\alpha)\in(0,\infty)$ converging to 0 when $\alpha\to0$ and $R\geq1$ such that for all $0<\alpha<1$ and $S>0$*

$$\limsup_{T\to\infty}\frac{1}{T}\log\#\mathcal{P}(K,K_R,\alpha,T,T+S)\leq h^{K^c}_{\mathcal{C}}(\varphi)\left(1+2\frac{b\alpha}{R}\right)+\varepsilon+\psi(\alpha).$$

where K_R is the R-neighbourhood of K.

From Theorem 5.26, we will deduce the following result.

Corollary 5.27. *Let $\varphi\colon M\to M$ be a H-flow. Then $h^{\infty}_{\mathrm{Gur}}(\varphi)\leq h^{\infty}_{\mathcal{C}}(\varphi)$.*

Proof. If $h^{\infty}_{\mathcal{C}}(\varphi)=\infty$, then there is nothing to prove. Assume now $h^{\infty}_{\mathcal{C}}(\varphi)<\infty$. Let $\varepsilon>0$. Fix K a compact subset with nonempty interior such that

$$h^{K^c}_{\mathcal{C}}(\varphi)\leq h^{\infty}_{\mathcal{C}}(\varphi)+\varepsilon\,.$$

Use Theorem 5.26 to obtain ψ and R. As $\lim_{\alpha\to0}h^{K_{R+1},\alpha}_{\mathrm{Gur}}(\varphi)\geq h^{\infty}_{\mathrm{Gur}}(\varphi)$ and since, by Fact 3.18, the function $\alpha\mapsto h^{K_{R+1},\alpha}_{\mathrm{Gur}}(\varphi)$ is non increasing, then for all $\alpha>0$ one has

$$h^{K_{R+1},\alpha}_{\mathrm{Gur}}(\varphi)\geq h^{\infty}_{\mathrm{Gur}}(\varphi).$$

Choose α such that $\alpha<1/2$, $2b\alpha/R<\varepsilon$ and $\psi(2\alpha)<\varepsilon$. From Lemma 5.25 with parameters $K\subset K_{R+1}$ and α we have

$$h^{K_{R+1},\alpha}_{\mathrm{Gur}}(\varphi)=\limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}^{\alpha}_{K_{R+1}}(T,T+S) \quad\leq\quad \limsup_{T\to+\infty}\frac{1}{T}\log\#\mathcal{P}(K,K_R,2\alpha,T,T+S)\,.$$

We now use Theorem 5.26 to obtain

$$\limsup_{T\to\infty}\frac{1}{T}\log\#\mathcal{P}(K,K_R,2\alpha,T,T+S)\leq h^{K^c}_{\mathcal{C}}(\varphi)(1+\varepsilon)+\varepsilon+\psi(2\alpha).$$

Therefore

$$h^{\infty}_{\mathrm{Gur}}(\varphi)\leq h^{\infty}_{\mathcal{C}}(\varphi)(1+\varepsilon)+3\varepsilon\,.$$

As ε can be choose arbitrarily small, we obtain $h^{\infty}_{\mathrm{Gur}}(\varphi)\leq h^{\infty}_{\mathcal{C}}(\varphi)$. □

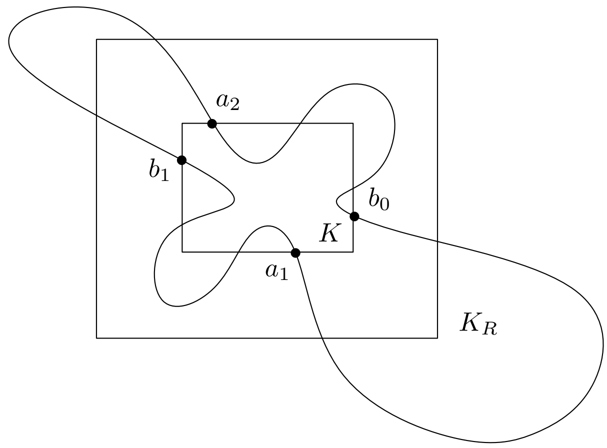

Figure 5.6: larges excursions

We will prove Theorem 5.26 by adapting the arguments of [GST23] to our context. We provide details when necessary, and refer to [GST23] for complements.

Proof of Theorem 5.26 . If $h_{\mathcal{C}}^{K^c}(\varphi) = \infty$, then there is nothing to prove. Assume now that $h_{\mathcal{C}}^{K^c}(\varphi) < \infty$. We start a given compact set $K \subset M$ with $\mathring{K} \neq \emptyset$ and a given $\varepsilon > 0$.

The heuristic of the proof is the following. We cut every periodic orbit $\gamma \in \mathcal{P}(K, K_R, \alpha, T, T+S)$ into chords joining points of ∂K, in such a way that a chord is either a *large excursion* outside K_R, i.e. a connected component of $\gamma \setminus K$ that intersects $(K_R)^c$, or a chord between points of ∂K staying inside K_R, between two large excursions. This decomposition will give a bound on the number of periodic orbits in terms of number of chords. More precisely, the number of large excursions, i.e. the chords going outside K_R will be controlled by the chord entropy at infinity, while the number of other chords, i.e. those remaining inside K_R, will be controlled by the Gurevic entropy.

The set of useful chords is quite involved and bounding its cardinal will require some work. This will be the main technical part of the proof.

Step 1. We quantitatively approximate chord entropy at infinity.

Apply Lemma 2.13 with $\nu = 1$ and $\tau_1 = S$: we obtain $\bar{\tau}_0 > 0$ and $\bar{\varepsilon} > 0$. Recall that $lip(\varphi) \geq 1$ is the Lipschitz constant of the map φ_τ, for all $\tau \in [-1, 1]$ (see point 1 of Definition 2.9). By the definition of $h_{\mathcal{C}}^{K^c}(\varphi)$ and since this quantity is finite, there exists $\overline{\delta} > 0$ such that $\overline{\delta} < \overline{\varepsilon}/(3lip(\varphi))$, $\overline{\delta} \leq b/2$ and, for all $0 < \delta \leq \overline{\delta}$,

$$\lim_{\eta \to 0} \sup_{x,y \in \partial K} \limsup_{T \to +\infty} \frac{\log \mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \eta, T, T+S, \delta)}{T} \in \left(h_{\mathcal{C}}^{K^c}(\varphi) - \frac{\varepsilon}{4}, h_{\mathcal{C}}^{K^c}(\varphi) + \frac{\varepsilon}{4} \right).$$

Choose $\delta = \overline{\delta}$. There exists $\overline{\eta} < \overline{\delta}$ such that, for all $0 < \eta \leq \overline{\eta}$,

$$\sup_{x,y \in \partial K} \limsup_{T \to +\infty} \frac{\log \mathcal{N}_{\mathcal{C}}^{K^c}(x, y, \eta, T, T+S, \overline{\delta})}{T} \in \left(h_{\mathcal{C}}^{K^c}(\varphi) - \frac{\varepsilon}{2}, h_{\mathcal{C}}^{K^c}(\varphi) + \frac{\varepsilon}{2} \right). \tag{19}$$

Choose $0 < \eta \leq \overline{\eta}$. As above, by the same argument as in the proof of Theorem 3.10, the above limsup does not depend on S. Therefore, the quantifiers η and δ do not depend either on S.

Step 2. We divide a periodic orbit in excursions.

Choose $M = M(\eta) \in \mathbb{N}$ a (minimal) number of balls $B(x_i, \eta)$ centered at points $x_i \in \partial K$ for $i = 1, \ldots, M$ so that

$$\partial K \subset \bigcup_{i=1}^{M} B(x_i, \eta).$$

Fix an arbitrary $R \geq 1$, whose value will be determined later on the proof, and some $\alpha \in (0, 1)$. We will divide each periodic orbit of $\mathcal{P}(K, K_R, \alpha, T, T+S)$ into suitable chords.

Step 2.a. Large excursions.

Let $\gamma \in \mathcal{P}(K, K_R, \alpha, T, T+S)$. A *large excursion* of γ outside K_R is a connected component of $\gamma \cap \overline{K^c}$ starting from ∂K, staying outside K except at the endpoints of the interval, and intersecting $(K_R)^c$. We will divide γ into $2N$ connected components, with N large excursions separated by N pieces of orbit that stay inside K_R. Choose a point $b_0 = b_N$ on γ at the beginning of such a large excursion. Following [GST23], we denote by $(a_i)_{1\le i\le N}$ on $\gamma\cap\partial K$ the endpoints of the large excursions, and starting points of components inside K_R, and $(b_i)_{0\le i\le N-1}$ on $\gamma \cap \partial K$, the starting points of the large excursions. With these notations, every large excursion goes from b_{i-1} to a_i for some $1 \le i \le N$, and each complement goes from a_i to b_i, for some $1 \le i \le N$. Parametrize $\gamma = (\gamma(t))_{t\in[0,\ell(\gamma)]}$ so that $b_0 = b_N = \gamma(0)$, $b_i = \gamma(\tau_i)$, with $\tau_0 = 0$ and $\tau_N = \ell(\gamma)$, and $a_i = \gamma(\sigma_i)$. We get a decomposition of $[0, \ell(\gamma)]$ into $2N$ intervals corresponding to N large excursions (τ_i, σ_{i+1}) for $i = 0, \dots, N-1$, and their N complements (σ_i, τ_i) for $i = 1, \dots, N$.

Step 2.b. Elementary observations on excursions. As in [GST23], let us do the following elementary but crucial observations.

1. By definition of $\mathcal{P}(K, K_R, \alpha, T, T+S)$, as $\ell(\gamma) \in [T, T+S]$,
$$\ell(\gamma \cap K) \le \sum_{i=1}^{N} (\tau_i - \sigma_i) \le \ell(\gamma \cap K_R) < \alpha \ell(\gamma) \le \alpha T + \alpha S\,.$$

2. For $1 \le i \le N$, $\gamma([\tau_{i-1}, \sigma_i])$ lies outside K, except at points $\gamma(\tau_{i-1})$ and $\gamma(\sigma_i)$ which belong to ∂K. Moreover, by the right hand side in (2), each large excursion spends a time at least $\frac{2R}{b}$ inside $K_R \setminus K$, where b is the constant in (2). Thus, since there are N large excursions, we get $\ell(\gamma \cap (K_R \setminus K)) \ge \frac{2R}{b} N$. By definition, as $\gamma \in \mathcal{P}(K, K_R, \alpha, T, T+S)$, we know that $\ell(\gamma \cap K_R^c) \ge (1-\alpha)T$. Thus, we deduce that
$$(1-\alpha)T + \frac{2R}{b} N \le \ell(\gamma \cap K_R^c) + \ell(\gamma \cap (K_R \setminus K)) = \ell(\gamma \cap K^c) \le \ell(\gamma) \le T + S\,. \qquad (20)$$
Set
$$\hat{\nu} = \hat{\nu}_{\alpha,R,T} = \frac{b\alpha}{R} T.$$
From (20), we get $N \le \frac{b(\alpha T+S)}{2R}$. Moreover, when $T \ge \frac{S}{\alpha}$, then $\frac{b(\alpha T+S)}{2R} \le \hat{\nu}$, so that
$$N \le \hat{\nu}\,.$$

We set $t_0 = \tau_0 = 0$ and for $i = 1, \dots, N$, $t_i = \lfloor \tau_i \rfloor$ and $s_i = \lfloor \sigma_i \rfloor$.

3. When $T \ge \max(\frac{S}{\alpha}, \frac{RS}{\alpha b})$, we get $\hat{\nu} \ge S \ge \alpha S$ and by point 2, $\hat{\nu} \ge N$. Therefore, by points 1 and 2, we have
$$\sum_{i=1}^{N} (t_i - s_i) \le \sum_{i=1}^{N} (\tau_i - \sigma_i) + N \le \alpha T + \alpha S + N \le \alpha T + 2\hat{\nu}\,.$$

4. For $T \ge \max(\frac{S}{\alpha}, \frac{RS}{\alpha b})$, we deduce from point 2 that
$$\sum_{i=1}^{N} (s_i - t_{i-1}) \le \sum_{i=1}^{N} (\sigma_i - \tau_{i-1}) + N \le T + S + N \le T + 2\hat{\nu}\,.$$

5. As $\tau_N = \ell(\gamma)$, we have
$$T - 1 \le t_N \le T + S.$$

Step 2.c. Construction of a map from periodic orbits to a set of excursions. We first define our set of excursions. Let $\mathcal{E}$ be defined as

$$\bigcup_{N=1}^{\hat{\nu}} \bigcup_{\substack{(t_1,\dots,t_N)\in\mathbb{N}^N \\ (s_1,\dots,s_N)\in\mathbb{N}^N \\ |t_N-T|\le S+1 \\ 0<s_1\le t_1<s_2\le\dots<s_N\le t_N}} \bigcup_{\substack{(j_i)_{i=1}^N\in\{1,\dots,M\}^N \\ (l_i)_{i=1}^N\in\{1,\dots,M\}^N}} \prod_{i=1}^N E_{K_R}\left(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta\right)$$
$$\times E^{K^c}\left(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta\right)$$

where each $E_{K_R}\left(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta\right)$ is a $E(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta)$-set of chords contained in K_R of maximal cardinality, i.e.,

$$\#E_{K_R}\left(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta\right) = \mathcal{N}_{\mathcal{C},K_R}\left(x_{j_i}, x_{l_i}, t_i - s_i - 1, t_i - s_i + 1, \delta\right)\,,$$

while each $E^{K^c}\left(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta\right)$ is a $E^{K^c}(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta)$-set of maximal cardinality, i.e.

$$\#E^{K^c}\left(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta\right) = \mathcal{N}_{\mathcal{C}}^{K^c}\left(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta\right).$$

We construct $f : \mathcal{P}(K, K_R, \alpha, T, T+S) \to \mathcal{E}$ as follows. With the above decomposition, we associate to the orbit γ with an arbitrary choice of origin such that $\gamma(0) \in \partial K$ the following family of $2N$ chords: the N chords of respective lengths $\tau_i - \sigma_i \in [t_i - s_i - 1, t_i - s_i + 1]$ from $a_i \in \partial K$ to $b_i \in \partial K$ and the N chords of respective lengths $\sigma_i - \tau_{i-1} \in [s_i - t_{i-1} - 1, s_i - t_{i-1} + 1]$ from b_i to a_{i+1} outside K. Each a_i belongs to some ball $B(x_{j_i}, \eta)$, while each b_i belongs to some ball $B(x_{l_i}, \eta)$. In particular, each $\gamma([\sigma_i, \tau_i])$ defines a point in $E_{K_R}\left(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta\right)$, while each $\gamma([\tau_i, \sigma_{i+1}])$ defines a point in $E^{K^c}\left(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta\right)$. Note that, if $R \geq 3b/2$, for every $1 \leq i \leq N$, we have $s_i - t_{i-1} \geq \sigma_i - \tau_{i-1} - 1 \geq 1$. This shows that the image of f is indeed contained in $\mathcal{E}$.

Step 2.d. The map f is almost injective.

For every $\gamma_0 \in \mathcal{P}(K, K_R, \alpha, T, T+S)$, we need to bound the cardinality of $f^{-1}(f(\gamma_0))$.

