Counterexamples in non-positive curvature

Yves Coudene, Barbara Schapira

09/04/2010

Université de Bretagne Occidentale, 6 av. Le Gorgeu, 29238 Brest cedex, France
LAMFA, Université Picardie Jules Verne, 33 rue St Leu, 80000 Amiens, France

Abstract

We give examples of rank one compact surfaces on which there exist recurrent geodesics that cannot be shadowed by periodic geodesics. We build rank one compact surfaces such that ergodic measures on the unit tangent bundle of the surface are not dense in the set of probability measures invariant by the geodesic flow. Finally, we give examples of complete rank one surfaces for which the non wandering set of the geodesic flow is connected, the periodic orbits are dense in that set, yet the geodesic flow is not transitive in restriction to its non wandering set. [1]

1 Introduction

Geodesic flows on negatively curved manifolds have been extensively studied both from the topological and the measurable point of view. Let M be a complete connected pinched negatively curved riemannian manifold, and denote by $g_t : T^1M \to T^1M$ the geodesic flow defined on the unit tangent bundle T^1M of M. The non-wandering set of g_t is denoted by Ω.

$$\Omega = \{\, v \in T^1M \mid \forall V \text{ neighborhood of } v,\ \exists\, t_n \to +\infty \text{ such that } g_{t_n}(V) \cap V \neq \emptyset \,\}$$

Recall from the Poincaré recurrence theorem that $\Omega = T^1M$ if M is compact or with finite volume.

Here are some basic properties of g_t.

• *Every vector in Ω can be shadowed by a periodic geodesic. As a consequence, periodic geodesics are dense in the non wandering set of the flow.*

• *Assuming $\Omega = T^1M$, there are ergodic measures with full support that are invariant by g_t. In fact ergodicity and full support are generic properties in the set of all invariant probability measures.*

• *The flow g_t is transitive in restriction to Ω as soon as Ω is connected.*

There has been several attempts to generalize these results to the case of geodesic flows defined on non-positively curved manifolds. The goal of this paper is to provide explicit counterexamples to each of the results stated above, in the context of rank one manifolds.

Let us recall what is a rank one manifold. A vector $v \in T^1M$ is a *rank one vector* if the only parallel Jacobi fields along the geodesic generated by v are proportional to the generator of the geodesic flow. A connected complete non-positively curved

[1] 37B10, 37D40, 34C28

manifold is said to be a *rank one manifold* if its tangent bundle admits a rank one vector. In that case, the set of rank one vectors is an open subset of T^1M.

A vector v on a non-positively curved surface has rank one if and only if there is a point of negative sectional curvature on the geodesic generated by v. Hence, a rank one surface is simply a non-positively curved surface with at least a point where the curvature is negative.

We shall build rank one surfaces M such that

• *the surface M is compact, and there exist recurrent rank one geodesics that cannot be shadowed by periodic geodesics;*

• *the surface M is compact, and ergodic measures are not dense in the set of all probability measures invariant by g_t;*

• *the set Ω is connected, the periodic orbits are dense in Ω, and the flow g_t is not transitive in restriction to Ω.*

All these counterexamples contain embedded flat cylinders. It is plausible that in dimension two, such cylinders are the only obstruction to the results stated above. In higher dimension however, the question is wide open.

The article is organized in three sections, each devoted to one of the counterexamples stated above.

2 The closing lemma

The first instance of a closing lemma appears in the work of J. Hadamard [Had98] on negatively curved surfaces embedded in $\mathbf{R}^3$. J. Hadamard showed that a piece of geodesic coming back close to its starting point is in fact shadowed by a periodic closed geodesic. As a consequence, the set of periodic closed geodesics is dense in the closure of the set of recurrent geodesics.

If the negatively curved surface is of finite volume, then the celebrated theorem of Poincaré ensures that the recurrent geodesics are in fact dense in the manifold. So periodic vectors are dense in the unit tangent bundle of the surface. Remarkably, the result of J. Hadamard is contemporary to the Poincaré recurrence theorem, and there were no examples of finite volume negatively curved surfaces embedded in $\mathbf{R}^3$ at that time. First examples of such surfaces are attributed to Vaigant in [GeomIII], although we were unable to locate the original reference. Yet J. Hadamard provided examples of negatively curved embedded surfaces in $\mathbf{R}^3$, and the closing lemma was one of the key ingredient in the study of recurrent geodesics on these surfaces.

Purely dynamical proofs of the closing lemma were given by D.V. Anosov, who extended that result to the class of systems that now bear his name. The closing lemma has become a classical tool in the study of systems exhibiting some form of hyperbolic behavior, and the name of Anosov is frequently associated with the theorem. In the context of negatively curved manifolds, the Anosov closing lemma reads as follows.

