THE GENERIC POINTS FOR THE HOROCYCLE FLOW ON A CLASS OF HYPERBOLIC SURFACES WITH INFINITE GENUS

OMRI SARIG AND BARBARA SCHAPIRA

Abstract. A point is called generic for a flow preserving an infinite ergodic invariant Radon measure, if its orbit satisfies the conclusion of the ratio ergodic theorem for every pair of continuous functions with compact support and non-zero integrals. The generic points for horocycle flows on hyperbolic surfaces of finite genus are understood, but there are no results in infinite genus. We give such a result, by characterizing the generic points for $\mathbb{Z}^d$–covers.

1. Introduction

Generic Points. Suppose $\phi^t : X \to X$ is a continuous flow on a second countable locally compact metric space X. Let $C_c(X)$ denote the set of all continuous functions with compact support. A point $x \in X$ is called *generic* for an invariant Radon measure m, if

(1) $m(X) < \infty$, and for all $f \in C_c(X)$, $\frac{1}{T}\int_0^T f(\phi^t x)dt \xrightarrow[T\to\infty]{} \frac{1}{m(X)}\int f dm$,
(2) $m(X) = \infty$, and for all $f, g \in C_c(X)$ with non-zero integrals,

$$\frac{\int_0^T f(\phi^t x)dt}{\int_0^T g(\phi^t x)dt} \xrightarrow[T\to\infty]{} \frac{\int f dm}{\int g dm}.$$

If m is ergodic and conservative, then m–almost every x is generic, because of the ratio ergodic theorem. The question is to identify this set of full measure.

Horocycle Flows. Suppose M is a connected hyperbolic surface, and let T^1M denote its unit tangent bundle (made of all tangent vectors of length one). The (stable) *horocycle* of $\omega \in T^1M$ is the set $\mathrm{Hor}(\omega)$ of all unit tangent vectors ω' such that $d(g^s\omega', g^s\omega) \xrightarrow[s\to\infty]{} 0$, where $g^s : T^1M \to T^1M$ is the geodesic flow, and d is the hyperbolic metric of T^1M. This is a smooth curve. The (stable) *horocycle flow* is the flow $h^t : T^1M \to T^1M$ which moves a vector $\omega \in T^1M$ along $\mathrm{Hor}(\omega)$ at unit speed, in the positive direction (to determine the orientation, lift to the universal cover $T^1\mathbb{D}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$).

If M has finite volume, then the generic points for the horocycle flow are understood thanks to the works of Furstenberg [**F**] (compact surfaces), and Dani & Smillie [**DS**] (surfaces of finite area). Burger [**Bu**] characterized the symmetrically generic points for a large class of hyperbolic surfaces of finite genus and infinite

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volume, where 'symmetrically generic' means that $\int_0^T$ is replaced by $\int_{-T}^T$. Schapira [**Scha**] characterized the symmetrically generic points for all hyperbolic surfaces of finite genus.

These results can be summarized as saying that all non-periodic, non-wandering horocycles are generic for the volume measure in case $vol(T^1M) < \infty$, or symmetrically generic for a certain singular infinite Radon measure which is carried by the non-wandering set of the horocycle flow in case $vol(T^1M) = \infty$ (see [**Bu**], [**Scha**] for details).

Ratner [**Ra2**] generalized the part of these results pertaining to *finite* invariant measures to all unipotent flows, whence to all horocycle flows on all hyperbolic surfaces. One of her results is that the horocycle flow on a hyperbolic surface of infinite volume (e.g. a surface with infinite genus) does not admit finite invariant measures except possibly trivial measures on periodic horocycles encircling cusps. But there could be globally supported *infinite* invariant Radon measures (Babillot & Ledrappier [**BL2**]). The problem of describing the generic points for these measures is completely open.

The purpose of this paper is to solve this problem for the simplest possible class of hyperbolic surfaces of infinite genus: $\mathbb{Z}^d$–covers of compact hyperbolic surfaces.

$\mathbb{Z}^d$**–Covers.** All surfaces in this paper are assumed to be connected.