Assume that $\gamma, \widetilde{\gamma}$ are in $f^{-1}(f(\gamma_0))$, i.e. they are orbits in $\mathcal{P}(K, K_R, \alpha, T, T+S)$ such that $f(\gamma) = f(\widetilde{\gamma})$. By definition of f, these orbits lead to the same $2N$-tuple of integers $(s_1, t_1, \dots, s_N, t_N)$, and the same $2N$-tuple of balls $B(x_{j_i}, \eta)$ and $B(x_{l_i}, \eta)$, and the same $2N$ tuple of chords.

We will need refined informations. Therefore, we partition $f^{-1}(\gamma_0)$ accordingly to the precise length of γ and to the precise length of the chords. In other words, we assume that $|\ell(\gamma) - \ell(\widetilde{\gamma})| \leq \bar{\tau}_0$, where $\bar{\tau}_0$ is given by Lemma 2.13. Denote by $(\sigma_i)_{i=1,\dots,N}, (\tau_i)_{i=0,\dots,N-1}$, resp. $(\widetilde{\sigma}_i)_{i=1,\dots,N}, (\widetilde{\tau}_i)_{i=0,\dots,N-1}$, the lengths of chords in the construction of $f(\gamma)$ (resp. $f(\widetilde{\gamma})$). We also assume that for every $1 \leq i \leq N$, we have $|\sigma_i - \widetilde{\sigma}_i| < \frac{\delta}{b}$ and for every $0 \leq i \leq N-1$, we have $|\tau_i - \widetilde{\tau}_i| < \frac{\delta}{b}$, where b is the constant in (2) Recall that $\tau_0 = \widetilde{\tau}_0 = 0$.

We will show that for every $s \in [0, T]$, $d(\gamma(s), \widetilde{\gamma}(s)) \leq \bar{\varepsilon}$. Thanks to lemma 2.13, we will deduce the desired result $\gamma = \widetilde{\gamma}$. To do so, we divide $[0, T]$ into sub-intervals where we control the distance between γ and $\widetilde{\gamma}$. For $i = 1, \dots, N$, set

$$S_i = [\sigma_i, \tau_i] \cap [\widetilde{\sigma}_i, \widetilde{\tau}_i] = [\max(\sigma_i, \widetilde{\sigma}_i), \min(\tau_i, \widetilde{\tau}_i)]$$

and for $i = 0, \dots, N-1$, set

$$B_i = [\tau_i, \sigma_{i+1}] \cap [\widetilde{\tau}_i, \widetilde{\sigma}_{i+1}] = [\max(\tau_i, \widetilde{\tau}_i), \min(\sigma_{i+1}, \widetilde{\sigma}_{i+1})].$$

As every large excursion has length $\tau_i - \sigma_i$ (resp $\widetilde{\tau}_i - \widetilde{\sigma}_i$ at least $2R/b$, for $R > \delta/2$, we know that $B_i \neq \emptyset$. Nonetheless, S_i may be empty if $\sigma_i \leq \tau_i < \widetilde{\sigma}_i \leq \widetilde{\tau}_i$ or $\widetilde{\sigma}_i \leq \widetilde{\tau}_i < \sigma_i \leq \tau_i$. In this case, we have $\max(\tau_i, \widetilde{\tau}_i) - \min(\sigma_i, \widetilde{\sigma}_i) \leq 2\frac{\delta}{b}$. Let $(R_j)_{j\in J}$ be the connected components of

$$[0, T] \setminus \left(\bigcup_{i=1}^N S_i \cup \bigcup_{i=0}^{N-1} B_i\right).$$

By construction, R_j can be of the following forms

$$R_j = (\min(\tau_i, \widetilde{\tau}_i), \max(\tau_i, \widetilde{\tau}_i))$$
$$R_j = (\min(\sigma_i, \widetilde{\sigma}_i), \max(\sigma_i, \widetilde{\sigma}_i))$$
$$R_j = (\min(\sigma_i, \widetilde{\sigma}_i), \max(\tau_i, \widetilde{\tau}_i))$$

and the last case can only happen if $S_i = \emptyset$. We then always have $\mathrm{Leb}(R_j) \leq 2\frac{\delta}{b} \leq 1$.

Let $s \in [0, T]$, then

- if $s \in S_i$, as $f(\gamma) = f(\widetilde{\gamma})$, the points $\gamma(\sigma_i)$ and $\widetilde{\gamma}(\widetilde{\sigma}_i)$ belong to the same ball $B(x_{j_i}, \eta)$ and lead to the same chord, so that $d(\gamma(s), \widetilde{\gamma}(s)) \leq 2\delta + b|\sigma_i - \widetilde{\sigma}_i| < 3\delta \leq \overline{\varepsilon}$
- if $s \in B_i$, by the same argument, we have $d(\gamma(s), \widetilde{\gamma}(s)) \leq 2\delta + b|\tau_i - \widetilde{\tau}_i| < 3\delta \leq \overline{\varepsilon}$
- if $s \in R_j = (r_j^-, r_j^+)$, as $r_j^+ - r_j^- \leq 1$, we have $d(\gamma(s), \widetilde{\gamma}(s)) \leq lip(\varphi)(\gamma(r_j^-), \widetilde{\gamma}(r_j^+)) < lip(\varphi)3\delta \leq \overline{\varepsilon}$.

By Lemma 2.13, we conclude that $\gamma = \widetilde{\gamma}$. Therefore, we proved that $f(\gamma) = f(\gamma')$, $|\ell(\gamma) - \ell(\gamma')| \leq \bar{\tau}_0$, $|\sigma_i - \widetilde{\sigma}_i| < \frac{\delta}{b}$ and $|\tau_i - \widetilde{\tau}_i| < \frac{\delta}{b}$ for every i implies $\gamma = \gamma'$. As two elements γ and γ' in $\mathcal{P}(K, K_R, \alpha, T, T+S)$ with $f(\gamma) = f(\gamma')$ are such that $|\ell(\gamma) - \ell(\gamma')| \leq S$, $|\sigma_i - \widetilde{\sigma}_i| \leq 2$ and $|\tau_i - \widetilde{\tau}_i| \leq 2$ for all i, we have at most $\left\lceil \frac{S}{\bar{\tau}_0} \left(\frac{2b}{\delta}\right)^{2N} \right\rceil$ elements in $f^{-1}(f(\gamma))$. We deduce that

$$\#\mathcal{P}(K, K_R, \alpha, T, T+S) \leq \left\lceil \frac{S}{\bar{\tau}_0} \left(\frac{2b}{\delta}\right)^{2\hat{\nu}} \right\rceil \#\mathcal{E}\,.$$

Step 3. Bound on $\#\mathcal{E}$.

Step 3.a. Bound on the number of chords. By the initial choice of η, δ and by (19), for every x_j, x_l there exists $T_{j,l} > 0$ such that for every $T \geq T_{j,l}$ we have

$$\mathcal{N}_{\mathcal{C}}^{K^c}(x_j, x_l, \eta, T, T+2, \delta) \leq e^{(h_{\mathcal{C}}^{K^c}(\varphi) + \varepsilon)T}\,.$$

Let $T_{\max} = \max_{j,l} T_{j,l} + 1 > 0$. Note that $T_{\max}$ does not depend on α.

Choose $R \geq \max\left(\frac{T_{\max} b}{2}, R_{\min}, \delta/2, 3b/2\right)$, where $R_{\min}$ is given by Proposition 5.8 applied with K, δ. Since the length of every large excursion satisfies $s_i - t_{i-1} \geq \frac{2R}{b}$ for every i, we have $s_i - t_{i-1} - 1 \geq T_{\max}$ for every $i = 1, \ldots, N$. Thus, for every $j, l = 1, \ldots, M$, we obtain

$$\mathcal{N}_{\mathcal{C}}^{K^c}(x_l, x_j, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta) \leq e^{(h_{\mathcal{C}}^{K^c}(\varphi) + \varepsilon)(s_i - t_{i-1})}\,.$$

Let

$$h = \limsup_{T \to \infty} \frac{1}{T} \log \#\{\gamma \in \mathcal{P}_K(T, T+S), \gamma \subset K_{R+1}\} < \infty\,,$$

where we use Lemma 4.14 to guarantee that h is finite. By the same argument as used in the proof of Theorem 3.10, the above limsup does not depend on S. Therefore, with $S = 3\tau_K$, we also have

$$h = \limsup_{T \to \infty} \frac{1}{T} \log\left((T + 3\tau_K)\#\{\gamma \in \mathcal{P}_K(T, T+3\tau_K), \gamma \subset K_{R+1}\}\right) < \infty\,.$$

Thus there exists $D' > 0$ such that for every $T > 0$

$$(T + 3\tau_K)\#\{\gamma \in \mathcal{P}_K(T, T+3\tau_K), \gamma \subset K_{R+1}\} \leq D' e^{(h+\varepsilon)T}\,. \tag{21}$$

By Proposition 5.8, there exists $D > 0$ and $\sigma > 0$ such that when $t_i - s_i$ is large enough

$$\begin{aligned}\mathcal{N}_{\mathcal{C},K_R}(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta) \leq & \mathcal{N}_{\mathcal{C},K_R}(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i - 1 + 3\tau_K, \delta)\\ \leq & D(t_i - s_i + \sigma + 3\tau_K)\\ & \#\{\gamma \in \mathcal{P}_K(t_i - s_i + \sigma, s_i + \sigma + 3\tau_K), \gamma \subset K_{R+1}\}\\ \leq & DD' e^{(t_i - s_i + \sigma)(h+\varepsilon)}.\end{aligned}$$

Up to increasing D, we may assume that the second inequality is also satisfied for small $t_i - s_i$ Then we obtain, for all $t_i - s_i$

$$\mathcal{N}_{\mathcal{C},K_R}(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, t_i - s_i + 1, \delta) \leq DD' e^{(t_i - s_i + \sigma)(h+\varepsilon)} \leq D_1 e^{(t_i - s_i)(h+\varepsilon)}.$$

Step 3.b. Bound on $\#\mathcal{E}$. From the previous step and by points 3 and 4 in Step 2.b, we have

$$\begin{aligned}\#\prod_{i=1}^{N} E_{K_R}(x_{j_i}, x_{l_i}, \eta, t_i - s_i - 1, &t_i - s_i + 1, \delta) \times E^{K^c}(x_{l_i}, x_{j_{i+1}}, \eta, s_i - t_{i-1} - 1, s_i - t_{i-1} + 1, \delta) \\ &\leq D_1^N \exp\left((h_{\mathcal{C}}^{K^c}(\varphi) + \varepsilon) \sum_{i=1}^{N} (s_i - t_{i-1}) + (h+\varepsilon) \sum_{i=1}^{N} (t_i - s_i) \right) \\ &\leq D_1^N \exp\left(h_{\mathcal{C}}^{K^c}(\varphi)(T + 2\hat{\nu}) + h(\alpha T + 2\hat{\nu}) + \varepsilon(T(1+\alpha) + 4\hat{\nu}) \right).\end{aligned}$$

Therefore $\#\mathcal{E}$ is bounded by

$$\sum_{N=1}^{\hat{\nu}} \sum_{\substack{(t_1,\dots,t_N)\in\mathbb{N}^N \\ (s_1,\dots,s_N)\in\mathbb{N}^N \\ |t_N - T| \leq S+1 \\ 0 < s_1 \leq t_1 < s_2 \leq \dots < s_N \leq t_N}} M^{2N} D_1^N \exp\left(h_{\mathcal{C}}^{K^c}(\varphi)(T + 2\hat{\nu}) + h(\alpha T + 2\hat{\nu}) + \varepsilon(T(1+\alpha) + 4\hat{\nu}) \right).$$

As in [GST23, Lemma 5.5], the number of terms in the second sum in the previous equation is bounded from above by the number of ordered integer decompositions of $T + S + 1$ of length $2N$, ie $\binom{T+S+1}{2N}$. As $N \leq \hat{\nu}$ (as soon as $T \geq S/\alpha$) and $1 + S \leq \hat{\nu}$ (as soon as $T \geq R(S+1)/b\alpha$) we have

$$\binom{T+S+1}{2N} \leq \binom{T+\hat{\nu}}{2N}.$$

Therefore we obtain

$$\#\mathcal{E} \leq (M^2 D_1)^{\hat{\nu}} \hat{\nu} \binom{T+\hat{\nu}}{2\hat{\nu}} \exp\left(h_{\mathcal{C}}^{K^c}(\varphi)(T + 2\hat{\nu}) + h(\alpha T + 2\hat{\nu}) + \varepsilon(T(1+\alpha) + 4\hat{\nu}) \right).$$

Step 4. Conclusion. From the previous step (recall that $\hat{\nu} = b\alpha T/R$), we obtain

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T} \log \#\mathcal{P}(K, K_R, \alpha, T, T+S) \leq & h_{\mathcal{C}}^{K^c}(\varphi)\left(1 + 2\frac{b\alpha}{R}\right) + h\left(\alpha + 2\frac{b\alpha}{R}\right) + \varepsilon\left(1 + \alpha + 4\frac{b\alpha}{R}\right) \\ &+ \frac{b\alpha}{R}\left[\log(M^2 D_1) + 2\log\left(\frac{b}{\delta}\right)\right] \\ &+ \limsup_{T\to\infty} \frac{1}{T} \log\left(\binom{T+\hat{\nu}}{2\hat{\nu}}\right).\end{aligned}$$

We then get

$$\limsup_{T\to\infty} \frac{1}{T} \log \#\mathcal{P}(K, K_R, \alpha, T, T+S) \leq h_{\mathcal{C}}^{K^c}(\varphi)\left(1 + 2\frac{b\alpha}{R}\right) + \varepsilon + \psi(\alpha),$$

with

$$\psi(\alpha) = \alpha h + \alpha\varepsilon + \frac{b\alpha}{R}\left[4\varepsilon + 2h + \log(M^2 D_1) + 2\log\left(\frac{2b}{\delta}\right)\right] + \limsup_{T\to\infty} \frac{1}{T} \log\left(\binom{T+\hat{\nu}}{2\hat{\nu}}\right).$$

As $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, we have

$$\frac{1}{T} \log\left(\binom{T+\hat{\nu}}{2\hat{\nu}}\right) \leq 2b\frac{\alpha}{R} \log\left(\frac{e(1 + b\frac{\alpha}{R})}{2b\frac{\alpha}{R}}\right),$$

so $\limsup_{T\to\infty} \frac{1}{T} \log\left(\binom{T+\hat{\nu}}{2\hat{\nu}}\right)$ converges to 0 as $\alpha \to 0$. This, together with the fact that h is finite, proves that ψ converges to 0 as $\alpha \to 0$. $\square$

6 Subadditivity properties

In Section 6.1, we construct the probability measure $m_{\max}$ that is the candidate to satisfy the conclusion of Theorem 1.1. The existence of such a measure is implied by the h_{Gur}-strongly positive recurrent hypothesis (see Definition 4.11): $h^{\infty}_{\text{Gur}}(\varphi) < h_{\text{Gur}}(\varphi)$. The SPR assumption is a sufficient but *a priori* not necessary condition for the existence of such a measure.

We go on with subadditivity statements, Propositions 6.4 and 6.5, that will be crucial in the proof of Theorem 1.1. Proposition 6.4 does not require either the construction of the measure nor the SPR assumption $h^{\infty}_{\text{Gur}}(\varphi) < h_{\text{Gur}}(\varphi)$, and could have been proven in Section 3. However, both statements are more relevant together: this is the reason why it is stated and proven here. Proposition 6.5 is much more subtle, and requires the existence of the measure $m_{\max}$. The end of the section is devoted to the proof of Proposition 6.5.

