Gibbs measures have positive entropy

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Abstract

We prove that Gibbs measures for the geodesic flow on negatively curved (possibly non compact) manifolds have positive entropy ([1]) ([2])

This short note is a complement to [PPS15, PS18, GST20]. We follow notations of [PPS15]. For a dynamical system, existence of Gibbs measures is a feature of a chaotic behaviour. In [PPS15], for the geodesic flow of noncompact negatively curved manifolds, a geometric construction of Gibbs measures in the spirit of Patterson-Sullivan is exposed, and many properties of these measures are proven, among which their mixing, as soon as they are finite. In [PS18], we characterize their finiteness, and in [GST20], we provide a sufficient condition for finiteness. However, we did not investigate the positiveness of entropy. We do it below.

Theorem 1. *Let M be a nonelementary negatively curved manifold, and $F : T^1M \to \mathbb{R}$ be a Hölder continuous potential whose Gibbs measure m_F is finite and normalized into a probability measure. Then it has positive Kolmogorov-Sinai entropy.*

Without loss of generality, we can normalize the potential F so that $P(F) = 0$ and $h(m_F) = -\int_{T^1M} F\, dm_F$. Thus, it is enough to show that $\int F\, dm_F < 0$.

Choose a large compact subset $K \subset T^1M$ with positive measure. The *Gibbs property* [PPS15,] says that there exists $C > 0$ such that for all $v \in K$ and $T > 0$ with $g^T v \in K$,

$$\frac{1}{C} \exp\left(\int_0^T F(g^t v)\, dt\right) \le m_F(B(v, 0, T, \epsilon)) \le C \exp\left(\int_0^T F(g^t v)\, dt\right). \tag{1}$$

Lemma 2. *For all $v \in K$, $m_F(B(v, 0, T, \varepsilon)) \to 0$ when $T \to +\infty$.*

Proof. By [PPS15, Lemma 3.17] every dynamical ball $B(v, T, \varepsilon)$ is included in a product of shadows

$$\mathcal{O}_{\pi(g^T v)}(B(\pi(v), 2\varepsilon)) \times \mathcal{O}_{\pi(v)}(B(\pi(g^T v), 2\varepsilon)) \times [-1, 1].$$

The definition of m_F as a product measure implies that up to a constant, depending on v but not on T, the measure $m_F(B(v, 0, T, \varepsilon))$ is bounded from above by $\mu_o^F(\mathcal{O}_{\pi(v)}(B(\pi(g^T v, 2\varepsilon))$. The sets $(\mathcal{O}_{\pi(v)}(B(\pi(g^T v, 2\varepsilon))))_{T>0}$ decrease to a point when $T \to +\infty$, and the measure μ_o^F has no atoms. This proves the lemma. □

The Gibbs property 1 implies immediately the following corollary.

Corollary 3. *For all $v \in K$*

$$\int_0^T F(g^t v)\, dt \to -\infty \quad when \quad T \to +\infty \quad with \quad g^T v \in K.$$

[1]Keywords: Negative curvature, geodesic flow, Gibbs measure, entropy.
[2]MSC Classification 37A25, 37A35, 37D35, 37D40.

The end of the proof consists in proving that the conclusion of the above corollary, true for all $K \subset T^1M$ and $v \in K$, implies $\int F\,dm_F < 0$.

Consider the first return map $\tau^K : T^1M \to \mathbb{R}_+ \cup \{\infty\}$ in K of the time 1-map g^1 of the geodesic flow, let m_F^K be the normalization of the restriction of m_F to K, and τ_N^K the $n-$th return time in K. Induce on the compact subset $K \subset T^1M$. Define $T(v) = g^{\tau_K(v)}(v)$, and $G(v) = \int_0^{\tau_K(v)} F(g^t v)\,dt$. We have

$$S_N G(v) = \int_0^{\tau_N(v)} F(g^s v) ds \quad \text{and} \quad \int_{T^1M} F\,dm_F = m_F(K) \times \int_K G(v)\,dm_F^K\,.$$

The above corollary implies that when $N \to +\infty$, $S_N G(v) \to -\infty$. The main Theorem follows from the lemma below.

Lemma 4 (Atkinson [Atk76])**.** *Let $(X, \mathcal{B}, m, T)$ be an ergodic system, with m finite probability measure. Let $G \in L^1(X, m)$. If $S_N G(v) \to -\infty$ almost surely, then $\int G\,dm < 0$.*

References

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