A hyperbolic surface M is called a *regular $\mathbb{Z}^d$–cover of a compact hyperbolic surface* M_0 (or just '$\mathbb{Z}^d$–cover'), if there is an onto map $p : M \to M_0$ such that (1) every $x \in M_0$ has a neighborhood V_x such that every connected component of $p^{-1}(V_x)$ is mapped isometrically by p onto V_x, (2) the group

$$\mathrm{Deck}(M, p) := \{D : M \to M : D \text{ is an isometry s.t. } p \circ D = D\}$$

is isomorphic to $\mathbb{Z}^d$ (elements of $\mathrm{Deck}(M, p)$ are called *deck transformations*), and (3) for every $x \in M$, there is $\widetilde{x} \in M$ such that $p^{-1}(x) = \{D(\widetilde{x}) : D \in \mathrm{Deck}(M, p)\}$.

The deck transformations act on T^1M by their differentials. We abuse notation and use the same notation for D and its differential.

Choose some connected fundamental domain $\widetilde{M_0}$ for the action of the group of deck transformations on T^1M. Then every $\omega \in T^1M$ can be associated with a unique $\xi(\omega) \in \mathbb{Z}^d$ such that $\omega \in D_{\xi(\omega)}[\widetilde{M_0}]$. We call $\xi(\omega)$ the *$\mathbb{Z}^d$–coordinate of* ω. It is useful to think of T^1M as of a $\mathbb{Z}^d$–array of copies of $\widetilde{M_0}$, tagged by their $\mathbb{Z}^d$–coordinates.

The *asymptotic cycle of a vector* $\omega \in T^1M$ is the following limit, if it exists:

$$\Xi(\omega) := \lim_{T\to\infty} \frac{1}{T}\xi_T(\omega), \text{ where } \xi_T(\omega) := \xi(g^T\omega)$$

(compare with [**Schw**]). Let

$$\mathfrak{C} := \overline{\mathrm{conv}}\left(\{\Xi(\omega) : \omega \in T^1M \text{ s.t. } \Xi(\omega) \text{ exists}\}\right) \subset \mathbb{R}^d,$$

where $\overline{\mathrm{conv}}$ denote the closure of the convex hull.

Babillot–Ledrappier Measures. The ergodic invariant Radon measures for horocycle flows on $\mathbb{Z}^d$–covers are known. To list them, fix a parametrization $\mathrm{Deck}(M, p) = \{D_\xi : \xi \in \mathbb{Z}^d\}$ such that $D_\xi \circ D_\eta = D_{\xi+\eta}$. Then

(1) For every homomorphism $\varphi : \mathbb{Z}^d \to \mathbb{R}$, there exists a unique (infinite) horocycle ergodic invariant Radon measure m_φ such that $m_\varphi \circ D_\xi = e^{\varphi(\xi)}m_\varphi$ for

all $\xi \in \mathbb{Z}^d$, and $m_\varphi(\widetilde{M_0}) = 1$ ([**BL2**], Theorem 1.2). We call these measures the *Babillot–Ledrappier measures.*

(2) There is a vector $\Xi_\varphi \in \mathbb{R}^d$ such that $\Xi(\omega) = \Xi_\varphi$ m_φ–almost everywhere; the vector Ξ_φ determines φ; and the set of all possible Ξ_φ is equal to the interior of $\mathfrak{C}$, $int(\mathfrak{C})$ ([**BL2**], Corollary 6.1 and [**BL1**], Proposition 1.1).

(3) Every ergodic invariant Radon measure for the horocycle flow on a $\mathbb{Z}^d$–cover of a compact hyperbolic surface is a constant times a Babillot–Ledrappier measure [**S**].

The (normalized) volume measure corresponds to $\varphi \equiv 0$, and its almost sure asymptotic cycle is $\Xi_0 = 0$.

Main Result. Suppose M is a connected regular $\mathbb{Z}^d$–cover of a compact hyperbolic surface M_0. The purpose of the paper is to prove:

Theorem 1.1. *A vector $\omega \in T^1M$ is generic for some horocycle ergodic invariant Radon measure m iff $\Xi(\omega) \in int(\mathfrak{C})$. In this case $\Xi(\omega) = \Xi_\varphi$ and $m = cm_\varphi$ for some uniquely determined homomorphism $\varphi : \mathbb{Z}^d \to \mathbb{R}$ and $c > 0$.*

We comment on the variable negative curvature case at the end of the paper.