6.1 Construction of the measures m_∞ and $m_{\max}$ through periodic measures

For every periodic orbit $\gamma = ((\varphi_t(x))_{t\in\mathbb{R}}, T) \in \text{Per}(\varphi_t)$, let μ_γ be the φ-invariant probability measure obtained by push-forward of the normalized Lebesgue-measure of the circle.

Assume that the H-flow φ is h_{Gur}-strongly positively recurrent, i.e., $h^{\infty}_{\text{Gur}}(\varphi) < h_{\text{Gur}}(\varphi)$. By Definition 3.16, for every small $\alpha > 0$, we can find a large compact set K_0, and $\varepsilon > 0$ small enough so that

$$\left|\limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}^{\varepsilon}_{K_0}(L, L + 5\tau_{K_0}) - h^{\infty}_{Gur}(\varphi)\right| \leq \alpha \,.$$

By Theorem 3.10, we have that

$$h_{Gur}(\varphi) = \lim_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_{K_0}(L, L + 5\tau_{K_0}) \,.$$

By Definition 4.11, choose $0 < \alpha < h_{\text{Gur}}(\varphi) - h^{\infty}_{\text{Gur}}(\varphi)$. It follows that

$$\begin{aligned}
\limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}^{\varepsilon}_{K_0}(L, L + 5\tau_{K_0}) &\leq h^{\infty}_{\text{Gur}}(\varphi) + \alpha \\
&< h_{\text{Gur}}(\varphi) \\
&= \lim_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_{K_0}(L, L + 5\tau_{K_0}) \,.
\end{aligned}$$

We get a strict inequality

$$\limsup_{L\to\infty} \frac{1}{L} \log \#\{\gamma \in \mathcal{P}_{K_0}(L, L + 5\tau_{K_0}), \ell(\gamma \cap K_0) < \varepsilon\ell(\gamma)\} < \lim_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_{K_0}(L, L + 5\tau_{K_0}) \,. \quad (22)$$

Let

$$m_{K_0,L} = \frac{1}{\#\mathcal{P}_{K_0}(L, L + 5\tau_{K_0})} \sum_{\gamma\in\mathcal{P}_{K_0}(L,L+5\tau_{K_0})} \mu_\gamma \quad (23)$$

be the invariant probability measure supported on the periodic orbits of $\mathcal{P}_{K_0}(L, L + 5\tau_{K_0})$.

A sequence $(\mu_n)_{n\in\mathbb{N}}$ of finite Borel measures on M converges to μ_∞ in the vague topology if, for every continuous map $f\colon M \to \mathbb{R}$ with compact support, one has $\int f\,d\mu_n \to \int f\,d\mu$, as $n \to +\infty$. Recall that the set $\mathcal{M}^{\leq 1}(\varphi)$ of φ-invariant measures μ such that $\mu(M) \leq 1$ is compact for the vague topology. Indeed, let $(\mu_n)_n$ be a sequence(M, d) such that $\mu_n(M) \leq 1$. As M is a locally compact metric space, there exists a countable family of functions $(\varphi_k)_{k\in\mathbb{N}}$ that are dense in the set $C_c(M)$ of continuous functions with compact support. Denote by $\|\varphi_k\|$ the maximum over M of the function φ_k. By compactness of $[-1, 1]$, one can find, for every fixed $k \in \mathbb{N}$, using a recursive definition, a strictly increasing map $\psi_k\colon \mathbb{N} \to \mathbb{N}$ such that, considering the subsequence $(\psi_1 \circ \psi_2 \circ \ldots \psi_k(n))_{n\in\mathbb{N}}$, the quantity $\int_M \frac{\varphi_k}{\|\varphi_k\|}\,d\mu_{\psi_1\circ\psi_2\circ\ldots\psi_k(n)}$ converges to some limit denoted by $\int_M \frac{\varphi_k}{\|\varphi_k\|}\,d\mu_\infty$. By Cantor's diagonal argument, defining $\psi(n) := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(n)$, we get a subsequence $(\mu_{\psi(n)})_{n\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $\int_M \frac{\varphi_k}{\|\varphi_k\|}\,d\mu_{\psi(n)} \to \int_M \frac{\varphi_k}{\|\varphi_k\|}\,d\mu_\infty$. By a standard density argument, for every $\varphi \in C_c(M)$, we show that $\int_M \frac{\varphi}{\|\varphi\|}\, d\mu_{\psi(n)}$ is a Cauchy sequence, and therefore converges towards a limit, denoted by $\int_M \frac{\varphi}{\|\varphi\|}\, d\mu_\infty$. Thus, for every $\varphi \in C_c(M)$, we obtain that

$$\int_M \varphi\, d\mu_{\psi(n)} = \|\varphi\| \int_M \frac{\varphi}{\|\varphi\|}\, d\mu_{\psi(n)} \to \|\varphi\| \int_M \frac{\varphi}{\|\varphi\|}\, d\mu_\infty = \int_M \varphi\, d\mu_\infty\,.$$

If every μ_n is φ-invariant, then the limit μ_∞ of the converging subsequence will be φ-invariant. Moreover, we can check that $\mu_\infty(M) \leq 1$. Therefore, $(m_{K_0,L})_{L>0}$ has at least one accumulation point, and all its accumulation points are φ-invariant and of mass at most 1.

Proposition 6.1. *Let $\varphi : M \to M$ be a H-flow being h_{Gur}-strongly positive recurrent. Then for every compact set K_0 big enough, any accumulation point m_∞ of the family $(m_{K_0,L})_{L>0}$ when $L \to +\infty$ is a nonzero finite invariant measure.*

Proof. Fix a compact set K_0 and $\varepsilon > 0$ such that (22) is satisfied. That is

$$\limsup_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}^{\varepsilon}_{K_0}(L, L+5\tau_{K_0}) < \lim_{L\to\infty} \frac{1}{L} \log \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})\,.$$

Let $L_n \to \infty$ be a sequence such that $(m_{K_0,L_n})_n$ converges to m_∞ in the vague topology. Let us show that

$$\liminf_{L\to\infty} m_{K_0,L}(K_0) \geq \varepsilon\,.$$

Split $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ into

$$\mathcal{P}^{good}_{K_0}(L, L+5\tau_{K_0}) = \{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}),\ \ell(\gamma \cap K_0) \geq \varepsilon \ell(\gamma)\}$$

and

$$\mathcal{P}^{bad}_{K_0}(L, L+5\tau_{K_0}) = \mathcal{P}_{K_0}(L, L+5\tau_{K_0}) \setminus \mathcal{P}^{good}_{K_0}(L, L+5\tau_{K_0}) = \mathcal{P}^{\varepsilon}_{K_0}(L, L+5\tau_{K_0})\,.$$

Inequality (22) ensures that $\limsup_{L\to\infty} \frac{1}{L} \log\left(\frac{\#\mathcal{P}^{bad}_{K_0}(L,L+5\tau_{K_0})}{\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\right) < 0$, so that

$$\lim_{L\to+\infty} \frac{\#\mathcal{P}^{bad}_{K_0}(L, L+5\tau_{K_0})}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} = 0\,. \tag{24}$$

We deduce easily that

$$\begin{aligned} m_{K_0,L}(K_0) &= \frac{1}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \left(\sum_{\gamma \in \mathcal{P}^{good}_{K_0}(L,L+5\tau_{K_0})} \frac{\ell(\gamma\cap K_0)}{\ell(\gamma)} + \sum_{\gamma \in \mathcal{P}^{bad}_{K_0}(L,L+5\tau_{K_0})} \frac{\ell(\gamma\cap K_0)}{\ell(\gamma)} \right) \\ &\geq \varepsilon \frac{\#\mathcal{P}^{good}_{K_0}(L, L+5\tau_{K_0})}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})}\,; \end{aligned}$$

thus, we obtain

$$\liminf_{L\to+\infty} m_{K_0,L}(K_0) \geq \varepsilon\,.$$

For every continuous map with compact support $\varphi \geq 1_{K_0}$, by definition of vague convergence, we get $\int \varphi dm_\infty = \lim_{n\to\infty} \int \varphi dm_{K_0,L_n} \geq \liminf_{n\to\infty} m_{K_0,L_n}(K_0) \geq \varepsilon$. Choosing a decreasing sequence φ_k of such maps, with $\lim_{k\to\infty} \varphi_k = 1_{K_0}$, and using the decreasing version of the monotone convergence theorem we obtain $m_\infty(K) = \lim \int \varphi_k dm_\infty \geq \varepsilon$. The proposition follows. □

Notation 6.2 (Measures)**.** From now, we denote by m_∞ an arbitrary fixed accumulation point of the family $(m_{K_0,L})_{L>0}$, and by $m_{\max}$ the invariant probability measure obtained by renormalizing m_∞.

Remark 6.3. The h_{Gur}-strongly positive recurrent assumption $h^\infty_{\mathrm{Gur}}(\varphi) < h_{Gur}(\varphi)$ is natural, and leads to many natural examples, as shown in [ST21, GST23] in the case of geodesic flows, or in [FSV25]. Moreover, it is stable under perturbations on compact sets.

6.2 The subadditivity statements

We will need rigorous versions of the following heuristic statements.

First, given two chords of respective lengths T_0 and T_1, we can concatenate them, and use the closing lemma to get a periodic orbit of length roughly $T_0 + T_1$. This process is essentially injective: we obtain a lower bound on the number of periodic orbits of length T_0+T_1 in terms of the number of chords of lengths T_0 and T_1. This lower bound is clearly still valid when we add a linear term T_0+T_1 on the right. This is properly stated in Proposition 6.4.

Second, we wish to say that a periodic orbit of length T_0+T_1 can be divided in two pieces of orbits of respective lengths T_0 and T_1, and therefore, bound the number of periodic orbits of length T_0+T_1 in terms of the numbers of chords of lengths T_0 and T_1, respectively. This is much more subtle, for two deep reasons. First, due to the lack of compactness, there is absolutely no reason that an arbitrary periodic orbit return in a compact set at the time T_0. The measure constructed in definition 6.2 in section 6.1 will be crucial in the argument, and allow us to say that "most", or more precisely a positive proportion, of periodic orbits of length T_0+T_1 come back to K_0. The second difficulty is that we need a strong subadditivity, with a linear term on the left : T_0+T_1 times the number of periodic orbits of length T_0+T_1 should be smaller than the number of chords of length T_0 times the number of chords of length T_1. This requires precise statements on the number of returns of a typical periodic orbits, see Lemmas 6.7 and 6.8 and Proposition 6.5.

Proposition 6.4 (Easy subadditivity)**.** *Let $\varphi\colon M\to M$ be a H-flow. Let K_0 be a compact set with nonempty interior. For every compact set $K\supset K_0$, every $0<\delta<1$ and every $0<\eta<1$, there exist $T_{\min}>0$, $S_1>0$ and $D_1>0$ such that for every $S\geq S_1$, all x_0,y_0,z_0 in K, and all $T_0,T_1\geq T_{\min}$, we have :*

$$\mathcal{N}_C(x_0,y_0,\eta,T_0,T_0+5\tau_{K_0},\delta)\times\mathcal{N}_C(y_0,z_0,\eta,T_1,T_1+5\tau_{K_0},\delta)\leq \\ D_1\times(T_0+T_1+S)\times\#\mathcal{P}_{K_0}(T_0+T_1+S,T_0+T_1+S+5\tau_{K_0})\,.$$

The following proposition is more difficult. It is stated under the assumption of existence of a nonzero measure $m_{\max}$ as in definition 6.2. This assumption is satisfied as soon as the flow is h_{Gur}-strongly positive recurrent, i.e., $h^{\infty}_{\mathrm{Gur}}(\varphi)<h_{\mathrm{Gur}}(\varphi)$, by Proposition 6.1.

Proposition 6.5 (Hard subadditivity)**.** *Let $\varphi\colon M\to M$ be a H-flow that is h_{Gur}-strongly positive recurrent. Let K_0 be a compact set with nonempty interior that satisfies inequality (24). Let $L_n\to+\infty$ be a sequence such that the sequence of measures $(m_{K_0,L_n})_n$ converges in the vague topology to a non zero measure m_∞. Let K be a compact subset such that $\overset{\circ}{K}\supset K_0$ and $m_\infty(\overset{\circ}{K})\geq\frac{3}{4}m_\infty(M)$. There exists $\varepsilon_1>0$ such that, for all $0<\eta<\delta<\frac{\varepsilon_1}{4}$ there exist constants $S_2>0$ and $D_2>0$ such that for every $T>5\tau_{K_0}$, there exists $k_0\in\mathbb{N}$, such that for every integer $n\geq k_0$, for all quadruples of points x_0, y_0, x_1 and y_1 in K, and every $S\geq S_2$, we have*

$$L_n\times\#\mathcal{P}_{K_0}(L_n,L_n+5\tau_{K_0})\leq D_2\times\mathcal{N}_C(x_0,y_0,\eta,T+S,T+S+5\tau_{K_0},\delta) \\ \times\mathcal{N}_C(x_1,y_1,\eta,L_n-T+S,L_n-T+S+5\tau_{K_0},\delta)\,.$$

Remark 6.6. We could have considered periodic orbits of periods T_0 and T_1, and stated a variant of the above propositions involving respectively $\#\mathcal{P}_{K_0}(T_0,T_0+5\tau_{K_0})$ and $\#\mathcal{P}_{K_0}(T_1,T_1+5\tau_{K_0})$ instead of the numbers of chords on the left side of the inequality of Proposition 6.4, and $\mathcal{P}_{K_0}(T_0+S,T_0+S+5\tau_{K_0})$, $\mathcal{P}_{K_0}(L_n-T_0+S,L_n-T_0+S+5\tau_{K_0})$ on the right side of the inequality of Proposition 6.4.

6.3 Proof of Proposition 6.4

Step 1. Choice of appropriate parameters and notations.

Let K, K_0, δ and η be as in the statement of the Proposition. Let $K_1=\overline{B(K,1)}$. The proof will follow from the application of Lemma 2.16 with parameters $K_0\subset K_1$, with $\nu=\tau_{K_0}/2$, $\delta/4$ and $N=2$. This lemma gives constants $\sigma>0$ and $T_{\min}>0$. Set $S_1=2\sigma+10\tau_{K_0}$. Let x_0, y_0 and z_0

be in K. Let $T_0, T_1 \geq T_{\min}$. Let E_0 and E_1 be respectively a $E(x_0, y_0, \eta, T_0, T_0 + 5\tau_{K_0}, \delta)$-set and a $E(y_0, z_0, \eta, T_1, T_1 + 5\tau_{K_0}, \delta)$-set of maximal cardinality. In particular

$$\#E_0 = \mathcal{N}_{\mathcal{C}}(x_0, y_0, \eta, T_0, T_0 + 5\tau_{K_0}, \delta) \quad \text{and} \quad \#E_1 = \mathcal{N}_{\mathcal{C}}(y_0, z_0, \eta, T_1, T_1 + 5\tau_{K_0}, \delta)\,.$$

Throughout the proof, we will see the elements in E_0 and E_1 as points in $B(x_0, \eta)$ and $B(y_0, \eta)$ or chords with their initial points in $B(x_0, \eta)$ and $B(y_0, \eta)$. In particular, the initial and final points of every chord is in K_1, since $\eta < 1$.

Step 2. Construction of a map f from chords to periodic orbits.