Theorem 1 (Anosov closing lemma [An67])
Let M be a compact negatively curved manifold, g_t the geodesic flow on T^1M. Then for all $\varepsilon > 0$, there exists $\delta > 0$ and $T_0 > 0$, such that for all $w \in T^1M$, we have:

for all $T \geq T_0$ such that $d\big(g_T(w), (w)\big) < \delta$, there exists a periodic vector w_0 of period l such that

$$|T - l| < \varepsilon$$
$$\forall t \in [0, T], \quad d\big(g_t(w), g_t(w_0)\big) < \varepsilon.$$

The closing lemma implies the density of the periodic orbits in the non-wandering set of the flow. The term "Anosov closing lemma" is sometimes used to describe that last property [Rob99]. We will see below examples of (non-uniform) hyperbolic systems for which the closing lemma is not satisfied, yet the periodic orbits are dense in the ambient space.

Note also that the compactness hypothesis on M allows for uniform δ, ε. We will need some local version of the closing lemma.

Definition
We say that the closing lemma is satisfied around a vector $v \in T^1M$ *if there is a neighborhood V of v such that:*

$\forall\,\varepsilon > 0,\ \exists\,\delta > 0,\ \exists\,T_0 > 0,\ \forall\,w \in V,\ \forall\,T \geq T_0,$
the conditions $g_T(w) \in V$ and $d\big(g_T(w), w\big) < \delta$ imply that there is a periodic vector w_0 of period l satisfying $|T - l| < \varepsilon$ and $d\big(g_t(w), g_t(w_0)\big) < \varepsilon$ for all $t \in [0, T]$.

We now turn to non-positive curvature. A version of the closing lemma is given by W. Ballmann, M. Brin, R. Spatzier in [BBS85], th. 4.5. The authors are mainly interested in higher rank manifolds, although their arguments hold in the rank one setting. They show that the closing lemma is satisfied around every vectors of minimal rank, and the proof carries to the rank one setting. As an example, the reader may want to convince himself by an elementary argument that the closing lemma indeed holds on a flat torus. In that example, all vectors have a rank equal to the dimension of the torus.

Let us now restrict the discussion to rank one manifolds. The following result is stated in [Eb96], th. 4.5.15. We refer also to [CS10] for a shorter proof on rank one manifolds.

Theorem 2 (Rank one closing lemma [Eb96])
Let $\tilde{M}$ be a rank one manifold. Then the closing lemma is satisfied around all rank one vectors.

This result implies that on a compact rank one manifold, periodic vectors are dense in the set of rank one vectors. That set is both open and dense, hence we see that periodic vectors are in fact dense in T^1M. This can be extended to rank one manifolds for which $\Omega = T^1M$.

Establishing the density of periodic orbits is the first step in the proof of the transitivity of the geodesic flow on compact connected rank one manifolds. We will use that property in the sequel, so let us state the following result due to P. Eberlein [Eb72].

Theorem 3 [Eb72]
Let M be a compact rank one manifold. Then the geodesic flow on T^1M is transitive: there exists a vector in T^1M with a dense orbit.

Without the compactness assumption on M, transitivity holds as soon as $\Omega = T^1M$ [Eb96] (1.9.15, 4.7.4, 4.7.3).

Let us now turn our attention to compact non-positively curved surfaces. We have seen that the closing lemma is satisfied on flat toruses. Also, on all other surfaces, there is an open dense set of vectors around which it is satisfied. Yet there are examples of surfaces on which there exists recurrent rank one trajectories that cannot be closed. In fact, that happens as soon as there is an embedded flat cylinder in the surface.

We say that a surface M contains an *embedded flat cylinder* if there is a compact subset of M which is isometric to the euclidean cylinder $[0, l] \times \mathbf{R}/(2\pi h\mathbf{Z})$, for some $l, h > 0$. Geodesics in the cylinder are straight lines. The periodic geodesics contained in the cylinder are parallel among themselves, and define a direction in the tangent bundle of the cylinder, that will be deemed as "vertical". This direction will be depicted vertically in the figures.

Note that all negatively curved surfaces carrying a simple closed geodesic can be flattened around that geodesic so as to obtain a surface with an embedded cylinder.

We can now state the counterexample to the closing lemma.