Using the hyperbolicity of the geodesic flow and a standard specification argument, it is easy to construct vectors ω without asymptotic cycles, and vectors ω whose asymptotic cycle exist and belong to $\partial\mathfrak{C}$. It is no problem to arrange for ω *not* to be almost minimizing (v is *almost minimizing* if there is a number C s.t. $\text{dist}(g^t v, v) > t - C$ for all $t \geq 0$). By [**E**] theorem 5.5, and the ergodicity of the volume measure [**BL2**], the horocycle of ω is dense. By theorem 1.1 this horocycle is not equidistributed for any ergodic invariant Radon measure. Thus there are many dense horocycles which are not equidistributed with respect to any measure. This should be contrasted with the finite genus case discussed above.

It is interesting that the mere existence of an asymptotic cycle is not enough for genericity, but that the value is important as well.

The Proof. The proof that every ω with asymptotic cycle in $int(\mathfrak{C})$ is generic uses harmonic analysis. It is a combination of methods from [**LS1**] for approximating the Birkhoff integral of a function by a symbolic dynamical quantity, and techniques from [**BL1**] (building on [**L**]) for finding the asymptotic behavior of this quantity.

The main contribution of the paper is the converse statement: every generic vector has asymptotic cycle in $int(\mathfrak{C})$. We do not know how to do this using harmonic analysis, because the harmonic analytic tools we have fail for $\omega \in T^1M$ s.t. $\xi_T(\omega)/T \to \partial\mathfrak{C}$. We use a different method based on a certain built-in approximate exchangeability structure for the Lebesgue measure on horocycles.

Notational Convention. $a = b \pm c$ means $|a - b| < c$ or $\|a - b\| < c$ depending on the context. $a = e^{\pm c} b$ means $e^{-c} \leq \frac{a}{b} \leq e^c$. Our error bounds c are always very generous, and rarely optimal.

2. Preparations: Symbolic Dynamics

Generalities. A *subshift of finite type* with set of *states* S and *transition matrix* $A = (t_{ij})_{S\times S}$ ($t_{ij} \in \{0, 1\}$) is the set

$$\Sigma := \{x = (x_i) \in S^{\mathbb{Z}} : \forall i \in \mathbb{Z},\, t_{x_i x_{i+1}} = 1\}$$

together with the action of the *left shift map* $\sigma : \Sigma \to \Sigma$, $\sigma(x)_k = x_{k+1}$, and the metric $d(x,y) = \sum_{k\in\mathbb{Z}} \frac{1}{2^{|k|}}(1-\delta_{x_k y_k})$. There is a *one–sided* version $\sigma : \Sigma^+ \to \Sigma^+$ obtained by replacing $\mathbb{Z}$ by $\mathbb{N}\cup\{0\}$.

A *cylinder* in Σ is a set of the form $[a_{-m},\dots,\dot{a}_0,\dots,a_n] := \{x \in \Sigma : x_{-m}^n = a_{-m}^n\}$, where the notation x_{-m}^n means $(x_{-m},\dots,x_n)$, and the dot designates the zeroth coordinate. Cylinders in Σ^+ are defined similarly (with $m=0$). A word $\underline{a}$ is called *admissible* if $[\underline{a}] \neq \varnothing$. The cylinders form a basis of clopen sets for the topology of Σ (or Σ^+). The length of a word $\underline{a}$ is denoted by $|\underline{a}|$.

A subshift of finite type is *topologically mixing* iff $\exists m$ s.t. all the entries of A^m are positive. Any topologically mixing subshift of finite type has a number $M_{br} \in \mathbb{N}$ and a collection of admissible words of length M_{br} $\mathfrak{B} := \{\underline{w}_{ab} : a,b \in S\}$ such that

$$[a, \underline{w}_{ab}, b] \neq \varnothing.$$

We fix such a collection, and refer to its elements as *'Markovian bridges'*.

Suppose F is a real valued function on Σ or Σ^+. The Birkhoff sums of functions F are denoted by F_n:

$$F_n := F + F\circ\sigma + \cdots + F\circ\sigma^{n-1}.$$

A function F is said *to depend only on non-negative coordinates* if $x_0^\infty = y_0^\infty$ implies $F(x) = F(y)$. The *variation* of such a function is

$$\mathrm{var}(F) := \sum_{n=1}^{\infty} \sup\{|F(x)-F(y)| : x_0^{n-1} = y_0^{n-1}\}.$$

If F is Hölder continuous, then this number is finite.