As $T_0 \geq T_{\min}$ and $T_1 \geq T_{\min}$, for every pair of chords $\beta_0 \in E_0$ and $\beta_1 \in E_1$ with lengths $\ell(\beta_0) \geq T_0$ and $\ell(\beta_1) \geq T_1$, and every $\hat{S} \geq \sigma$, Lemma 2.16 provides a periodic orbit γ with length in

$$\left[\ell(\beta_0) + \ell(\beta_1) + 2\hat{S} - \tau_{K_1} - \frac{\tau_{K_0}}{2}, \ell(\beta_0) + \ell(\beta_1) + 2\hat{S} + \tau_{K_1} + \frac{\tau_{K_0}}{2}\right]$$

that intersects the interior of K_0. Recall that, since $K_0 \subset K_1$, we have $\tau_{K_1} \leq \tau_{K_0}$. Thus, the periodic orbit γ has length in

$$\left[\ell(\beta_0) + \ell(\beta_1) + 2\hat{S} - \frac{3}{2}\tau_{K_0}, \ell(\beta_0) + \ell(\beta_1) + 2\hat{S} + \frac{3}{2}\tau_{K_0}\right]\,.$$

In particular, for

$$\hat{S} = \frac{1}{2}\left(S + (T_0 - \ell(\beta_0)) + (T_1 - \ell(\beta_1)) + \frac{3}{2}\tau_{K_0}\right)$$

(notice that $\hat{S} \geq \sigma$ as $S \geq S_1 = 2\sigma + 10\tau_{K_0}$, $T_0 - \ell(\beta_0) \geq -5\tau_{K_0}$ and $T_1 - \ell(\beta_1) \geq -5\tau_{K_0}$) we have

$$T_0 + T_1 + S \leq \ell(\beta_0) + \ell(\beta_1) + 2\hat{S} - \frac{3}{2}\tau_{K_0}$$

and

$$\ell(\beta_0) + \ell(\beta_1) + 2\hat{S} + \frac{3}{2}\tau_{K_0} \leq T_0 + T_1 + S + 5\tau_{K_0}\,.$$

Then $\gamma \in \mathcal{P}_{K_0}(T_0 + T_1 + S, T_0 + T_1 + S + 5\tau_{K_0})$.

Thus for every $S \geq S_1$, the above construction defines a map

$$f : E_0 \times E_1 \to \mathcal{P}_{K_0}(T_0 + T_1 + S, T_0 + T_1 + S + 5\tau_{K_0})\,.$$

Observe that the above construction gives a parametrization of γ with an origin s_0. More precisely, by Lemma 2.16, there exists $s_0 \in \mathbb{R}$ and $\tau \in [\hat{S} - \tau_{K_1}, \hat{S} + \tau_{K_1}]$ such that,

- for all $s \in [0, \ell(\beta_0)]$, we have $d(\gamma(s_0 + s), \varphi_s(\beta_0)) < \frac{\delta}{4}$,
- for all $s \in [0, \ell(\beta_1)]$, we have $d(\gamma(s_0 + s + \ell(\beta_0) + \tau), \varphi_s(\beta_1)) < \frac{\delta}{4}$.

Step 3. Bound on $f^{-1}(\gamma)$.

Consider $\beta_0, \beta_0' \in E_0, \beta_1, \beta_1' \in E_1$ such that $(\beta_0, \beta_1) \neq (\beta_0', \beta_1')$ and $f(\beta_0, \beta_1) = f(\beta_0', \beta_1') = \gamma$. Let s_0 (resp. s_0') be the origin of γ from the construction $\gamma = f(\beta_0, \beta_1)$ (resp. $\gamma = f(\beta_0', \beta_1')$). Assume as a first case that $\beta_0 \neq \beta_0'$. Since they belong to a $E(x_0, y_0, \eta, T_0, T_0 + 5\tau_{K_0}, \delta)$-set, there exist $0 \leq u \leq T_0 \leq \min(\ell(\beta_0), \ell(\beta_1))$, such that

$$d(\varphi_u(\beta_0), \varphi_u(\beta_0')) \geq \delta.$$

However, by construction, γ satisfies

$$d(\gamma(s_0 + u), \varphi_u(\beta_0)) \leq \delta/4 \quad \text{and} \quad d(\gamma(s_0' + u), \varphi_u(\beta_0')) \leq \delta/4\,.$$

Therefore

$$d(\gamma(s_0 + u), \gamma(s_0' + u)) \geq \delta/2\,.$$

As the flow satisfies (2), we have

$$d(\gamma(s_0+u),\gamma(s_0'+u)) \leq b|s_0'-s_0|\,,$$

so that

$$|s_0'-s_0| \geq \frac{\delta}{2b}\,.$$

As the length of γ is at most $T_0+T_1+S+5\tau_{K_0}$, it follows that

$$\#\{(\beta_0',\beta_1'), f(\beta_0',\beta_1')=f(\beta_0,\beta_1) \text{ and } \beta_0'\neq\beta_0\} \leq\ (T_0+T_1+S+5\tau_{K_0})\times\frac{2b}{\delta}\,.$$

Assume now that $\beta_0=\beta_0'$ and $\beta_1\neq\beta_1'$. The construction of the periodic orbit γ gives us two constants τ and τ' in $[\hat{S}-\tau_{K_0},\hat{S}+\tau_{K_0}]$ and $[\hat{S}'-\tau_{K_0},\hat{S}'+\tau_{K_0}]$. Observe that, as $|\hat{S}-\hat{S}'|=\frac{1}{2}|\ell(\beta_1)-\ell(\beta_1')|\leq 5\tau_{K_0}/2$, we have $|\tau'-\tau|\leq 5\tau_{K_0}$.

Assume that $|\tau'-\tau|\leq\frac{\delta}{4b}$. Since E_1 is a $E(y_0,z_0,\eta,T_1,T_1+5\tau_{K_0},\delta)$-set, there exists $0\leq u\leq T_1$ such that

$$d(\varphi_u(\beta_1),\varphi_u(\beta_1'))\geq\delta\,.$$

Arguing as in the first case, we obtain that

$$d(\gamma(s_0+\ell(\beta_0)+\tau+u),\gamma(s_0'+\ell(\beta_0)+\tau'+u))\geq\frac{\delta}{2}\,,$$

and so, using (2), we get

$$d(\gamma(s_0+\ell(\beta_0)+\tau+u),\gamma(s_0'+\ell(\beta_0)+\tau+u))\geq\frac{\delta}{2}-b|\tau'-\tau|\geq\frac{\delta}{4}\,.$$

Using again (2), we deduce that

$$|s_0'-s_0|\geq\frac{\delta}{4b}\,.$$

We then conclude that

$$\#\{(\beta_0',\beta_1'), f(\beta_0',\beta_1')=f(\beta_0,\beta_1) \text{ and } \beta_1'\neq\beta_1\} \leq\ (T_0+T_1+S+5\tau_{K_0})\times\frac{4b}{\delta}\times 5\tau_{K_0}\times\frac{4b}{\delta}.$$

Thus

$$\#f^{-1}(\gamma)\leq\ (T_0+T_1+S+5\tau_{K_0})\times\left(\frac{2b}{\delta}+5\tau_{K_0}\left(\frac{4b}{\delta}\right)^2\right)\,.$$

By choosing $D_1>0$ such that

$$(T_0+T_1+S+5\tau_{K_0})\times\left(\frac{2b}{\delta}+5\tau_{K_0}\left(\frac{4b}{\delta}\right)^2\right)\leq D_1(T_0+T_1+S)\,,$$

this concludes the proof of Proposition 6.4.

6.4 Proof of Proposition 6.5

6.4.1 Strategy of the proof

As usual, we want to construct an almost injective map, specifically a map from periodic orbits of period L_n intersecting K_0 to chords of length (almost) T and endpoints x_0,y_0 and chords of length (almost) L_n-T and endpoints x_1,y_1.

For every periodic orbit of length L that intersects K_0, the naive idea is to consider a piece of orbit of length T starting and ending in K_0 and to cut the periodic orbit at the beginning and the end of this piece of orbit. We then obtain two arcs, one of length T and one of length $L-T$. Using transitivity and the shadowing property, we obtain arcs from x_0 to y_0 and from x_1 to y_1.

Unfortunately, there is no guarantee that such arcs exist. The difficulty of the proof consists in finding enough periodic orbits on which enough points of K_0 return to K_0 after a time T. To find these arcs, we will choose a compact set $K \supset K_0$ with large measure $m_{\max}(K) > 3/4$ so that for every $T > 0$, $m_{\max}(K \cap \varphi_{-T}(K)) > 1/2$. Set $A = K \cap \varphi_{-T}(K)$. We want to find enough intersections between A and $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$.

As $m_\infty(A) > 0$, for L large enough, a positive proportion of periodic orbits in $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ spend a positive proportion of their time in A. This is proved in Lemma 6.7.

In Lemma 6.8, we prove that if $\mu_\gamma(A) \geq \alpha' > 0$ then we control the number of points in A for some appropriate discretization of γ of length T_0.

Lemmas 6.7 and 6.8 give us Proposition 6.9 where we prove that there are indeed enough periodic orbits with enough point in K_0.

6.4.2 How often a typical periodic orbit comes back

The existence of a nonzero measure $m_{\max}$ leads to the first important lemma.

Lemma 6.7. *Let $\varphi\colon M \to M$ be a flow. Let K_0 be a compact set with nonempty interior. Let A be a Borel set, $L > 0$ and $\alpha > 0$ such that $m_{K_0,L}(A) \geq \alpha > 0$. For every $0 < \alpha' < \alpha$, we have*

$$\#\left\{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}) :\ \mu_\gamma(A) \geq \alpha'\right\} \geq \frac{\alpha - \alpha'}{1-\alpha'} \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}).$$

Proof. Let $n = \#\{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}), \mu_\gamma(A) < \alpha'\}$. Then

$$0 < \alpha \leq m_{K_0,L}(A) \leq \frac{\alpha' n + \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) - n}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})}.$$

Therefore

$$\alpha \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) \leq \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) - (1-\alpha')n.$$

so that $n \leq \frac{1-\alpha}{1-\alpha'} \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$. The result follows. □

Lemma 6.8. *Let $\varphi\colon M \to M$ be a flow. Let K_0 be a compact set with nonempty interior. Let A be a Borel set and $\alpha' > 0$. For all $L > T_0 > 5\tau_{K_0} > 0$, for every periodic orbit $\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ such that $\mu_\gamma(A) \geq \alpha'$, for any parametrization $\gamma : [0, \ell(\gamma)] \to M$ of γ there exists a real number $s \in [0, T_0]$ such that*

$$\#\left\{i \in \left\{0, \dots, \left\lfloor \frac{L}{T_0} \right\rfloor - 1\right\}, \gamma(s + iT_0) \in A\right\} \geq \left\lfloor \alpha' \left\lfloor \frac{L}{T_0} \right\rfloor \right\rfloor - 3.$$

Proof. Set $k = \left\lfloor \frac{L}{T_0} \right\rfloor$ and $k' = \left\lfloor \alpha' \left\lfloor \frac{L}{T_0} \right\rfloor \right\rfloor - 3$. Set $L = kT_0 + r$ with $0 \leq r < T_0$. By assumption, since $\mu_\gamma(A) \geq \alpha'$, we have $\ell(\gamma \cap A) \geq \alpha' \ell(\gamma)$. As $L \leq \ell(\gamma) \leq L + 5\tau_{K_0}$, we deduce that

$$\mathrm{Leb}\left(\{u \in [0, kT_0], \gamma(u) \in A\}\right) \geq \alpha' \ell(\gamma) - r - 5\tau_{K_0} \geq \alpha' k T_0 - T_0 - 5\tau_{K_0}.$$

For every $s \in [0, T_0]$, let $J(s) = \{0 \leq i \leq \left\lfloor \frac{L}{T_0} \right\rfloor - 1, \gamma(s + iT_0) \in A\}$. Suppose that for every $s \in [0, T_0]$, $\#J(s) \leq l$, for some integer $0 \leq l \leq k$.

We now prove that

$$\mathrm{Leb}\left(\{u \in [0, kT_0], \gamma(u) \in A\}\right) \leq lT_0. \tag{25}$$

Write $u = s + iT_0$, with $s \in [0, T_0]$ and $0 \leq i \leq k-1$. We have

$$\begin{aligned}\{u \in [0, kT_0], \gamma(u) \in A\} &= \bigsqcup_{i=0}^{k-1} \{s + iT_0, s \in [0, T_0], \gamma(s + iT_0) \in A\} \\ &= \bigsqcup_{J \in \mathcal{P}(\{0, \cdots, k-1\})} \left(\{s \in [0, T_0], J(s) = J\} + JT_0\right).\end{aligned}$$

As, for all $J \in \mathcal{P}(\{0, \cdots, k-1\})$, we have

$$\mathrm{Leb}\left(\{s \in [0, T_0], J(s) = J\} + JT_0\right) = \#J \times \mathrm{Leb}\left(\{s \in [0, T_0], J(s) = J\}\right),$$

as the subsets $\{s \in [0,T_0], J(s) = J\}$ form a measurable partition of $[0,T_0]$ and as $\#J(s) \leq l$, the inequality (25) follows.

Therefore, $\alpha' k T_0 - T_0 - 5\tau_{K_0} \leq l T_0$, and, since $5\tau_{K_0} < T_0$, we have $l \geq \alpha' k - 2$. For $k' = \lfloor \alpha' k \rfloor - 3$, we have $k' < \alpha' k - 2$. Therefore, $J(s)$ is not bounded by k' for all s and there exists $s \in \mathbb{R}$ such that

$$\#\left\{i \in \{0,\ldots,k-1\},\ \gamma(s+iT_0) \in A\right\} \geq k' .$$

□

In Subsection 3.3.1, we have defined $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ as the set of periodic orbits $((\varphi_t(x))_{t\in\mathbb{R}}, T)$, where (x,T) is a periodic point (that is $\varphi_T(x) = x$). Let $\mathcal{P}'_{K_0}(L, L+5\tau_{K_0})$ be the set of primitive periodic orbits in $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$, i.e., the set of periodic orbits $((\varphi_t(x))_{t\in\mathbb{R}}, T)$, where (x,T) is a periodic point and $\varphi_t(x) \neq x$ for every $t \in (0,T)$. The following statement will follow from the fact that:

- almost all periodic orbits are simple;
- if $m_{K_0,L}(A) \geq \alpha$, then a positive proportion of periodic orbits γ satisfy $\mu_\gamma(A) \geq \alpha'$ (Lemma 6.7);
- if $\mu_\gamma(A) \geq \alpha'$, then there exists an appropriate discretization of γ with step T_0 having a positive proportion of points in A (Lemma 6.8).