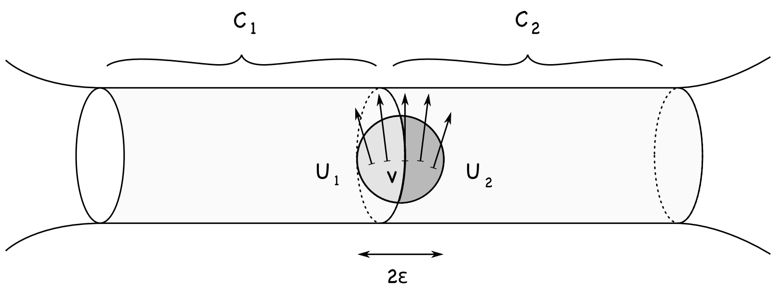
Theorem 4
Let M be a complete surface with an embedded flat cylinder. We assume that the geodesic flow is transitive on T^1M. Then the closing lemma is not satisfied around vectors generating closed geodesics contained inside the cylinder.

No hypothesis on the curvature of M is needed. The transitivity assumption rules out the case of a flat torus, where the closing lemma holds everywhere. Transitivity is satisfied in the rank one setting, so we get:

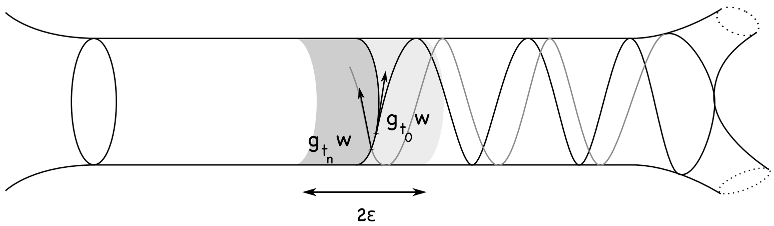
Corollary
The Anosov closing lemma does not hold on compact rank one surfaces admitting an embedded cylinder.

Proof
Let v be a periodic vector whose trajectory is contained in the interior of the cylinder. The trajectory of v bounds two connected components of the cylinder, denoted by C_1 and C_2.

Given $\varepsilon > 0$ and $\theta \in]0, \pi/2[$, we consider the open set U_1 consisting in vectors whose base points are in C_1, at distance less than ε from the base point of v, and whose angle with the vertical direction belongs to $] - \theta, 0[$. We also consider the open set U_2 of vectors with base points in C_2 close to v, and with angles belonging to $]0, \theta[$.

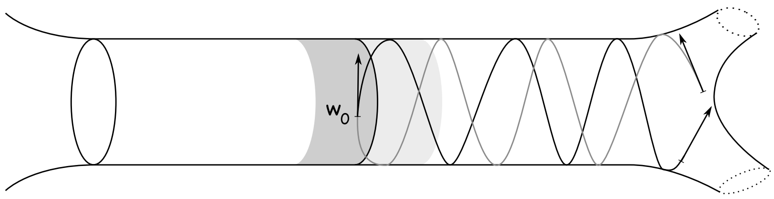


Recall that the geodesic flow is transitive. There is a vector $w \in T^1M$ with a dense orbit. The orbit of w crosses the closed geodesic generated by v infinitely many times. The geodesic generated by w enters U_2 at time t_0 and then enters U_1 infinitely often. This means that we can find times $t_n > 0$ arbitrarily large, so that $d(g_{t_0}w, v)$ is less than $\theta + \varepsilon$ and $d(g_{t_n}w, g_{t_0}w)$ is less than $2\theta + 2\varepsilon$. See next figure.



Orbits in the cylinder are straight lines. Hence an orbit of the geodesic flow entering the cylinder by one side must leave the cylinder by the other side.

If the closing lemma is true in the neighborhood of v, we can find a periodic orbit w_0 with period close to t_n, that starts close to $g_{t_0}(w)$ and follows the orbit of $g_{t_0}(w)$ until time t_n. The orbit $g_t(w_0)$ must leave the cylinder by the right side for $t > 0$, since it follows the orbit of $g_{t_0}(w)$ for positive t. It must also leave the cylinder by the right side for $t < 0$, since it follows the orbit of $g_{t_n}(w)$ for negative t. This is in contradiction with a closing lemma around v.



This ends the proof of the result.

The same argument would show that there is no local product structure in the neighbourhood of v: it is impossible to find an orbit close to v and $g^{t_0}w$, negatively asymptotic to the orbit of $g^{t_0}w$ and positively to the orbit of $g^{t_n}w$ (see [CS10] for definitions).

Finally, we remark that the closing lemma holds around $g_{t_0}w$ (see [CS10]). The orbit of $g_{t_0}w$ itself must come back sufficiently close to $g_{t_0}w$ however, so as to be directed toward the same boundary of the cylinder as w, for the closure to happen.

3 Genericity

The closing lemma is a key ingredient in the understanding of the generic properties of measures on T^1M invariant by the geodesic flow.

Let X be a complete metric space. Recall that a property is *generic* if the set of points satisfying that property contains a countable intersection of open dense sets. A countable intersection of generic sets is again generic. From the Baire category theorem, a generic set must be dense.