Let $T : X \to X$ a map on some space X, and G a group, assumed for simplicity to be abelian. The *skew-product* over $T : X \to X$ with the *cocycle* $f : X \to G$ is the map $T_f : X \times G \to X \times G$, $T(x,\xi) := \big(T(x), \xi + f(x)\big)$.

The *suspension semi-flow* over $T : X \to X$ and *height function* $r^* : X \to [0,\infty)$ is the semi-flow $\varphi^s : X_{r^*} \to X_{r^*}$, where

$$X_{r^*} := \{(x,t) : x \in X, 0 \le t < r^*(x)\}$$

and $\varphi^s(x,t) := \big(T^n x, t+s-r_n^*(x)\big)$ where n is chosen s.t. $0 \le t+s-r_n^*(x) < r^*(T^n x)$. If T is invertible, then this semi-flow has a unique extension to a flow. The suspension (semi)-flow can be identified with the (semi)-flow $(x,t) \mapsto (x,t+s)$ on $(X\times\mathbb{R})/\sim$ (respectively $X\times\mathbb{R}^+/\sim$) where $\sim$ is the orbit relation of the skew-product T_{-r^*}. Both descriptions shall be used below.

Symbolic Dynamics for the Geodesic Flow. Let $p : M \to M_0$ be a $\mathbb{Z}^d$–cover of a compact connected orientable hyperbolic surface M_0. We describe the geodesic flow on $g^s : T^1M \to T^1M$ as a suspension flow, whose base is a skew–product, whose base is a subshift of finite type. This description is well–known [**PS1**],[**Po**],[**BL1**]. It can be obtained by a lifting argument from the Bowen–Series symbolic dynamics of $g^s : T^1M_0 \to T^1M_0$ given by [**BS**], [**Se1**],[**Se2**], as in.

Lemma 2.1. *Fix $i : T^1M_0 \to T^1M$ 1-1 with image $\widetilde{M_0}$ and s.t. $dp \circ i = id$. There exist a topologically mixing two–sided subshift of finite type (Σ, T), a Hölder continuous function $r : \Sigma \to \mathbb{R}$ which depends only on the non-negative coordinates, a function $f : \Sigma \to \mathbb{Z}^d$ s.t. $f(x) = f(x_0,x_1)$, a Hölder function $h : \Sigma \to \mathbb{R}$, and a Hölder continuous map $\pi : \Sigma\times\mathbb{Z}^d\times\mathbb{R} \to T^1M$ with the following properties:*

(1) *$r^* := r + h - h \circ \sigma$ is non-negative, and there exists a constant n_0 such that $\inf r^*_{n_0} > 0$, where $r^*_{n_0} := \sum_{k=0}^{n_0-1} r^* \circ \sigma^k$.*
(2) *$\pi : (\Sigma \times \{0\})_{r^*} \to \widetilde{M_0}$ is a surjective finite-to-one map, where*

$$(\Sigma \times \{0\})_{r^*} = \{(x, 0, t) : 0 \le t < r^*(x)\}.$$

$\exists \omega_i \in T^1 M$, $i \in \mathbb{N}$, s.t. every ω outside the union of the weak stable and weak unstable leaves of ω_i, $i \in \mathbb{N}$, has exactly one preimage. See [**Se1**].
(3) *If $Q_{\xi_0,t_0}(x, \xi, t) = (x, \xi + \xi_0, t + t_0)$, then $\pi \circ Q_{(\xi_0,t_0)} = [g^{t_0} \circ D_{\xi_0}] \circ \pi$ for all $(\xi_0, t_0) \in \mathbb{Z}^d \times \mathbb{R}$;*
(4) *$\pi \circ T_{(f,-r^*)} = \pi$ where $T_{(f,-r^*)}(x, \xi, t) = \big(\sigma x, \xi + f(x), t - r^*(x)\big)$;*
(5) *Suppose $\omega = \pi(x, \xi, t), \omega' = \pi(x', \xi', t')$. Then*

$$\exists p, q \ge 0 \ s.t \begin{cases} x_p^\infty = (x')_q^\infty \\ t - t' = h(x) - h(x') + r_p(x) - r_q(x') \\ \xi - \xi' = f_q(x') - f_p(x) \end{cases} \Rightarrow \exists s \ s.t. \ \omega' = h^s(\omega).$$