Proposition 6.9. *Let $\varphi\colon M \to M$ be a H-flow such that $h_{\mathrm{Gur}}(\varphi) > 0$. Let K_0 be a compact set with nonempty interior. Let A be a Borel set, $0 < \alpha' < \alpha$ and $L > 0$ such that $m_{K_0,L}(A) \geq \alpha > 0$. There exists $N > 0$ such that for all $L > T_0 > 5\tau_{K_0}$, with $\frac{L}{T_0} \geq N$, we have*

$$\#\left\{\gamma \in \mathcal{P}'_{K_0}(L, L+5\tau_{K_0}),\ \exists s \in [0,T_0],\ \#\Big\{i \in \{0,\ldots,\lfloor L/T_0 \rfloor - 1\}, \gamma(s+iT_0) \in A\Big\} \geq \frac{\alpha'}{2}\frac{L}{T_0}\right\}$$
$$\geq 0.99 \times \frac{\alpha - \alpha'}{1-\alpha'} \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) .$$

Proof. Choose N large enough so that, for $\frac{L}{T_0} \geq N$, we have $k' = \left\lfloor \alpha' \left\lfloor \frac{L}{T_0} \right\rfloor \right\rfloor - 3 \geq \frac{\alpha'}{2}\frac{L}{T_0}$. First, observe that a non primitive periodic orbit γ is a multiple of a primitive periodic one, that has length at most $\ell(\gamma)/2$. It follows that the number of non-simple periodic orbits in $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ is bounded above by $\mathcal{P}_{K_0}\left(\frac{L+5\tau_{K_0}}{2}\right)$. By Corollary 3.11 (whose hypothesis are satisfied because φ is a H-flow with $h_{\mathrm{Gur}}(\varphi) > 0$), it follows that

$$\lim_{L\to+\infty} \frac{\#\mathcal{P}'_{K_0}(L, L+5\tau_{K_0})}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} = 1 .$$

Up to choose a bigger N, we can assume that, if $L > NT_0 > N\,5\tau_{K_0}$, we have

$$\frac{\#\mathcal{P}'_{K_0}(L, L+5\tau_{K_0})}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \geq 1 - 0.01 \times \frac{\alpha-\alpha'}{1-\alpha'} .$$

Let $n_\gamma(s) = \#\{i \in \{0,\ldots,\lfloor L/T_0 \rfloor - 1\}, \gamma(s+iT_0) \in A\}$. Now, by lemmas 6.7 and 6.8, we have the inequalities

$$\begin{aligned}
\#\left\{\gamma \in \mathcal{P}'_{K_0}(L, L+5\tau_{K_0}), \exists s \in \mathbb{R}, n_\gamma(s) \geq \frac{\alpha'}{2}\frac{L}{T_0}\right\} \geq& \#\left\{\gamma \in \mathcal{P}'_{K_0}(L, L+5\tau_{K_0}), \exists s \in \mathbb{R}, n_\gamma(s) \geq k'\right\} \\
\geq& \#\left\{\gamma \in \mathcal{P}'_{K_0}(L, L+5\tau_{K_0}),\ \mu_\gamma(A) \geq \alpha'\right\} \\
\geq& \#\left\{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}),\ \mu_\gamma(A) \geq \alpha'\right\} \\
&- 0.01 \times \frac{\alpha-\alpha'}{1-\alpha'} \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) \\
\geq& \frac{\alpha-\alpha'}{1-\alpha'} \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) \\
&- 0.01 \times \frac{\alpha-\alpha'}{1-\alpha'} \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) \\
=& 0.99 \times \frac{\alpha-\alpha'}{1-\alpha'} \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0}) .
\end{aligned}$$

□

6.4.3 Proof of Proposition 6.5

Recall that m_∞ is the limit, in the vague topology, of a sequence $(m_{K_0,L_n})_{n\in\mathbb{N}}$ when $L_n \to \infty$, and $m_{\max}$ is the probability measure obtained by renormalization of m_∞.

Step 1. Setting the parameters. Choose some large compact set K such that $\overset{\circ}{K} \supset K_0$ and $m_{\max}(\overset{\circ}{K}) > 3/4$. In particular, for every $T > 0$, we obtain that $m_{\max}(K \cap \varphi_{-T}(K)) > 1/2$. Fix $T > 5\tau_{K_0}$. Let $A = K \cap \varphi_{-T}(K)$. Let $\alpha = \frac{m_\infty(A)}{2} > 0$. In particular, there exists k_0 such that for all $n \geq k_0$, we have $m_{K_0,L_n}(A) \geq \alpha$. Choose $0 < \alpha' < \alpha$. Lemma 2.13 applied with $\nu = 1$ and $\tau_1 = 5\tau_{K_0}$ provides us with constants τ_0 and ε_1 which will be used below. Fix η and δ such that $0 < \eta < \delta < \varepsilon_1/4$. Fix $T_0 = 6\tau_{K_0}$.

The finite exact shadowing, i.e., Proposition 2.6, applied with the compact set $\overline{B(K,1)}$, with $\delta = \frac{\eta}{2}$ and $N = 3$, gives us a constant $0 < \rho < \frac{\eta}{2}$. The uniform transitivity, i.e., Lemma 2.14, applied with the compact set K and with $\delta = \rho$, provides a constant $\sigma \geq 0$. Let $S_2 = 2\sigma + 2\tau_K + 10\tau_{K_0}$. Fix some $S \geq S_2$.

Let x_0, y_0, x_1, y_1 be four arbitrary points in K. Let E_0 be a $E(x_0, y_0, \eta, T+S, T+S+5\tau_{K_0}, \delta)$-set of maximal cardinality and E_1 be a $E(x_1, y_1, \eta, L_n - T + S, L_n - T + S + 5\tau_{K_0}, \delta)$-set of maximal cardinality. In particular, we have

$$\#E_0 = \mathcal{N}_C(x_0, y_0, \eta, T+S, T+S+5\tau_{K_0}, \delta) \quad \text{and} \quad \#E_1 = \mathcal{N}_C(x_1, y_1, \eta, L_n-T+S, L_n-T+S+5\tau_{K_0}, \delta)\,.$$

Step 2. Some preliminary estimates on the number of cutting points.
Denote by $\widetilde{P}_{K_0}(L_n)$ the set

$$\left\{\gamma \in \mathcal{P}'_{K_0}(L_n, L_n + 5\tau_{K_0})\,,\, \exists s \in [0, T_0], \#\{i \in [0, \dots, \lfloor L/T_0 \rfloor - 1], \gamma(s + iT_0) \in A\} \geq \frac{\alpha'}{2}\frac{L}{T_0}\right\}.$$

From Proposition 6.9, since $m_{K_0,L_n}(A) \geq \alpha > 0$, we know that

$$\#\widetilde{P}_{K_0}(L_n) \geq 0.99 \times \frac{\alpha - \alpha'}{1 - \alpha'} \times \#\mathcal{P}_{K_0}(L_n, L_n + 5\tau_{K_0})\,.$$

Without loss of generality, until the end of the proof, we reparametrize each $\gamma \in \widetilde{\mathcal{P}}_{K_0}(L_n)$ so that $s = 0$. Let $\mathcal{D}_\gamma$ be the set of times of the form iT_0, such that $\gamma(iT_0) \in A$. By the definition of $\widetilde{\mathcal{P}}_{K_0}(L_n)$, we have

$$\#\mathcal{D}_\gamma \geq \frac{\alpha'}{2} \times \frac{L_n}{T_0} = \frac{\alpha'}{12\tau_{K_0}} \times L_n\,.$$

Therefore

$$\# \bigsqcup_{\gamma \in \widetilde{P}_{K_0}(L_n)} \{\gamma\} \times D_\gamma \geq 0.99 \times \frac{\alpha - \alpha'}{1 - \alpha'} \times \frac{\alpha'}{12\tau_{K_0}} \times L_n \times \#\mathcal{P}_{K_0}(L_n, L_n + 5\tau_{K_0})\,. \tag{26}$$

Step 3. A map F from periodic orbits to chords.

The intuitive idea is the following. For each point $\gamma(iT_0) \in A$, as $A = K \cap \varphi_{-T}(K)$, we know that $\varphi_T(\gamma(iT_0)) \in K$. Cutting the orbit γ at these two points we get two chords of respective lengths T and $\ell(\gamma) - T \in [L_n - T, L_n - T + 5\tau_{K_0}]$. Using transitivity, we add some small arcs to obtain chords from x_0 to y_0 and from x_1 to y_1.

Let $\gamma \in \widetilde{\mathcal{P}}_{K_0}(L_n)$ and $i \in D_\gamma$. Using transitivity and finite exact shadowing, we can build a chord β^0 with length $\ell(\beta^0) \in [T + S, T + S + 4\tau_K]$ that goes from $B(x_0, \eta)$ at time 0, to $B(\gamma(iT_0), \eta/2)$ at time $t_{x_0} \in [\frac{S}{2}, \frac{S}{2} + 2\tau_K]$, then follows exactly γ at a distance at most $\eta/2$ during a time T, and then goes from $B(\gamma(iT_0+T), \eta/2)$ to $B(y_0, \eta)$ at time $\ell(\beta^0)$. For the remaining part of the proof, we denote by $\beta^0_{x_0}$ the restriction of β^0 to the interval $[0, t_{x_0}]$, β^0_γ the restriction of β^0 to the interval $[t_{x_0}, t_{x_0} + T]$, and $\beta^0_{y_0}$ the restriction of β^0 to the interval $[t_{x_0} + T, \ell(\beta^0)]$.

Similarly, we get a chord β^1 from $B(x_1, \eta)$ to $B(y_1, \eta)$ of length $\ell(\beta^1) \in [L_n - T + S, L_n - T + S + 4\tau_K]$ that goes from $B(x_1, \eta)$ at time 0, to $B(\gamma(iT_0 + T), \eta/2)$ at some time $t_{x_1} \in [\frac{S}{2}, \frac{S}{2} + 2\tau_K]$, then follows exactly γ at a distance at most $\eta/2$ during a time $\ell(\gamma) - T$, and then goes from $B(\gamma(iT_0), \eta/2)$ to

$B(y_1,\eta)$ at a time $\ell(\beta^1) \in [L_n - T + S, L_n - T + S + 4\tau_K]$. Observe that, in such a construction, the last part of the chord is obtained from a path built by transitivity whose length depends on the lengths of the chords of the first part of the construction, similarly to what is done in the proof of Proposition 5.7: this enables to have a more precise control on the length of the final chord. As above, we denote by $\beta^1_{x_1}$ the restriction of β^1 to $[0, t_{x_1}]$, β^1_γ the restriction of β^1 to $[t_{x_1}, t_{x_1} + \ell(\gamma) - T]$ and $\beta^1_{y_1}$ the restriction of β^1 to $[t_{x_1} + \ell(\gamma) - T, \ell(\beta^1)]$. Now, in order to define the image of γ by F, we consider the pair of chords $(\bar\beta_0, \bar\beta_1) \in E_0 \times E_1$ that is the closest to the pair (β^0, β^1).

Therefore, we obtain a map

$$F\colon \bigcup_{\gamma \in \widetilde{\mathcal{P}}_{K_0}(L_n)} \{\gamma\} \times \mathcal{D}_\gamma \to E_0 \times E_1 \,.$$

Step 4. The map F is almost injective.

Assume that (γ, iT_0) and $(\widetilde\gamma, \widetilde{i}T_0)$ lead to the same pair $(\bar\beta_0, \bar\beta_1) \in E_0 \times E_1$, where $\gamma, \widetilde\gamma \in \widetilde{\mathcal{P}}_{K_0}(L_n)$ and $i \in \mathcal{D}_\gamma, \widetilde{i} \in \mathcal{D}_{\widetilde\gamma}$. First, assume that

$$|\ell(\gamma) - \ell(\widetilde\gamma)| \leq \tau_0 \,, \tag{27}$$

where τ_0 is defined in Step 1.

The above construction associates to (γ, iT_0) (resp. $(\widetilde\gamma, \widetilde{i}T_0)$) two chords divided in three parts $\beta^0 = (\beta^0_{x_0}, \beta^0_\gamma, \beta^0_{y_0})$ and $\beta^1 = (\beta^1_{x_1}, \beta^1_\gamma, \beta^1_{y_1})$ (resp. $\widetilde\beta^0 = (\widetilde\beta^0_{x_0}, \widetilde\beta^0_{\widetilde\gamma}, \widetilde\beta^0_{y_0})$ and $\widetilde\beta^1 = (\widetilde\beta^1_{x_1}, \widetilde\beta^1_{\widetilde\gamma}, \widetilde\beta^1_{y_1})$) that are δ-close to the chord $\bar\beta_0$, during a time at least $T + S$, and to the chord $\bar\beta_1$, during a time at least $L_n - T + S$, respectively. Assume also that

$$\max\left\{ \left|\ell(\widetilde\beta^0_{x_0}) - \ell(\beta^0_{x_0})\right|, \left|\ell(\widetilde\beta^0_{y_0}) - \ell(\beta^0_{y_0})\right|, \left|\ell(\widetilde\beta^1_{x_1}) - \ell(\beta^1_{x_1})\right|, \left|\ell(\widetilde\beta^1_{y_1}) - \ell(\beta^1_{y_1})\right| \right\} \leq \frac{\delta}{b} \,. \tag{28}$$

By construction, for every $u \in [0, T]$, we have

$$d(\bar\beta_0(\ell(\beta^0_{x_0}) + u), \beta^0(\ell(\beta^0_{x_0}) + u)) \leq \delta$$

and

$$d(\beta^0(\ell(\beta^0_{x_0}) + u), \gamma(iT_0 + u)) \leq \eta/2.$$

Therefore, for every $u \in [0, T]$,

$$d(\overline\beta_0(\ell(\beta^0_{x_0}) + u), \gamma(iT_0 + u)) \leq \delta + \eta/2.$$

Similarly, for every $u \in [0, T]$,

$$d(\overline\beta_0(l(\widetilde\beta^0_{x_0}) + u), \widetilde\gamma(\widetilde{i}T_0 + u)) \leq \delta + \eta/2.$$

As, by (2),

$$d(\overline\beta_0(\ell(\beta^0_{x_0}) + u), \overline\beta_0(\ell(\widetilde\beta^0_{x_0}) + u)) \leq b\left|\ell(\widetilde\beta^0_{x_0}) - \ell(\beta^0_{x_0})\right| \leq \delta \,,$$

for every $u \in [0, T]$, we get, since $\eta < \delta < \frac{\varepsilon_1}{4}$,

$$d(\gamma(iT_0 + u), \widetilde\gamma(\widetilde{i}T_0 + u)) \leq 4\delta < \varepsilon_1 \,.$$

On the other part of the orbit, the same reasoning gives for every $u \in [0, L_n - T]$

$$d(\gamma(iT_0 + T + u), \widetilde\gamma(\widetilde{i}T_0 + T + u)) \leq 4\delta < \varepsilon_1 \,.$$

Therefore γ and $\widetilde\gamma$ are ε_1-close on an interval of length L_n. Lemma 2.13 implies that $\gamma = \widetilde\gamma$ and $i = \widetilde{i}$.

This proves that for every choice of length of $\ell(\gamma) \in [L_n, L_n + 5\tau_{K_0}]$ up to τ_0 as in (27), and every choice respectively of $\ell(\beta^0_{x_0}), \ell(\beta^0_{y_0}), \ell(\beta^1_{x_1}), \ell(\beta^1_{y_1})$ in $[\frac{S}{2}, \frac{S}{2} + 2\tau_K]$ up to δ/b as in (28), there is at most one pair (γ, iT_0) leading to $(\bar\beta^0, \bar\beta^1)$. As a consequence, there are at most $\left(\left\lfloor \frac{5\tau_{K_0}}{\tau_0} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{2\tau_K\, b}{\delta} \right\rfloor + 1\right)^4$

pairs (γ, iT_0) with γ a periodic orbit in $\widetilde{\mathcal{P}}_{K_0}(L_n)$ and $iT_0 \in \mathcal{D}_\gamma$ that lead to the pair of chords $(\bar\beta_0, \bar\beta_1)$. Therefore

$$\# \bigcup_{\gamma \in \widetilde{\mathcal{P}}_{K_0}(L_n)} (\{\gamma\} \times \mathcal{D}_\gamma) \leq \left(\left\lfloor \frac{5\tau_{K_0}}{\tau_0} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{2\tau_K\, b}{\delta} \right\rfloor + 1\right)^4 \times \# E_0 \times \# E_1 \,. \tag{29}$$

Step 5. Conclusion.