The first use of the Baire category theorem in the study of geodesics is again due to J. Hadamard [Had98]. This allowed him to show that on a negatively curved surface embedded in $\mathbf{R}^3$ with many periodic geodesics, there are geodesics that accumulate on infinitely many periodic geodesics.

The following theorem of K. Sigmund, proven in the context of transitive Anosov flows, can be seen as some quantitative strengthening of J. Hadamard result.

Theorem 5 [Si72]
Let M be a connected compact negatively curved manifold. Then the Dirac masses on periodic orbits are dense in the set of all probability measures on T^1M invariant by the geodesic flow. Moreover, ergodicity and full support are generic properties in the set of invariant probability measures.

Note that there are indeed many explicit examples of ergodic probability measures of full support, when the negatively curved manifold is compact. The volume, the measure of maximal entropy, more generally the equilibrium state associated to a Hölder potential, are all Bernoulli measures of full support.

We extended K. Sigmund's result to the rank one setting in [CS10], freeing ourselves from any compactness assumption. We denote by $\mathcal{R}_1$ the open set of rank one vectors in T^1M, and $\Omega \cap \mathcal{R}_1$ the set of non-wandering rank one vectors on T^1M. Our theorem can be stated as follows.

Theorem 6 [CS10]
Let M be a connected complete non-positively curved manifold. We assume that the geodesic flow on T^1M admits more than two rank one periodic orbits. Then the set of ergodic measures on $\Omega \cap \mathcal{R}_1$, with full support in $\Omega \cap \mathcal{R}_1$, is generic in the set of all probability measures defined on $\Omega \cap \mathcal{R}_1$.

Strictly speaking, we proved that theorem for a subset $\Omega_1 \subset \Omega \cap \mathcal{R}_1$ that contains the set of recurrent rank one vectors. Let us explain why it is also true for $\Omega \cap \mathcal{R}_1$. Note that the set of recurrent rank one vectors is dense in $\Omega \cap \mathcal{R}_1$, this follows from the rank one closing lemma. From the Poincaré recurrence theorem, an invariant probability measure must give full measure to the set of recurrent vectors. As a consequence, an invariant measure supported by $\mathcal{R}_1$ gives full measure to Ω_1. The set of invariant measures on $\Omega \cap \mathcal{R}_1$ and Ω_1 thus can be identified.

The assumption that there are more than two rank one periodic orbits rules out the case of a hyperbolic cylinder. On such a cylinder, the non-wandering set is made of two opposite periodic geodesics; it is not connected and the flow is not transitive on that set.

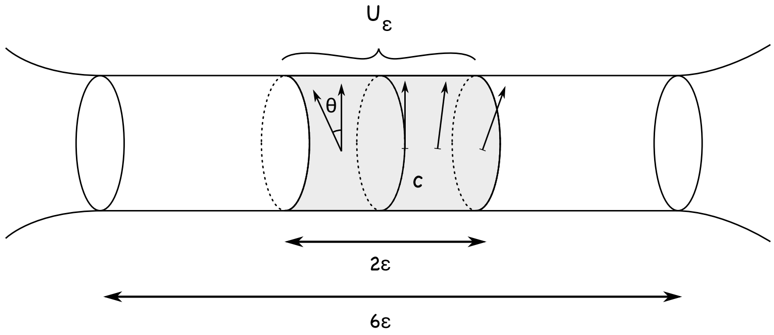
Corollary [CS10]
Under the assumptions of the previous theorem, there exists an ergodic probability measure invariant by the geodesic flow, whose support contains all rank one periodic orbits.

So if we assume that M is a connected complete rank one manifold satisfying $\Omega = T^1M$, then there is always an invariant probability measure of full support on T^1M. The existence of such measure is non trivial even in the case of a surface with constant negative curvature.

The question of whether ergodicity and full support are generic properties in the set of all invariant measures on T^1M was left open in [CS10]. One of the difficulties was the lack of a closing lemma valid for all vectors in T^1M. Now, refining on the counterexample of the previous section, we shall show that ergodicity is not a generic property in general.

Theorem 7
Let M be a compact riemannian surface with an embedded flat cylinder. Then the Dirac measures supported by the periodic geodesics in the interior of the cylinder are not in the closure of the invariant ergodic probability measures with full support.