(6) *Suppose $\omega = \pi(x, \xi, t)$, $0 \le t < r^*(x)$. For every s, all but at most countably many points $\omega' \in g^s \operatorname{Hor}(\omega)$ have a unique representation $\omega' = \pi(x', \xi', t')$ such that $0 \le t' < r^*(x')$ and such that $\exists p, q$ with $(x')_p^\infty = x_q^\infty$.*
(7) *Nonarithmeticity* [**Sh**]*: $\overline{\langle(-r_n(x), f_n(x)) : T^n x = x, \ n \in \mathbb{N}\rangle} = \mathbb{R} \times \mathbb{Z}^d$. (See also* [**C**]*.)*

Symbolic Coordinates. Recall the definition of ω_i from lemma 2.1 (2), and choose once and for all representations $\omega_i = \pi(x_i, \xi_i, t_i)$ such that $0 \le t_i < r^*(x_i)$. We say that $(x, \xi, t) \in \Sigma \times \mathbb{Z}^d \times \mathbb{R}$ is a set of *symbolic coordinates* for $\omega \in T^1 M$, if

(1) $\omega \notin \bigcup_{i=1}^\infty \bigcup_{s \in \mathbb{R}} g^s \operatorname{Hor}(\omega_i)$, $\omega = \pi(x, \xi, t)$, and $0 \le t < r^*(x)$;
(2) $\omega \in g^s \operatorname{Hor}(\omega_i)$, $\omega = \pi(x, \xi, t)$, $0 \le t < r^*(x)$, and $x_p^\infty = (x_i)_q^\infty$ for some p, q.

Some vectors have more than one set of symbolic coordinates, but by lemma 2.1 parts 2 and 6 for every ω, the set of $\omega' \in \operatorname{Hor}(\omega)$ with more than one set of symbolic coordinates is at most countable. In particular, for every ω the Birkhoff integral $\int_0^T f(h^t\omega)dt$ is determined by the t's for which $h^t(\omega)$ has a unique symbolic representation.

We may therefore safely ignore the vectors with more than one symbolic representation, and call $x = x(\omega)$ the Σ*–coordinate*, $\xi = \xi(\omega)$ the $\mathbb{Z}^d$*–coordinate*, and $t = t(\omega)$ the $\mathbb{R}$*–coordinate* (of ω). This definition of $\xi(\cdot)$ agrees with the definition of the $\mathbb{Z}^d$–coordinate from the introduction.

The Symbolic Description of the Babillot-Ledrappier Measures. Suppose a flow $\phi^t : X \to X$ has a Poincaré section K with section map $T_K : K \to K$. If we represent the flow as a suspension over the section, then any ϕ–invariant measure can be identified with the product of a T_K–invariant measure μ_K and the Lebesgue measure on $\mathbb{R}$, restricted to the suspension space. Thus any ϕ–invariant measure is determined by the measure it induces on a Poincaré section.

The horocycle flow has a Poincaré section which can be naturally coded as $\Sigma^+ \times \mathbb{Z}^d \times \mathbb{R}$. We describe the Babillot–Ledrappier measures in terms of the measures they induce on this section.

Let S be the set of states of Σ, and fix some $P : S \to S$ such that $(P(a), a)$ is admissible. Define $\rho' : \Sigma^+ \times \mathbb{Z}^d \times \mathbb{R} \to \Sigma \times \mathbb{Z}^d \times \mathbb{R}$ by $\rho'(x^+, \xi, s) = (x, \xi, s + h(x))$ where $x_0^\infty = (x^+)_0^\infty$, and $x_k = P(x_{k+1})$ for $k < 0$. By [**BM**] (see also [**S**])

(1) $\rho := \pi \circ \rho'$ maps $\Sigma^+ \times \mathbb{Z}^d \times \mathbb{R}$ onto a Poincaré section for the horocycle flow on T^1M;
(2) if $\exists p, q$ s.t. $(x^+)_p^\infty = (y^+)_q^\infty$, $\xi - \xi' = f_q(x^+) - f_p(y^+)$, $s - s' = r_p(x) - r_q(x')$, then $\rho(x^+, \xi, s)$ and $\rho(y^+, \xi', s')$ lie on the same horocycle.

There is a slight inconvenience in that ρ is countably-to-one (because of lemma 2.1 part (4)). We shall deal with this problem by working with restrictions of ρ to sets where it is one-to-one.