This allows to conclude the proof of Proposition 6.5. Indeed, by (26) and (29),

$$\begin{aligned} L_n \times \#\mathcal{P}_{K_0}(L_n, L_n + 5\tau_{K_0}) \leq & \frac{1}{0.99} \frac{1-\alpha'}{\alpha - \alpha'} \times \frac{12\tau_{K_0}}{\alpha'} \times \# \bigcup_{\gamma \in \widetilde{\mathcal{P}}_{K_0}(L_n)} (\{\gamma\} \times \mathcal{D}_\gamma) \\ \leq & \frac{1}{0.99} \frac{1-\alpha'}{\alpha - \alpha'} \times \frac{12\tau_{K_0}}{\alpha'} \times \left(\left\lfloor \frac{5\tau_{K_0}}{\tau_0} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{2\tau_K b}{\delta} \right\rfloor + 1\right)^4 \times \# E_0 \times \# E_1 \\ = & D_2 \times \mathcal{N}_C(x_0, y_0, \eta, T+S, T+S+5\tau_{K_0}, \delta) \\ & \times N_C(x_1, y_1, \eta, L_n - T + S, L_n - T + S + 5\tau_{K_0}, \delta) \end{aligned}$$

with

$$D_2 = \frac{1}{0.99} \frac{1-\alpha'}{\alpha - \alpha'} \times \frac{12\tau_{K_0}}{\alpha'} \times \left(\left\lfloor \frac{5\tau_{K_0}}{\tau_0} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{2\tau_K\, b}{\delta} \right\rfloor + 1\right)^4 .$$

7 The measure maximizes the entropy

In this section, K_0 is a compact set with nonempty interior as in section 6.1 and $K \supset K_0$ is a larger compact set, such that $K_0 \subset \overset{\circ}{K}$.

In the first section, we prove uniform estimates on $m_{K_0,L}(B(x,T,\varepsilon))$ for $x \in K$. In the second section, we take the limit when $L \to +\infty$ and obtain uniform estimates on the measure $m_{\max}(B(x,T,\varepsilon))$, for every $x \in K$ (for every compact set $K \subset M$). In the last section, we finally prove our main theorem.

7.1 Relation between $m_{K_0,L}$ and the number of chords

The heuristics of this section is the following. Recall that the measure $m_{K_0,L}$ is defined in (23) as the average of the periodic invariant probability measures μ_γ, where γ varies over all periodic orbits in $\mathcal{P}_{K_0}(L, L+5\tau_{K_0})$, i.e., those intersecting K_0, with length in $[L, L+5\tau_K]$. Given some dynamical ball $B(x,T,\varepsilon)$ for $x \in M$, the measure $m_{K_0,L}(B(x,T,\varepsilon))$ satisfies therefore

$$\begin{aligned} m_{K_0,L}(B(x,T,\varepsilon)) &= \frac{1}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \sum_{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \mu_\gamma(B(x,T,\varepsilon)) \\ &\simeq \frac{1}{L \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \sum_{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \ell(\gamma \cap B(x,T,\varepsilon)) \,. \end{aligned}$$

Now, the heart of the argument is the proof that the last sum is comparable to the number $\mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, \eta, L-T, \delta)$ of chords from $\varphi_T(x)$ to x, so that

$$m_{K_0,L}(B(x,T,\varepsilon)) \simeq \frac{1}{L \times \#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \times \mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, L-T, \delta) \,.$$

The idea of the proof consists in the following remark. Given a chord starting at a point z of length roughly $L-T$ from a neighbourhood of $\varphi_T(x)$ to a neighbourhood of x, and using the closing lemma as in Lemma 2.16, we can build a periodic orbit in $\mathcal{P}_{K_0}(L, L+5\tau_K)$ following first $(\varphi_s(x))_{0\leq s\leq T}$ and afterwards the chord $(\varphi_s(z))_{0\leq s\leq L-T}$, and intersecting $B(x,\varepsilon,T)$. Conversely, given a periodic orbit $\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_K)$ with an origin $w \in B(x,\varepsilon,T)$, we can cut γ at 0 and T, and get a chord of length roughly $L-T$ from a neighbourhood of $\varphi_T(x)$ to a neighbourhood of x. The difficulty of the argument is to show that the above constructions are almost one-to-one, or more precisely, that each preimage of the corresponding maps between chords and periodic orbits has bounded cardinality. The rigorous details corresponding to the above heuristics are provided in Lemmas 7.3 (lower bound) and 7.5 (upper bound). To this purpose, we will need to notice in Lemma 7.2 that each return of γ in $B(x,T,\varepsilon)$ has a bounded length and in Lemma 7.4 that these returns are not too close one from another.

We start with an easy but useful observation. Recall that b is defined in Equation (2).

Lemma 7.1. *For all $\varepsilon>0$, $T>0$, $x\in M$, and $y\in B(x,\frac{\varepsilon}{2},T)$, for every $0\le s<\frac{\varepsilon}{2b}$, we have $\varphi_s(y)\in B(x,\varepsilon,T)$.*

Proof. For every $s\in[0,\frac{\varepsilon}{2b}[$, and every $\tau\in[0,T]$, we have

$$d(\varphi_{\tau+s}(y),\varphi_\tau(x))\le d(\varphi_{\tau+s}(y),\varphi_\tau(y))+d(\varphi_\tau(y),\varphi_\tau(x))\le |s|b+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

so that $\varphi_s(y)\in B(x,\varepsilon,T)$, as desired. □

In Lemmas 7.2 and 7.4, given a set C, and a point $y\in C$, we consider the connected component of 0 in $\{s\in\mathbb{R},\varphi_s(y)\in C\}$. It is an interval that we denote by $J_C(y)=(J_C^{\min}(y),J_C^{\max}(y))$. In Lemma 7.2, we prove that for a dynamical ball $B(x,\varepsilon,T)$, and any $y\in B(x,\varepsilon,T)$, the size of the interval $J_{B(x,\varepsilon,T)}(y)$ is bounded.

Lemma 7.2. *Let $K\subset M$ be compact and $\Delta>0$. There exist $\varepsilon_{K,\Delta}>0$ such that for every $0<\varepsilon\le\varepsilon_{K,\Delta}$, $x\in K$, $T\ge1$ and $y\in B(x,\varepsilon,T)$, we have*

$$\mathrm{Leb}\left(J_{B(x,\varepsilon,T)}(y)\right)=J^{\max}_{B(x,\varepsilon,T)}(y)-J^{\min}_{B(x,\varepsilon,T)}(y)\le \mathrm{Leb}\left(J_{B(x,\varepsilon,1)}(y)\right)\le\Delta\,.$$

Proof. First note that for all $T\ge1$, $x\in M$ and $\varepsilon>0$, we have $B(x,\varepsilon,T)\subset B(x,\varepsilon,1)$ and therefore $J_{B(x,\varepsilon,T)}(y)\subset J_{B(x,\varepsilon,1)}(y)$. For every $x\in K$, there exists $\tau_0=\tau_0(x)>0$, $r=r(x)>0$ and a flow-bow Ω centered at x diffeomorphic to $B(0,r)\times(-\tau_0,\tau_0)$ (where $B(0,r)$ is a ball of radius r in $\mathbb{R}^{\dim(M)-1}$). As Ω is a neighborhood of x, there exists $\varepsilon(x)$ such that $B(x,\varepsilon(x),1)\subset\Omega$. For every $0<\varepsilon\le\varepsilon(x)$ and every $y\in B(x,\varepsilon,1)$, we have $B(x,\varepsilon,1)\subset B(x,\varepsilon(x),1)\subset\Omega$. Therefore, $\mathrm{Leb}\left(J_{B(x,\varepsilon,1)}(y)\right)\le 2\tau_0$. We may reduce τ_0 so that $\tau_0\le\Delta/2$. As K is compact, one can choose τ_0 and ε uniformly in x. This concludes the proof of the lemma. □

We are now able to prove the first key lemma of this section.

Lemma 7.3. *Let K_0 be a compact set with nonempty interior as in Section 6.1, and $K\supset K_0$ be a larger compact set. For every $\delta>0$, there exists $\varepsilon_{K,\delta}>0$ and $T'_{\min}>1$ such that for every $0<\varepsilon<\varepsilon_{K,\delta}$, there exist $\sigma=\sigma_{K,\delta,\varepsilon}>0$ and $\eta_{K,\delta,\varepsilon}>0$ such that for every $0<\eta<\eta_{K,\delta,\varepsilon}$, all L,T such that $L\ge T+T'_{\min}$ and $T\ge T'_{\min}$ and $x\in K$ such that $\varphi_T(x)\in K$, we have*

$$\frac{\varepsilon}{4b}\times\frac{\mathcal{N}_{\mathcal{C}}(\varphi_T(x),x,\eta,L-T-\sigma_{K,\delta,\varepsilon},L-T-\sigma_{K,\delta,\varepsilon}+5\tau_{K_0},\delta)}{L\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\le m_{K_0,L}(B(x,\varepsilon,T)).$$

Proof. The heuristics of the proof is the following. Thanks to Lemma 2.17, we can glue any chord of length roughly $L-T$ starting close to $\varphi_T(x)$ and arriving close to x to the orbit $(\varphi_t(x))_{0\le t\le T}$ to get a periodic orbit of length L intersecting $B(x,\varepsilon,T)$. Doing it with enough care will guarantee that this intersection point lies inside $B(x,\varepsilon/2,T)$, and therefore the orbit spends a time at least $\frac{\varepsilon}{2b}$ inside $B(x,\varepsilon,T)$. This will allow to get the desired bound. Let us start the rigorous argument.

Step 1. Choice of constants. Choose $T'_{\min}$ so that for $L\ge T'_{\min}$, we have $\frac{1}{L+5\tau_{K_0}}\ge\frac{1}{2L}$. Fix $\delta>0$. Then, Lemma 7.2 applied on K with $\Delta=\delta/2b$ gives us a constant $\varepsilon_{K,\delta}>0$.

Fix $0<\varepsilon<\varepsilon_{K,\delta}$ and set $\delta'=\min(\delta/4,\varepsilon/2,1)$. Lemma 2.17 applied with $K_0\subset\overline{B(K,1)}$, $\delta=\delta'>0$, $\nu=\tau_{K_0}$ and $N=2$, gives us $\eta_{K_0,\delta,\varepsilon}\le1$, $\sigma>0$ and $T_{\min}>0$ such that the following holds. Let $\sigma_{K,\delta,\varepsilon}:=\sigma+3\tau_{K_0}$. First, increase $T'_{\min}$ so that $T'_{\min}\ge T_{\min}$ and $T'_{\min}\ge\sigma_{K,\delta,\varepsilon}$. Then, for every $0<\eta<\eta_{K_0,\delta,\varepsilon}$, L,T such that $L\ge T+T'_{\min}$ and $T\ge T'_{\min}$ and any chord z from $B(\varphi_T(x),\eta)$ to $B(x,\eta)$ with length $\ell(z)$ such that

$$L-T-\sigma_{K,\delta,\varepsilon}\le\ell(z)\le L-T-\sigma_{K,\delta,\varepsilon}+5\tau_{K_0}\,,$$

let

$$S=\sigma_{K,\delta,\varepsilon}-\tau_{K_0}+(L-T-\sigma_{K,\delta,\varepsilon}+5\tau_{K_0}-\ell(z))=L-T-\ell(z)+2\tau_{K_0}\geq\sigma\,;$$

By Lemma 2.17, there exists a periodic orbit γ with length

$$\ell(\gamma)\in[T+\ell(z)+S-2\tau_{K_0},T+\ell(z)+S+2\tau_{K_0}]=[L,L+4\tau_{K_0}]\subset[L,L+5\tau_{K_0}]$$

that intersects $\mathring{K}_0$, δ'-shadows first $(\varphi_t(x))_{0\leq t\leq T}$ and then $(\varphi_t(z))_{0\leq t\leq\ell(z)}$. In particular, $\gamma(0)\in B(x,\delta')$. Observe that $\gamma\in\mathcal{P}_{K_0}(L,L+5\tau_{K_0})$.

Step 2. A first lower bound for $m_{K_0,L}(B(x,\varepsilon,T))$. Denote by $\mathcal{J}(L+5\tau_{K_0})$ the set of intervals included in $[0,L+5\tau_{K_0}]$. For each periodic orbit $\gamma\in\mathcal{P}_{K_0}(L,L+5\tau_{K_0})$, choose a parametrization such that $\gamma(0)\notin B(x,\varepsilon,T)$. Denote by $\Theta_{K_0,L}(x,\varepsilon,T)$ the set of pairs $(\gamma,I)\in\mathcal{P}_{K_0}(L,L+5\tau_{K_0})\times\mathcal{J}(L+5\tau_{K_0})$ such that for every $\gamma\in\mathcal{P}_{K_0}(L,L+5\tau_{K_0})$ (with its parametrization and associated origin), and every $s\in I$, $\gamma(s)\in B(x,\varepsilon,T)$. Moreover, assume I is maximal for this property. In a similar way, let $\Theta'_{K_0,L}(x,\varepsilon,T)$ be set of pairs $(\gamma,I)\in\Theta_{K_0,L}(x,\varepsilon,T)$ such that $\gamma(I)\cap B(x,\varepsilon/2,T)\neq\emptyset$.

Observe that for $L\geq T'_{\min}$, we have

$$\begin{aligned}
m_{K_0,L}(B(x,\varepsilon,T))&=\frac{1}{\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\sum_{\gamma\in\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\mu_\gamma(B(x,\varepsilon,T))\\
&=\frac{1}{\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\sum_{(\gamma,I)\in\Theta_{K_0,L}(x,\varepsilon,T)}\frac{\mathrm{Leb}(I)}{\ell(\gamma)}\\
&\geq\frac{1}{\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\sum_{(\gamma,I)\in\Theta'_{K_0,L}(x,\varepsilon,T)}\frac{\mathrm{Leb}(I)}{\ell(\gamma)}\\
&\geq\frac{1}{\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\sum_{(\gamma,I)\in\Theta'_{K_0,L}(x,\varepsilon,T)}\frac{\mathrm{Leb}(I)}{2L}\\
&\geq\frac{1}{\#\mathcal{P}_{K_0}(L,L+5\tau_{K_0})}\times\frac{\varepsilon}{4bL}\times\#\Theta'_{K_0,L}(x,\varepsilon,T)\,.
\end{aligned}$$

where the second lower bound follows from the inequalities $L\leq\ell(\gamma)\leq L+5\tau_{K_0}$ and $\frac{1}{L+5\tau_{K_0}}\geq\frac{1}{2L}$, and the third lower bound comes from Lemma 7.1.