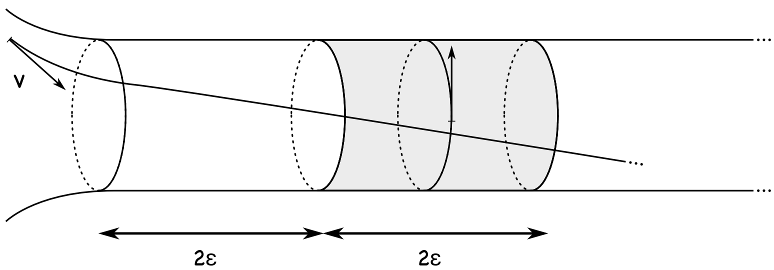
Proof
Let $c : \mathbf{R} \to M$ be a periodic geodesic lying inside the cylinder. Let us consider a tubular neighborhood around c which is isometric to $]-3\,\varepsilon, 3\,\varepsilon[\times S^1$ and let $\theta \in]0, \frac{\pi}{2}[$. We denote by $U_\varepsilon \subset T^1M$ the neighborhood of c containing all unit vectors tangent to $]-\varepsilon, \varepsilon\,[\times S^1$ whose angle with the vertical direction belongs to $]-\theta, \theta[$. The set U_ε is depicted below.



The orbits of the geodesic flow in the cylinder are straight lines. As a result, a trajectory that enters U_ε at time t_0 and leaves U_ε at time t_1, must have spent a time at least $t_1 - t_0$ in the left cylinder $]-3\varepsilon, -\varepsilon[\times S^1$ or in the right cylinder $]\varepsilon, 3\varepsilon[\times S^1$. Hence, the quantity

$$\frac{1}{T}\,\lambda\Big(\{\, t \in [0,T] \mid g_t(v) \in U_\varepsilon \,\}\Big)$$

is bounded from above by $\frac{1}{2}$ for all $T \geq 0$ and all $v \in T^1M$ that do not belong to the cylinder. See picture below.



Let μ be an ergodic measure with full support. If it is close enough to the Dirac measure supported by c, then $\mu(U_\varepsilon) > \frac{1}{2}$. Now we use the ergodic theorem. We can find some vector v outside the cylinder such that:

$$\frac{1}{T}\,\lambda\Big(\{\, t \in [0,T] \mid g_t(v) \in U_\varepsilon \,\}\Big) \xrightarrow[n\to\infty]{} \mu(U_\varepsilon)$$

This gives a contradiction and the theorem is proven.

Remarks
- Here again, there is no condition on the curvature of the surface.
- The compactness assumption on M plays no essential role in the proof.
- The same method can be applied in any dimension to build examples of rank one manifolds on which ergodicity is not a generic property.

Corollary
Let M be a compact rank one surface with an embedded flat cylinder. Then ergodic measures are not dense in the set of all invariant probability measures on T^1M. Hence, ergodicity is not a generic property in T^1M.

Proof
Recall that the set of ergodic probability measures is a G_δ set. Also, the set of measures with full support is a G_δ-dense set if the periodic orbits are dense in the ambiant space, we refer to [Si72] or [CS10] for a proof.
So, if the set of ergodic measures is dense, then the ergodic measures with full support are dense by the Baire category theorem, and the (ergodic !) Dirac measures in the cylinder can be approached by these measures, in contradiction with the previous theorem. This proves the corollary.

The reader may want an explicit example of a measure that is not in the closure of the ergodic measures. A non trivial convex combination of Dirac measures supported inside the cylinder does the job.

Let us give a quantitative version of the previous theorem. First, recall that the riemannian metric on M induces a natural riemannian metric on TM. This metric on TM is uniquely caracterized by the following properties:

- the canonical projection from TM to M is a riemannian submersion,
- the metric induced in the fibers is euclidean,
- horizontal and vertical distributions are orthogonal.

This metric induces in turn a metric on the submanifold T^1M. As an example, the metric obtained on the unit tangent bundle T^1C of a flat two dimensional cylinder C is just the euclidean metric on the product $T^1C \simeq C \times S^1$.

The Prohorov metric on the set of Borel probability measures is defined by

$$\rho(\mu, \nu) = \inf\{\, \varepsilon > 0 \mid \forall\, A \subset X,\ \mu(A) \leq \nu(V_\varepsilon(A)) + \varepsilon \,\}$$

where $V_\varepsilon(A)$ is the ε-neighborhood of A,

$$V_\varepsilon(A) = \{y \in X \mid d(x, A) < \varepsilon\}.$$

The metric ρ is bounded by one. We refer to [Bi99] for its basic properties.

Theorem 8
Let M be a compact riemannian surface with an embedded euclidean cylinder isometric to $[0, l] \times \mathbf{R}/(2\pi h\mathbf{Z})$, $l, h > 0$. Let μ be an ergodic probability measure of full support, invariant by the geodesic flow, δ the Dirac measure on a closed geodesic contained in the interior of the cylinder. The distance on M between that closed geodesic and the boundary of the cylinder is denoted by d. Then

$$\rho(\delta, \mu) \geq \min\Big(d,\ \frac{l}{l+2}\Big)$$

This gives a bound of order $1 - \frac{2}{l}$ when l is big.