By (1), any measure m on T^1M can be identified (locally, on subsets of the suspension space on which ρ is one-to-one) with $\mu \times dt$, where μ is some measure on $\Sigma^+ \times \mathbb{Z}^d \times \mathbb{R}$. By (2), if μ is invariant for the equivalence relation

$$(x^+, \xi, s) \sim (y^+, \xi', s') \iff \begin{cases} (x^+)_p^\infty & = & (y^+)_q^\infty \\ \xi - \xi' & = & f_q(x^+) - f_p(y^+) \\ s - s' & = & r_p(x) - r_q(x'), \end{cases} \tag{2.1}$$

then μ is invariant for the Poincaré map of the horocycle flow[1], and therefore gives rise to a flow invariant measure on T^1M (see [**BL2**] for details).

This was the way Babillot and Ledrappier constructed their family of infinite invariant measures for the horocycle flow: they constructed an infinite family of measures μ_φ which are invariant under the equivalence relation (2.1).

The construction uses the thermodynamic formalism. We review some necessary facts (see e.g. [**PP**]). Let $\sigma : \Sigma^+ \to \Sigma^+$ denote the *one*–sided version of Σ. The *topological pressure* of a continuous function $\varphi : \Sigma^+ \to \mathbb{R}$ is the number $P_{top}(\varphi) := \sup\{h_\mu(\sigma) + \int \varphi d\mu\}$ where the supremum ranges over all shift invariant probability measures on Σ^+. Define $P : \mathbb{R}^d \to \mathbb{R}$ implicitly by $u \mapsto P(u)$ where $P(u) = P$ is the root of

$$P_{top}(-Pr + \langle u, f\rangle) = 1.$$

This defines a C^∞–diffeomorphism from $\mathbb{R}^d$ onto $int(\mathfrak{C})$ [**BL2**]. This diffeomorphism can be defined canonically, and does not depend on the particular choice of the coding, see [**BL2**].

Any homomorphism $\varphi : \mathbb{Z}^d \to \mathbb{R}$ can be identified with a vector $u_\varphi \in \mathbb{R}^d$ by $\varphi(\cdot) = \langle u_\varphi, \cdot\rangle$. Define the operator $L_{-P(u_\varphi),u_\varphi} : C(\Sigma^+) \to C(\Sigma^+)$ by

$$(L_{-P(u_\varphi),u_\varphi}F)(x) = \sum_{\sigma y = x} e^{-P(u_\varphi)r(y) + \langle u_\varphi, f(y)\rangle} F(y).$$

Ruelle's Perron-Frobenius Theorem says that there exists a probability measure ν_φ and a positive Hölder continuous function Ψ_φ on Σ^+ such that

$$L_{-P(u_\varphi),u_\varphi}\Psi_\varphi = \Psi_\varphi \ , \ L^*_{-P(u_\varphi),u_\varphi}\nu_\varphi = \nu_\varphi \ , \text{ and } \int \Psi_\varphi d\nu_\varphi = 1.$$

[1]Here we use the general fact that a measure is invariant for a transformation if and only if it is invariant for the corresponding orbit relation. (A Borel measure m is *invariant* for a Borel equivalence relation $\sim$, if $m \circ \kappa|_{\mathrm{dom}(\kappa)} = m|_{\mathrm{image}(\kappa)}$ for all partial Borel isomorphisms $\kappa : \mathrm{dom}(\kappa) \to \mathrm{image}(\kappa)$ such that $\kappa(x) \sim x$ for all $x \in \mathrm{dom}(\kappa)$.)

The Babillot–Ledrappier measure m_φ (on T^1M) is given locally by $\mu_\varphi \times dt$, where μ_φ (a measure on the Poincaré section) is given in symbolic coordinates by

$$d\mu_\varphi(x,\xi,s) := \text{const}\, e^{-P(u_\varphi)s+\langle u_\varphi,\xi\rangle} d\nu_\varphi d\xi ds. \tag{2.2}$$

This formula can be used to deduce the following important consequences:

(1) m_φ is quasi–invariant under the geodesic flow, and $m_\varphi \circ g^s = e^{-P(u_\varphi)s} m_\varphi$;
(2) $\Xi_\varphi = (\nabla P)(u_\varphi)$ (corollary 6.1 in [**BL2**]);
(3) $u_\varphi = -(\nabla H)(\Xi_\varphi)$, where $H(\cdot)$ is minus the Legendre transform of $P(\cdot)$.