Step 3. Going from chords to periodic orbits. Let E be a $E(\varphi_T(x),x,\eta,L-T-\sigma_{K,\delta,\varepsilon},L-T-\sigma_{K,\delta,\varepsilon}+5\tau_{K_0},\delta)$-set of maximal cardinality, i.e., so that

$$\#E=\mathcal{N}_{\mathcal{C}}(\varphi_T(x),x,\eta,L-T-\sigma_{K,\delta,\varepsilon},L-T-\sigma_{K,\delta,\varepsilon}+5\tau_{K_0},\delta)\,.$$

Applying Lemma 2.17 as described above, we can associate to each z in E a periodic orbit γ in $\mathcal{P}_{K_0}(L,L+5\tau_{K_0})$. Moreover, we know that the origin s_0 given by the construction is such that $\gamma(s_0)\in B(x,\delta',T)\subset B(x,\varepsilon/2,T)$. This point in $B(x,\varepsilon/2,T)$ gives us an interval J with $(\gamma,J)\in\Theta'_{K_0,L}(x,\varepsilon,T)$ and therefore a map

$$\theta:E\to\Theta'_{K_0,L}(x,\varepsilon,T)\,.$$

Step 4. The map θ is injective. Assume that two chords z_1,z_2 in E have same image (γ,J). For γ we use the parametrization from the definition of $\Theta_{K_0,L}(x,\varepsilon,T)$. The origins s_1 and s_2 associated to the construction of $\theta(z_1)$ and $\theta(z_2)$ satisfy $s_1,s_2\in J$ and $\gamma(s_1),\gamma(s_2)\in B(x,\varepsilon/2,T)$. By Lemma 7.2,as $\varepsilon\leq\varepsilon_{K,\delta}$, we have $|J|\leq\delta/2b$ (see Step 1). Therefore $|s_1-s_2|\leq\delta/2b$. An elementary computation gives, for every $0\leq s\leq\min(\ell(z_1),\ell(z_2))$, where $\ell(z_1),\ell(z_2)$ denote the length of the chords of z_1,z_2 respectively,

$$\begin{aligned}
d(\varphi_{s+T+s_1}(z_1),\varphi_{s+T+s_2}(z_2))&\leq d(\varphi_{s+T+s_1}(z_1),\gamma(s+T+s_1))+d(\gamma(s+T+s_1),\gamma(s+T+s_2))\\
&\quad+\;d(\gamma(s+T+s_2),\varphi_{s+T+s_2}(z_2))\\
&<\delta'+\delta/2+\delta'\leq\delta\,.
\end{aligned}$$

As E is a $(\delta,L-T-\sigma_{K,\delta,\varepsilon})$-separating set, we deduce that $z_1=z_2$, so that θ is injective. Therefore

$$\#E\leq\#\Theta'_{K_0,L}(x,\varepsilon,T).$$

Conclusion. The above arguments show that

$$\begin{aligned} m_{K_0,L}(B(x,\varepsilon,T)) &\geq \frac{\varepsilon}{4b} \times \frac{1}{L\,\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \times \#\Theta'_{K_0,L}(x,\varepsilon,T) \\ &\geq \frac{\varepsilon}{4b} \times \frac{1}{L\,\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \times \#E \\ &= \frac{\varepsilon}{4b} \times \frac{\mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, \eta, L-T-\sigma_{K,\delta,\varepsilon}, L-T-\sigma_{K,\delta,\varepsilon}+5\tau_{K_0}, \delta)}{L\,\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})}\,. \end{aligned}$$

□

Our next goal is to bound $m_{K_0,L}(B(x,\varepsilon,T))$ from above. We will need to bound from above the total amount of time that a periodic orbit $\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ spends in $B(x,\varepsilon,T)$. We know from Lemma 7.2 that each interval of time that such an orbit spends in $B(x,\varepsilon,T)$ has a bounded length. Lemma 7.4 is the second technical lemma. It allows to say that, up to increasing slightly $B(x,\varepsilon,T)$ to make it more smooth, the distance between two such intervals is bounded from below by a uniform constant, which allows to bound as desired the total amount of time in $B(x,\varepsilon,T)$.

Lemma 7.4 (No immediate return)**.** *Let $K \subset M$ be a compact set. There exist $\Delta_K > 0$ and $\alpha_K > 0$ such that for every $0 < \varepsilon \leq \alpha_K$, for every $x \in K$ and $T \geq 1$, there exists a set $C(x,\varepsilon,T)$ satisfying $B(x,\varepsilon,T) \subset C(x,\varepsilon,T)$, such that for every $y \in B(x,\varepsilon,T)$, and every $0 < s < \Delta_K$,*

$$\varphi_{J^{\max}_{C(x,\varepsilon,T)}(y)+s}(y) \notin C(x,\varepsilon,T), \quad \varphi_{J^{\min}_{C(x,\varepsilon,T)}(y)-s}(y) \notin C(x,\varepsilon,T) \quad \textit{and} \quad \mathrm{Leb}\left(J_{C(x,\varepsilon,T)}(y)\right) \leq 2\Delta_K\,.$$

In other words, $C(x,\varepsilon,T)$ contains portion of orbits of length at most $2\Delta_K$, after exiting $C(x,\varepsilon,T)$ an orbit remains outside $C(x,\varepsilon,T)$ for a time bounded below by Δ_K and a similar property is satisfied in the past.

Proof. For every $x \in K$, we can find $0 < r(x) < 1$ and $0 < \tau_0(x) < \tau_K/4$ such that x admits a flow-box neighbourhood $\Omega(x)$ diffeomorphic to $B(0,r(x)) \times (-\tau_0(x), \tau_0(x))$. Let $\psi : \Omega \to B(0,r(x)) \times (-\tau_0(x), \tau_0(x))$ be the associated flow-box chart. There exists $\varepsilon_0(x) > 0$ small enough such that $B(x,\varepsilon_0(x),1)$ is included in a flow box of half height. More precisely, for every $T \geq 1$,

$$\psi(B(x,\varepsilon_0(x),T)) \subset \psi(B(x,\varepsilon_0(x),1)) \subset B(0,r(x)) \times (-\tau_0(x)/2, \tau_0(x)/2)\,.$$

Fix $0 < \varepsilon \leq \varepsilon_0(x)$. It is not clear to us whether $\psi(B(x,\varepsilon,T))$ is convex (at least in the direction of the flow) or not so an orbit may exit $B(x,\varepsilon,T)$ for a very short time. To avoid these technical problems, we fill $B(x,\varepsilon,T)$ in the direction as the flow, and define $C(x,\varepsilon,T)$ as

$$\psi^{-1}\left(\{(z,\tau) \in B(0,r(x)) \times (-\tau_0(x)/2, \tau_0(x)/2),\ \exists \tau' \in (-\tau_0(x)/2, \tau_0(x)/2), (z,\tau') \in \psi(B(x,\varepsilon,T))\}\right).$$

Then, by construction, $\psi(C(x,\varepsilon,T)) \subset B(0,r(x)) \times (-\tau_0(x)/2, \tau_0(x)/2)$ so that for every $0 < s < \tau_0(x)/2$ and every $y \in B(x,\varepsilon,T)$,

- $\varphi_{J^{\max}_{C(x,\varepsilon,T)}(y)+s}(y) \cap C(x,\varepsilon,T) = \emptyset$;
- $\varphi_{J^{\min}_{C(x,\varepsilon,T)}(y)-s}(y) \cap C(x,\varepsilon,T) = \emptyset$;
- $\mathrm{Leb}\left(J_{C(x,\varepsilon,T)}\right) \leq \tau_0(x)$

As K is compact, r, τ_0 and ε_0 can be chosen uniformly in x. The result follows with $\Delta_K = \tau_0/2$ and $\alpha_K = \varepsilon_0$. □

The upper bound for $m_{K_0,L}(B(x,\varepsilon,T))$ is proven in the second key lemma.

Lemma 7.5. *Let K_0 be a compact set with nonempty interior as in Section 6.1, and $K \supset K_0$ be a larger compact set. There exist $\varepsilon_{K_0,K} > 0$, $\delta_{K_0,K} > 0$ and $D_{K_0,K} > 0$ such that for every $0 < \varepsilon < \varepsilon_{K_0,K}$ and every $0 < \delta < \delta_{K_0,K}$, for every $x \in K$ and for all $1 \le T < L$ such that $\varphi_T(x) \in K$, the following inequality holds*

$$m_{K_0,L}(B(x,\varepsilon,T)) \le D_{K_0,K} \frac{\mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, \varepsilon, L-T, L-T+5\tau_{K_0}, \delta)}{L\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} + \frac{\#\mathcal{P}_{K_0}\left(\frac{L+5\tau_{K_0}}{2}\right)}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})}.$$

Proof. The proof follows the same lines as the one of Lemma 7.3.

Step 1. Choice of constants. Lemma 7.4 gives us constants Δ_K and $\alpha_K > 0$ associated with K. Lemma 2.13 applied with $\nu = \min(\Delta_K/3, 1)$ and $\tau_1 = 5\tau_{K_0} \ge 1$ gives us $\tau_0 > 0$ and $\varepsilon_1 > 0$. Let $\varepsilon_{K_0,K} = \min(\alpha_K, \varepsilon_1)$ and $\delta_{K_0,K} = \frac{\varepsilon_1}{2}$. Fix $\varepsilon < \varepsilon_{K_0,K}$ and $\delta < \delta_{K_0,K}$. Fix $x \in K$. Fix $1 \le T \le L$ such that $\varphi_T(x) \in K$.

Step 2. First upper bound for $m_{K_0,L}(B(x,\varepsilon,T))$. Denote by $\mathcal{J}(L+5\tau_{K_0})$ the set of intervals included in $[0, L+5\tau_{K_0}]$. For each periodic orbit $\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})$, choose a parametrization with an origin $\gamma(0) \notin C(x,\varepsilon,T)$. Denote by $\widetilde{\Theta}_{K_0,L}(x,\varepsilon,T)$ the set of pairs $(\gamma, I) \subset \mathcal{P}_{K_0}(L, L+5\tau_{K_0}) \times \mathcal{J}(L+5\tau_{K_0})$ such that for every $\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})$ (with its parametrization and associated origin), and every $s \in I$, we have $\gamma(s) \in C(x,\varepsilon,T)$, I is maximal for this property and $\gamma(I) \cap B(x,\varepsilon,T) \neq \emptyset$. Note that for all $(\gamma, I) \in \widetilde{\Theta}_{K_0,L}(x,\varepsilon,T)$, by Lemma 7.4, we have $\mathrm{Leb}(I) \le 2\Delta_K$.

In the proof it will be important to focus on primitive orbits. Recall that $\mathcal{P}'_{K_0}(L, L+5\tau_{K_0})$ is the subset of primitive orbits and observe that if γ is not primitive, then there exists a primitive periodic orbit with length at most $\ell(\gamma)/2$ with the same image. Therefore,

$$\#\left(\mathcal{P}_{K0}(L, L+5\tau_{K_0}) \setminus \mathcal{P}'_{K_0}(L, L+5\tau_{K_0})\right) \le \#\mathcal{P}_{K_0}\left(\frac{L+5\tau_{K_0}}{2}\right).$$

We then have

$$\begin{aligned}
m_{K_0,L}(B(x,\varepsilon,T) =& \frac{1}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \sum_{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \mu_\gamma(B(x,\varepsilon,T)) \\
\le& \frac{1}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \sum_{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}), \gamma \text{ primitive}} \mu_\gamma(C(x,\varepsilon,T)) \\
&+ \frac{1}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \sum_{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}), \gamma \text{ non-primitive}} \mu_\gamma(B(x,\varepsilon,T)) \\
\le& \frac{1}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} \sum_{(\gamma,I) \in \widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0})} \frac{2\Delta_K}{L} \\
&+ \frac{\#\{\gamma \in \mathcal{P}_{K_0}(L, L+5\tau_{K_0}), \gamma \text{ non-primitive}\}}{\#P_{K_0}(L, L+5\tau_{K_0})} \\
\le& \frac{2\Delta_K \times \#\widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0})}{L\,\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})} + \frac{\#\mathcal{P}_{K_0}\left(\frac{L+5\tau_{K_0}}{2}\right)}{\#\mathcal{P}_{K_0}(L, L+5\tau_{K_0})}
\end{aligned}$$

Step 3. From $\widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0})$ to chords. Every pair $(\gamma, I) \in \widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0})$ gives us a set $\{\gamma(s), s \in I\}$ of points of γ inside $C(x,\varepsilon,T)$, with at least some $s_0 \in I$ with $\gamma(s_0) \in B(x,\varepsilon,T)$. Following γ from $\gamma(s_0+T)$ to $\gamma(s_0+\ell(\gamma)) = \gamma(s_0)$ defines a chord with length $\ell(\gamma) - T \in [L-T, L+5\tau_{K_0} - T)$ from $B(\varphi_T(x,\varepsilon))$ to $B(x,\varepsilon)$.

Let E be a $E(\varphi_T(x), x, \varepsilon, L-T, L-T+5\tau_{K_0}, \delta)$-set with maximal cardinality. In particular, $\#E = \mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, \varepsilon, L-T, L-T+5\tau_{K_0}, \delta)$.

For each pair $(\gamma, I) \in \widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0})$, choose a point $z \in E$, such that $\gamma(s_0+T) \in B(z, \delta, L-T)$. This gives us a map $\theta : \widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0}) \to E$.

Step 4. Control the (lack of) injectivity of θ. Let (γ_1, I_1) and (γ_2, I_2) be two pairs that lead to the same point $z \in E$. In particular, there exist $s_1 \in I_1, s_2 \in I_2$ such that $\gamma_1(s_1), \gamma_2(s_2) \in B(x,\varepsilon,T)$, so, for every $0 \le s \le T$,

$$d(\gamma_1(s_1+s), \varphi_s(x)) \le \varepsilon \quad \text{and} \quad d(\gamma_2(s_2+s), \varphi_s(x)) \le \varepsilon\,.$$

Moreover, as $\theta(\gamma_1, I_1) = \theta(\gamma_2, I_2) = z$, for every $0 \le s \le L - T$, we have

$$d(\gamma_1(s_1+T+s), \gamma_2(s_2+T+s)) \le d(\gamma_1(s_1+T+s), \varphi_s(z)) + d(\varphi_s(z), \gamma_2(s_2+s+T)) \le 2\delta.$$

Therefore, for all $0 \le s \le L$, we have

$$d(\gamma_1(s_1+s), \gamma_2(s_2+s)) \le \varepsilon_1.$$

If $0 \le \ell(\gamma_1) - \ell(\gamma_2) \le \tau_0$, using Lemma 2.13 (see Step 1) and the fact that γ_1 and γ_2 are primitive, we deduce that $\gamma_1 = \gamma_2$ and there exists $u \in [-\nu, \nu]$ such that and $s_2 = s_1 + u$. By Lemma 7.4, we deduce that $I_1 = I_2$ (otherwise, we would have $|s_1 - s_2| = |u| \ge \Delta_k$, contradicting $|u| \le \nu = \min\left(\frac{\Delta_k}{3}, 1\right)$).

Cutting the interval $[L, L+5\tau_{K_0}]$ into intervals of length τ_0, we deduce that the number of elements $(\gamma, I) \in \widetilde{\Theta}_{K_0}(L, L+5\tau_{K_0})$ leading to the same chord is bounded from above by $5\tau_{K_0}/\tau_0$.

The result of the lemma follows with $D_{K_0,K} = 10\Delta_K \tau_{K_0}/\tau_0$. □

7.2 Estimation of the measure of dynamical balls

In this section, we gather all the inequalities proven in Lemma 7.3, Lemma 7.5, Proposition 6.4 and Proposition 6.5 to obtain the following strong inequalities.

Let K_0 be a compact set as in Section 6.1 Let m_∞ be any accumulation point of the family $(m_{K_0,L})_L$ and $m_{\max}$ be the probability measure obtained after renormalizing m_∞. In particular, $m_\infty(K_0) > 0$.

Choose $K \supset \overset{\circ}{K} \supset K_0$ such that $m_\infty(\overset{\circ}{K}) > \frac{3}{4} m_\infty(M)$.

Choose an increasing sequence $(L_k)_k$ such that $L_k \to +\infty$ as $k \to +\infty$ and $m_{K_0,L_k} \overset{*}{\rightharpoonup} m_\infty$. Fix some point $y_0 \in K_0$.