The bound does not depend on the height h of the cylinder, nor on the geometry of M outside the cylinder, nor on the measure μ. Here again we do not need any assumption on the curvature of M.

For the geodesic in the middle of the cylinder, we have $d = l/2$ and the lower bound is equal to $\frac{l}{l+2}$.

Proof
We take for A the closed orbit in T^1M supporting δ. Consider a vector v in the cylinder, denote by θ the angle between v and the vertical direction, and r the distance between the base point of v and the closed geodesic. The vector v is in $V_\varepsilon(A)$ if and only if $\sqrt{\theta^2 + r^2}$ is less than ε.

We now ensure that $V_\varepsilon(A)$ is contained in the cylinder by assuming $\varepsilon < d$. This unfortunately rules out the case of a set A given by a closed geodesic on the boundary of the flat cylinder. Without further assumption on the geometry of M, there is no way to guarantee that $V_\varepsilon(A)$ does not contain arbitrarily long pieces of geodesics when A is a bounding geodesic.

Let us consider a vector with a dense trajectory, that enters the cylinder with an angle θ with respect to the vertical direction. It spends a time $l/\sin(\theta)$ in the cylinder. If the trajectory crosses $V_\varepsilon(A)$, that is if θ is less than ε, it spends a time $2\sqrt{\varepsilon^2 - \theta^2}/\sin(\theta)$ in $V_\varepsilon(A)$. So the fraction of time spent in $V_\varepsilon(A)$ during the travel through the cylinder is given by

$$\frac{2\sqrt{\varepsilon^2 - \theta^2}}{l}$$

This can be bounded independently of θ by $2\varepsilon/l$, and this bound is best when the vector enters the cylinder close to the vertical direction.

Let μ an ergodic probability measure with full support. By the ergodic theorem, we can find some vector v with dense trajectory such that

$$\frac{1}{T}\,\lambda\Big(\{\, t \in [0,T] \mid g_t(v) \in V_\varepsilon(A)\,\}\Big) \xrightarrow[n\to\infty]{} \mu(V_\varepsilon(A))$$

This gives $\mu(V_\varepsilon(A)) \leq 2\,\varepsilon/l$. Let us denote by δ the Dirac measure on the periodic orbit under consideration. Coming back to the definition of the Prohorov distance, the equation

$$1 = \delta(A) \leq \mu(V_\varepsilon(A)) + \varepsilon$$

implies

$$1 \leq \frac{2}{l}\,\varepsilon + \varepsilon$$

that is,

$$\frac{l}{l+2} \leq \varepsilon$$

This estimate is greater than, and asymptotic to, $1 - 2/l$ when l is big.

Recall that we assumed $\varepsilon \leq d$. Thus we get

$$\rho(\delta,\mu) \geq \min\Big(d, \frac{l}{l+2}\Big)$$

This ends the proof of the theorem.

Let us mention a curious consequence of the previous theorem. Let us assume that M is a compact rank one surface. This implies that there are invariant ergodic probability measures of full support. Given a closed geodesic in the cylinder, we may consider the set of ergodic measures of full support that are closest to the Dirac measure on the closed geodesic, with respect to the Prohorov distance. This set is a non-empty compact set, and its elements should be related to the closed geodesic in some way.

These results lead to the following problem:

Characterize the non-positively curved compact manifolds on which ergodic measures of full support are dense in the set of all invariant probability measures.

Recall that full support is a generic property in the set of all invariant probability measures, as soon as the periodic geodesics are dense in T^1M. Also the set of ergodic measures is always a G_δ set. The question really is about whether ergodicity is dense or not. The existence of embedded euclidean cylinder seems to be the only obstruction to genericity in dimension two. In higher dimension, the question appears to be pretty elusive.

4 Transitivity

We now consider the question of the transitivity of the geodesic flow in restriction to the non-wandering set Ω of the flow. We have mentioned that if Ω is equal to T^1M, then the flow is indeed transitive. When there are wandering vectors, we have the following result.

Theorem 9 [CS10]
Let M be a rank one manifold. We assume that there are at least three rank one periodic geodesics on T^1M. Then the restriction of the geodesic flow to the closure of the set of rank one periodic orbits is transitive.