(The double minus sign in (3) is a convention from [**BL2**].)

Symbolic Local Manifolds. Suppose ω has symbolic coordinates (x,ξ,t) (i.e. $0 \le s+h(x) < r^*(x)$), and write $t = s+h(x)$. The *symbolic local stable manifold* of $\omega = \pi(x,\xi,s+h(x))$ is defined by

$$W^{ss}_{\text{loc}}(\omega) := \pi\{(y,\xi,s+h(y)) : y_0^\infty = x_0^\infty\}.$$

This is a subset of $\text{Hor}(\omega)$, because of lemma 2.1 part (5).

If $W^{ss}_{\text{loc}}(\omega_1), W^{ss}_{\text{loc}}(\omega_2)$ intersect with positive measure, then they are equal up to sets of length zero. Indeed, for almost every ω in the intersection $\exists p$ s.t. $\omega = \pi(x,\xi(\omega_1),s(\omega_1)+h(x))$, $x_0^\infty = x(\omega_1)_0^\infty$ and $\omega = \pi(\sigma^p x,\xi(\omega_2),s(\omega_2)+h(\sigma^p x))$, $x_p^\infty = x(\omega_2)_0^\infty$. This forces $x(\omega_1)_p^\infty = x(\omega_2)_0^\infty$, $\xi(\omega_2) = \xi(\omega_1) + f_p(x(\omega_1))$, $s(\omega_2) = s(\omega_1) - r_p(x(\omega_1))$ (lemma 2.1 part (1)). It follows from lemma 2.1 parts (4) and (1) that $W^{ss}_{\text{loc}}(\omega_1) = W^{ss}_{\text{loc}}(\omega_2)$ up to sets of measure zero.

Lemma 2.2. [**BL2**, prop. 4.5] *Let ℓ_ω denote the hyperbolic length measure on the horocycle of ω. If $\omega = \pi(x,\xi,s+h(x))$, then $\ell_\omega[W^{ss}_{\text{loc}}(\omega)] = e^{-s}\psi(x_0,x_1,\ldots)$ where $\psi : \Sigma^+ \to \mathbb{R}$ is Hölder continuous and positive. (In fact $\psi = \Psi_0$.)*

3. Proof that a Generic Vector has Asymptotic Cycle in *int*($\mathfrak{C}$)

In this section we prove

Proposition 3.1. *Let M be a $\mathbb{Z}^d$–cover of a compact hyperbolic surface. If ω_0 is generic for some horocycle ergodic invariant Radon measure m, then there exists a homomorphism $\varphi : \mathbb{Z}^d \to \mathbb{R}$ and $c > 0$ such that $m = cm_\varphi$ and $\Xi(\omega) = \Xi_\varphi$.*

The proof is as follows. Suppose ω_0 is generic for some horocycle ergodic invariant Radon measure m. By [**S**], there is a homomorphism $\varphi : \mathbb{Z}^d \to \mathbb{R}$ and $c > 0$ such that $m = cm_\varphi$. Of course if ω_0 is generic for cm_φ, then it is generic for m_φ, therefore we may assume w.l.o.g. that $m = m_\varphi$. We saw above that $\Xi(\omega) = \Xi_\varphi$ m_φ–a.e., for some $\Xi_\varphi \in \text{int}(\mathfrak{C})$. We aim at showing that $\Xi(\omega_0) = \Xi_\varphi$.

Recall that $\widetilde{M_0} \subset T^1M$ is the set of vectors whose $\mathbb{Z}^d$–coordinate is equal to zero, and define the following objects:

$$\begin{aligned} A_T &:= \{h^t(\omega_0) : 0 \le t \le T\}, \\ \lambda_T &:= \text{the normalized length measure on } A_T \cap \widetilde{M_0} \qquad (3.1) \\ &= \frac{1}{\int_0^T 1_{\widetilde{M_0}}(h^t(\omega_0))dt} \int_0^T 1_{\widetilde{M_0}}(h^t\omega_0)\delta_{h^t(\omega_0)}dt \quad (\delta_x := \text{Dirac at } x), \end{aligned}$$

$$\Lambda_n(N,\varepsilon_0) := \left\{\omega \in \widetilde{M_0} : \omega = \pi(x,\underline{\xi},s) \text{ and } \left\|\frac{f_N(\sigma^{nN}x)}{r_N^*(\sigma^{nN}x)} - \Xi_\varphi\right\| < \varepsilon_0\right\}. \tag{3.2}$$