Theorem 7.6. *There exist $\delta_{\text{end}} > 0$, such that for every $0 < \delta < \delta_{\text{end}}$, there exists $\varepsilon_{\text{end},\delta} > 0$ such that for every $0 < \varepsilon < \varepsilon_{\text{end},\delta}$ there exists $\eta_{\text{end},\delta,\varepsilon}$ such that for every $0 < \eta < \eta_{\text{end},\delta,\varepsilon}$, there exist positive constants $S^-, S^+, D^-, D^+ > 0$, and $T_{\text{end}} > 0$ such that the following holds. For every $x \in K$ and $T > T_{\text{end}}$ such that $\varphi_T(x) \in K$, we have*

$$\frac{D^-}{\mathcal{N}_{\mathcal{C}}(x, y_0, \eta, T+S^-, T+S^-+5\tau_{K_0}, \delta)} \le m_\infty(B(x, \varepsilon, T)) \le \frac{D^+}{\mathcal{N}_{\mathcal{C}}(x, y_0, \varepsilon, T-S^+, T-S^++5\tau_{K_0}, \delta)}\,. \tag{30}$$

Before proving it, let us emphasize the strength of this statement. Usually, in ergodic theory, invariant measures satisfy almost sure properties. The above inequalities hold for **every** $x \in K$, and are therefore more geometric than ergodic. This strong uniform property will allow us to conclude that $m_{\max}$ is a measure of maximal entropy.

Proof. The proof follows easily from the preceding work, and in particular from Propositions 6.4 and 6.5, and from Lemmas 7.3 and 7.5, as soon as parameters are carefully chosen. We start with this choice.

Step 1. Choice of constants. Set $\delta_{\text{end}} = \min(\frac{\varepsilon_1}{4}, \delta_{K_0,K}, 1)$ where ε_1 is given by Proposition 6.5 and $\delta_{K_0,K}$ by Lemma 7.5. Choose an arbitrary $0 < \delta < \delta_{\text{end}}$.

Set $\varepsilon_{\text{end},\delta} = \min(\varepsilon_{K_0,K}, \varepsilon_{K,\delta})$, where $\varepsilon_{K_0,K}$ is given by Lemma 7.5 and $\varepsilon_{K,\delta}$ by Lemma 7.3 applied with parameter δ. Fix $0 < \varepsilon < \varepsilon_{\text{end},\delta}$.

Lemma 7.3 applied with $\varepsilon/2$ gives constants $\sigma_{K,\delta,\varepsilon/2}$ and $\eta_{K,\delta,\varepsilon/2}$. Set $\eta_{\text{end},\delta,\varepsilon} = \min(\delta, \eta_{K,\delta,\varepsilon/2}, 1)$. Choose $0 < \eta < \eta_{\text{end},\delta,\varepsilon}$.

Propositions 6.4 (with parameters ε and δ) and 6.5 (with parameters η and δ) give constants $S_1, D_1, S_2, D_2 > 0$. Set $T_{\text{end}} = \max(T_{\min} + S_1, 5\tau_{K_0}, T'_{\min})$, where $T_{\min}$ is given by Proposition 6.4 and $T'_{\min}$ by Lemma 7.3. Set $S^+ = S_1$ and $S^- = 2S_2 + \sigma_{K,\delta,\varepsilon/2}$, $D^+ = D_{K_0,K} D_1$ and $D^- = \frac{\varepsilon}{8bD_2}$.

Let $T \ge T_{\text{end}}$. Let $x, y_0 \in K$.

Step 2. Upper bound. By Lemma 7.5 and Proposition 6.4 applied with $T_0 = L_k - T$, $T_1 = T - S_1$ and $S = S_1$, as soon as $L_k > T + T_{\min}$, we have

$$\begin{aligned} m_{K_0,L_k}(B(x,\varepsilon,T)) \leq & \quad D_{K_0,K} \frac{\mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, \varepsilon, L_k - T, L_k - T + 5\tau_{K_0}, \delta)}{L_k \# \mathcal{P}_{K_0}(L_k, L_k + 5\tau_{K_0})} + \frac{\#\mathcal{P}_{K_0}(\frac{L_k+5\tau_{K_0}}{2})}{\#\mathcal{P}_{K_0}(L_k, L_k+5\tau_{K_0})} \\ \leq & \quad \frac{D_{K_0,K}\, D_1}{\mathcal{N}_{\mathcal{C}}(x, y_0, T - S_1, T - S_1 + 5\tau_{K_0})} + \frac{\#\mathcal{P}_{K_0}(\frac{L_k+5\tau_{K_0}}{2})}{\#\mathcal{P}_{K_0}(L_k, L_k+5\tau_{K_0})} . \end{aligned}$$

Using that $B(x,\varepsilon,T)$ is open and from Corollary 3.11, we get

$$m_\infty(B(x,\varepsilon,T)) \leq \liminf_{k\to+\infty} m_{K_0,L_k}(B(x,\varepsilon,T)) \leq \frac{D_{K_0,K}\, D_1}{\mathcal{N}_{\mathcal{C}}(x, y_0, T - S_1, T - S_1 + 5\tau_{K_0})} .$$

The upper bound follows with $D^+ = D_{K_0,K} D_1$.

Step 3. Lower bound. Apply Lemma 7.3 with parameter $\varepsilon/2$ and Proposition 6.5 with $S = S_2$ and $T = T + S_2 + \sigma_{K,\delta,\varepsilon/2} > 5\tau_{K_0}$, and $x_1 = \varphi_T(x), y_1 = x_0 = x$. Then, we get, for k big enough,

$$\begin{aligned} m_{K_0,L_k}(B(x,\varepsilon/2,T)) & \geq \frac{\varepsilon}{8b} \times \frac{\mathcal{N}_{\mathcal{C}}(\varphi_T(x), x, \eta, L_k - T - \sigma_{K,\delta,\varepsilon/2}, L_k - T - \sigma_{K,\delta,\varepsilon/2} + 5\tau_{K_0}, \delta)}{L_k \#\mathcal{P}_{K_0}(L_k, L_k+5\tau_{K_0})} \\ & \geq \frac{\varepsilon}{8b} \times \frac{1}{D_2} \times \frac{1}{\mathcal{N}_{\mathcal{C}}(x, y_0, \eta, T + 2S_2 + \sigma_{K,\delta,\varepsilon/2}, T + 2S_2 + \sigma_{K,\delta,\varepsilon/2} + 5\tau_{K_0}, \delta)} . \end{aligned}$$

As $\limsup_{L_k\to\infty} m_{K_0,L_k}(B(x,\varepsilon/2,T)) \leq m_\infty(\overline{B(x,\varepsilon/2,T)}) \leq m_\infty(B(x,\varepsilon,T))$, the lower bound follows with $D^- = \frac{\varepsilon}{8bD_2}$ and $S^- = 2S_2 + \sigma_{K,\delta,\varepsilon/2}$. □

7.3 Computation of entropies of $m_{\max}$

Theorem 7.7. *Let φ be a H-flow on M such that $h^\infty_{\mathrm{Gur}}(\varphi) < h_{\mathrm{Gur}}(\varphi)$. Let $m_{\max}$ be the probability measure obtained after renormalizing an accumulation point of the family of measures distributed on periodic orbits of increasing length, see Section 6.1. Then*

$$\underline{h}_{\mathrm{BK}}(m_{\max}) = \overline{h}_{\mathrm{BK}}(m_{\max}) = h_{\mathrm{KS}}(m_{\max}) = h_{\mathrm{Kat}}(m_{\max}) = h_{\mathrm{Gur}}(\varphi) .$$

Proof. As at the beginning of Section 7.2, choose K_0, and $L_n \to \infty$ such that $m_{K_0,L_n} \overset{*}{\rightharpoonup} m_\infty$, and $K \supset \overset{\circ}{K} \supset K_0$ such that $m_\infty(\overset{\circ}{K}) > \frac{3}{4} m_\infty(M)$. Fix $\delta < \min(\alpha_0/2, \delta_{\mathrm{end}})$ where δ_{end} comes from Theorem 7.6 and α_0 from Proposition 5.12. Fix $\varepsilon < \min(\varepsilon_{\mathrm{end},\delta}, \alpha_0/4)$, where $\varepsilon_{\mathrm{end},\delta}$ comes from Theorem 7.6 wih parameter δ, such that ε is small enough compatibly with Proposition 4.1,. Let $x \in K$.

Theorems 7.6 and Proposition 5.12 give us

$$\limsup_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T} \log m_\infty\left(B(x,T,\varepsilon)\right) = \liminf_{\substack{T\to\infty \\ \varphi_T(x)\in K}} -\frac{1}{T} \log m_\infty\left(B(x,T,\varepsilon)\right) = h_{\mathrm{Gur}}(\varphi).$$

Therefore

$$\underline{h}_{BK}(m_{\max}) = \overline{h}_{BK}(m_{\max}) = h_{\mathrm{Gur}}(\varphi).$$

From Proposition 4.1 we also have

$$\int_K \limsup_{\substack{T\to+\infty \\ \varphi_T(x)\in K}} -\frac{1}{T} \log(m_\infty(B(x,\varepsilon,T))\, dm_{\max} \leq h_{\mathrm{KS}}(m_{\max}) .$$

Thus we obtain the lower bound $h_{\mathrm{KS}}(m_{\max}) \geq h_{\mathrm{Gur}}(\varphi)$.

Corollary 4.4 gives the inequality $h_{\mathrm{KS}}(m_{\max}) \leq h_{\mathrm{Gur}}(\varphi)$ so that $h_{\mathrm{KS}}(m_{\max}) = h_{\mathrm{Gur}}(\varphi)$. We know by Theorem 4.3 that $h_{\mathrm{Kat}}(m_{\max}) \leq h_{\mathrm{Gur}}(\varphi)$.

It only remains to prove the inequality $h_{\mathrm{Kat}}(m_{\max}) \geq h_{\mathrm{Gur}}(\varphi)$. By definition of Katok entropy, for every $0 < \nu < 1$, there exist $\alpha_\nu > 0$, $\varepsilon_\nu > 0$ and $T_\nu > 0$ such that, for every $0 < \alpha < \alpha_\nu$, $0 < \varepsilon < \varepsilon_\nu$

and $T \geq T_\nu$, the minimal cardinality $M(T, \varepsilon, \alpha, m_{\max})$ of a set of (ε, T)-dynamical balls covering a set of measure at least α satisfies

$$M(T, \varepsilon, \alpha, m_{max}) \leq e^{(h_{\mathrm{Kat}}(m_{\max})+\nu)T} .$$

Even if it means to increase K, we can assume that $m_{\max}(K) \geq 1 - \frac{\alpha}{4}$. By invariance of $m_{\max}$, we get $m_{\max}(K \cap \varphi_{-T}(K)) \geq 1 - \frac{\alpha}{2}$. Without loss of generality, we can assume that T is large enough and $\varepsilon > 0$ small enough so that Theorem 7.6 holds with parameters K, δ (where δ is some small enough constant), 2ε and T. We may also assume that Proposition 5.12 holds with parameters K, $C = 5\tau_{K_0}$, δ and $\eta = 2\varepsilon$.

Since $m_{\max}(K \cap \varphi_{-T}(K)) \geq 1 - \frac{\alpha}{2}$, we can use Lemma 4.9 to obtain a separating $(T, 2\varepsilon, \frac{\alpha}{2}, m_{\max})$-spanning set $E' \subset K \cap \varphi_{-T}(K)$ with $\#E' \leq M(T, \varepsilon, \alpha, m_{\max})$. Therefore

$$\begin{aligned}\frac{\alpha}{2} \leq m_{\max}\left(\cup_{x\in E'} B(x, 2\varepsilon, T)\right) &\leq \sum_{x\in E'} m_{\max}(B(x, 2\varepsilon, T)) \\ &\leq M(T, \varepsilon, \alpha, m_{\max}) \times \max_{x\in E'} m_{\max}(B(x, 2\varepsilon, T)) \\ &\leq e^{(h_{\mathrm{Kat}}(m_{\max})+\nu)T} \times \max_{x\in E'} m_{\max}(B(x, 2\varepsilon, T)) .\end{aligned}$$

Using Theorem 7.6 we get

$$m_{\max}(B(x, 2\varepsilon, T)) = \frac{1}{m_\infty(M)} m_\infty(B(x, 2\varepsilon, T)) \leq \frac{1}{m_\infty(M)} \frac{D^+}{\mathcal{N}_{\mathcal{C}}(x, y_0, 2\varepsilon, T - S^+, T - S^+ + 5\tau_{K_0}, \delta)} ,$$

so that

$$\frac{\alpha}{2D^+} \times m_\infty(M) \times \mathcal{N}_{\mathcal{C}}(x, y_0, 2\varepsilon, T - S^+, T - S^+ + 5\tau_{K_0}, \delta) \leq e^{(h_{\mathrm{Kat}}(m_{\max})+\nu)T} .$$

Thus,

$$\frac{1}{T} \log\left(\frac{\alpha}{2D^+} \times m_\infty(M)\right) + \frac{1}{T} \log(\mathcal{N}_{\mathcal{C}}(x, y_0, 2\varepsilon, T - S^+, T - S^+ + 5\tau_{K_0}, \delta)) \leq h_{\mathrm{Kat}}(m_{\max}) + \nu .$$

Taking the limit superior when $T \to +\infty$ of the above inequality, by Proposition 5.12, we obtain $h_{\mathrm{Gur}}(\varphi) \leq h_{\mathrm{Kat}}(m_{max}) + \nu$. As ν can be chose arbitrarily small, we have $h_{\mathrm{Gur}}(\varphi) \leq h_{\mathrm{Kat}}(m_{max})$. This concludes the proof of the theorem. □

8 Notations

- $B(x, \varepsilon, T)$ - p. 15
- $\mathcal{C}(x, y, \eta, T^-, T^+)$ - p. 31
- $\mathcal{C}^{K^c}(x, y, \eta, T^-, T^+)$ - p. 39
- $E(x, y, \eta, T^-, T^+, \delta)$ - p. 31
- $E^{K^c}(x, y, \eta, T^-, T^+, \delta)$-set - p. 39
- $E_K(x, y, \eta, T^-, T^+, \delta)$-set - p. 37
- $h_{\mathcal{C}}(x, y, \eta, \delta)$ - p. 37
- $h_{\mathcal{C}}(\varphi)$ - p. 39
- $h_{\mathcal{C}}^{K^c}(x, y, \eta, \delta)$ - p. 40
- $h_{\mathcal{C}}^{K^c}(\eta, \delta)$ - p. 40
- $h_{\mathcal{C}}^{K^c}(\varphi)$ - p. 40
- $h_{\mathcal{C}}^{\infty}(\varphi)$ - p. 41
- $\underline{h}_{\mathrm{BK}}(\mu)$ - p. 16
- $\overline{h}_{\mathrm{BK}}(\mu)$ - p. 16
- $h_{\mathrm{Gur}}(\varphi)$ - p. 17
- $h_{\mathrm{Kat}}(\mu)$ - p. 16
- $h_{\mathrm{KS}}(\mu)$ - p. 15
- $h_{\mathrm{var}}(\varphi)$ - p. 15
- $h_{\mathrm{Gur}}^{K,\alpha}$ - p. 22
- $h_{\mathrm{Gur}}^{\infty}$ - p. 22
- $h_{\mathrm{var}}^{\infty}$ - p. 23
- $K_{-\eta}$ - p. 39
- ℓ - p. 7

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