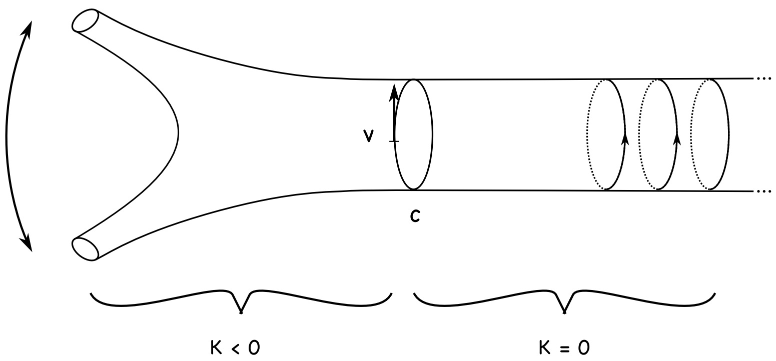
So the density of rank one periodic vectors in the non-wandering set is sufficient to get the transitivity of the flow. We will see below examples of surfaces where rank one periodic orbits are not dense in Ω.

Here again, the requirement that there are more than two periodic orbits rules out the case of an hyperbolic cylinder. On the unit tangent bundle of a hyperbolic cylinder, there are exactly two periodic geodesic orbits, corresponding to the same geometric geodesic on the surface, with its two opposite orientations. As a consequence, the set Ω is disconnected and transitivity does not hold on Ω

Is the connectedness of Ω is sufficient to guarantee the existence of a non wandering vector whose orbit is dense in Ω ? We shall answer that question by the negative.

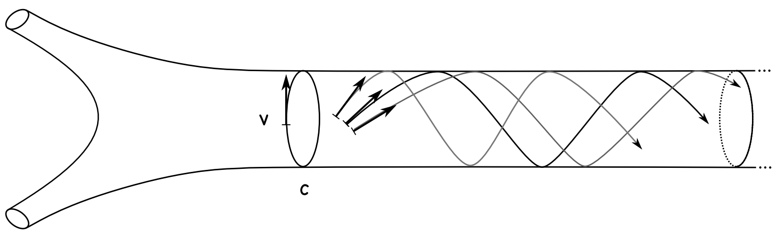
Let M be a connected complete, non-positively curved surface, admitting a single end isometric to a half flat cylinder $[0, \infty[\times(\mathbf{R}/\mathbf{Z})$. Let v be a periodic vector generating a geodesic whose projection c on M bounds the half cylinder. We assume that the curvature is negative outside the flat end. So the surface is separated in two parts by the curve c. One part is compact and contains (the projection on M of) all rank one periodic orbits of the geodesic flow, the other is flat and contains two continua of rank two periodic orbits.

An example is depicted below. We take a negatively curved pant, glue two ends together and flatten the last end so as to paste a flat cylinder. The negatively curved part is diffeomorphic to a once punctured torus bounded by c.



Theorem 10
The non wandering set $\Omega \subset T^1M$ is connected, periodic orbits are dense in Ω and the geodesic flow is not transitive in restriction to Ω.

Proof
The cylinder contains two sets of opposite rank two periodic orbits. We will call these orbits "vertical", in accordance to the figures. All other geodesics in the cylinder go to infinity either for positive or negative time. Any non periodic vector in the cylinder generates a geodesic that goes to infinity either for positive or negative time. As a consequence, the only non wandering orbits intersecting the flat cylinder are the vertical ones. See figure below.



This shows that rank one periodic orbits are contained in the negatively curved part of the surface. By the rank one closing lemma, they are dense in the set of non wandering rank one vectors. We see that the non-wandering set is composed of the closure of the rank one periodic orbits, together with the rank two periodic orbits contained in the cylinder.

Also, the flow is not transitive in restriction to Ω, since a non wandering vector entering the flat cylinder is periodic, hence cannot be dense.

It remains to show that Ω is connected. The flow is transitive in restriction to the closure of the rank one periodic orbits, that is, in restriction to the subset of Ω contained in the negatively curved part of the surface. This implies that this subset is connected. In particular, a periodic rank one vector and its opposite are in the same connected component of the non wandering set.

In order to prove the connectedness of Ω, it is sufficient to show that the vector v on the boundary of the flat half cylinder is in the closure of the rank one periodic vectors. Indeed, this will imply that $-v$ is also in that closure. Now, all rank two periodic orbits in the half cylinder can be connected by a path to v or $-v$. So the next lemma ends the proof.

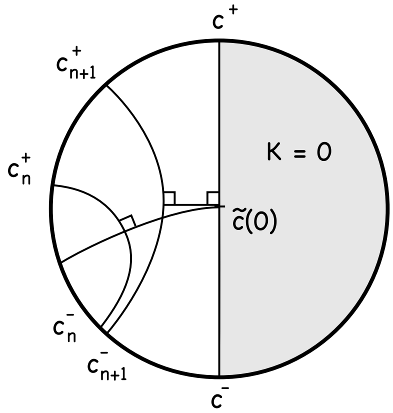
Lemma
The geodesic bounding the half cylinder is in the closure of the rank one periodic geodesics.

Proof of the lemma

The proof makes use of the action of the fundamental group Γ of M on the boundary of its universal cover $\tilde{M}$. Let us recall how the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$ is defined.

We consider the space of half geodesics $r : \mathbf{R}_+ \to \tilde{M}$. Two half geodesics r_1, r_2 are said to be asymptotic if they stay at a bounded distance from each other: there exists $C \geq 0$ such that $d(r_1(t), r_2(t)) < C$ for all $t \geq 0$. The boundary $\partial\tilde{M}$ is obtained by identifying asymptotic half geodesics. We now fix some origin $x_0 \in \tilde{M}$. The boundary $\partial\tilde{M}$ is homeomorphic to $T^1_{x_0}\tilde{M}$ via the map associating to each vector of $T^1_{x_0}\tilde{M}$ the half geodesic starting from that vector.

The limit set $\Lambda\Gamma$ is defined as the closure in $\partial\tilde{M}$ of the orbit of x_0 under Γ. It is contained in $\partial\tilde{M}$ and it does not depend on the choice of the origin x_0. G. Link, M. Peigné and J. C. Picaud showed that the end points of the lifts of the rank one periodic vectors are dense in $\Lambda\Gamma \times \Lambda\Gamma$ [LPP06]. As a side note, we remark that this does not imply that the rank one periodic orbits are dense in Ω.



Let $\tilde{c}$ be a lift of the periodic geodesic c that lies on the boundary of the negatively curved part. We denote by c^- and c^+ its two end points in $\Lambda\Gamma$. Let $\tilde{c}_n$ a sequence of geodesics in $\tilde{M}$ associated to rank one periodic geodesics on M, and whose end points c_n^- and c_n^+ tend to c^- and c^+. Passing to a subsequence, we can assume that the convergence is in fact monotonous in the neighborhood of c^- and c^+. Let us parameterize $\tilde{c}_n$ so that the distance from $\tilde{c}_n$ to $\tilde{c}(0)$ is realized at $\tilde{c}_n(0)$.

The geodesics $\tilde{c}_n$ and $\tilde{c}$ are separated by the geodesic $\tilde{c}_{n+1}$. This implies that the sequence $d(\tilde{c}_n(0), \tilde{c}(0))$ is decreasing. Let w be an accumulation point of the sequence $c'_n(0)$. The geodesic generated by w has c^- and c^+ as its end points, so it bounds a flat strip together with $\tilde{c}$. Moreover, it is in the closure of the (lifts of the) rank one periodic geodesics, so it must be equal to $\tilde{c}$. This proves the lemma.

References

[An67] D.V. Anosov. Geodesic flows on closed riemannian manifolds with negative curvature. *Proc. Steklov Inst. Math* **90**, (1967).

[BBS85] W. Ballmann; M. Brin; R. Spatzier Structure of manifolds of nonpositive curvature. II *Ann. of Math.* (2) **122** (1985), no. 1, 205-235.

[Bi99] P. Billingsley. Convergence of probability measures. 2nd ed. Wiley Series in Probability and Statistics. Chichester: Wiley. x, 277 p. (1999).

[GeomIII] Geometry. III. Theory of surfaces. A translation of Current problems in mathematics. Fundamental directions. Vol. 48 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989. Translation by E. Primrose. Translation edited by Yu. D. Burago and V. A. Zalgaller. Encyclopaedia of Mathematical Sciences, 48. Springer-Verlag, Berlin, 1992. vi+256 pp.

[CS10] Y. Coudene, B. Schapira. Generic measures for hyperbolic flows on noncompact spaces. *Israel J. Maths* **178** (2010).

[Eb72] P. Eberlein. Geodesic flows on negatively curved manifolds I. *Ann. Math. II Ser.* **95** (1972), 492-510.

[Eb96] P. Eberlein. Geometry of nonpositively curved manifolds. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996. vii+449 pp.

[Had98] J. Hadamard. Les surfaces à courbures opposées et leurs lignes géodésiques (1898) dans Oeuvres. Tome 2, 729-775, Paris: Editions du Centre National de la Recherche Scientifique. 2296 p. (1968).

[Kn02] G. Knieper. Hyperbolic dynamics and Riemannian geometry. Handbook of dynamical systems, Vol. 1A, 453–545, North-Holland, Amsterdam, 2002.

[LPP06] G. Link, M. Peigné, J.C. Picaud. Sur les surfaces non-compactes de rang un. *L'enseignement Mathématique* (2) **52** (2006), no. 1-2.

[Rob99] C. Robinson. Dynamical systems. Stability, symbolic dynamics, and chaos. Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1999. xiv+506 pp. I

[Si72] K. Sigmund. On the space of invariant measures for hyperbolic flows, *Amer. J. Math.* **94** (1972), 31–37.