Recall that n_0 is a constant such that $\min r^*_{n_0} > 0$ (lemma 2.1 part (1)). The key to the proof of proposition 3.1 is the following lemma:

Lemma 3.2 (Key Lemma)**.** *There are $C_0, T_0 > 0$ such that for every $\varepsilon_0 > 0$ there are constants $N(\varepsilon_0) > n_0$, $K(\varepsilon_0) > 0$, and $\tau(\varepsilon_0)$ such that for all $T > \tau(\varepsilon_0)$*

$$\sup_n \lambda_T \left\{\omega : r^*_{nN(\varepsilon_0)}(x(\omega)) < \ln \tfrac{T}{T_0} - K(\varepsilon_0) \text{ and } \omega \notin \Lambda_n(N(\varepsilon_0), \varepsilon_0)\right\} \leq C_0\varepsilon_0. \quad (3.3)$$

The proof is somewhat technical, so we postpone it, and first explain how to use the lemma to prove proposition 3.1.

Fix ε_0 and set $N = N(\varepsilon_0)$, $K = K(\varepsilon_0)$, $\Lambda_n := \Lambda_n(N(\varepsilon_0), \varepsilon_0)$, and $T^* := \ln(T/T_0)$. Abbreviate $\{r^*_{nN} < T^* - K\} := \{\omega \in T^1M : r^*_{nN}(x(\omega)) < T^* - K\}$.

Working in the symbolic model for T^1M, define the following functions, which we view as random variables on A_T: $A_k := f_N \circ \sigma^{kN}\, 1_{\{r^*_{kN} < T^* - K\}}$ and $B_k := r^*_N \circ \sigma^{kN} 1_{\{r^*_{kN} < T^* - K\}}$. Then for all T large enough

$$\begin{aligned}
\mathbb{E}_{\lambda_T}\!\left(\frac{\sum_i A_i}{\sum_i B_i}\right) &= \mathbb{E}_{\lambda_T}\left(\sum_i \frac{B_i}{\sum_j B_j}\left[\left(\frac{A_i}{B_i} - \Xi_\varphi\right) 1_{\{r^*_{iN} < T^* - K\}}\right]\right) + \Xi_\varphi \\
&= \Xi_\varphi \pm \sum_i \left\|\frac{B_i}{\sum_j B_j}\right\|_\infty \left\|\left(\frac{A_i}{B_i} - \Xi_\varphi\right) 1_{\{r^*_{iN} < T^* - K\}}\right\|_1 \\
&= \Xi_\varphi \pm \sum_{i=1}^{[(T^*/\min r^*_{n_0}) + n_0]/N} \left(\frac{N \max r^*}{T^* - K - N \max r^*}\right) \left\|\left(\frac{A_i}{B_i} - \Xi_\varphi\right) 1_{\{r^*_{iN} < T^* - K\}}\right\|_1.
\end{aligned}$$

We estimate the L^1-norm by breaking the support of λ_T into $\widetilde{M_0} \cap \Lambda_i$ and $\widetilde{M_0} \setminus \Lambda_i$:

$$\begin{aligned}
&\left\|\left(\frac{A_i}{B_i} - \Xi_\varphi\right) 1_{\{r^*_{iN} < T^* - K\}}\right\|_1 \\
\leq\ & \varepsilon_0 \lambda_T(\widetilde{M_0} \cap \Lambda_i) + \left(\frac{N \max \|f\|_\infty}{\lfloor N/n_0 \rfloor \min r^*_{n_0}} + \|\Xi_\varphi\|\right) \lambda_T(\widetilde{M_0} \cap \{r^*_{iN} < T^* - K\} \setminus \Lambda_i) \\
\leq\ & \varepsilon_0 + \left(\frac{N \max \|f\|_\infty}{\lfloor N/n_0 \rfloor \min r^*_{n_0}} + \|\Xi_\varphi\|\right) \sup_i \lambda_T(\widetilde{M_0} \cap \{r^*_{iN} < T^* - K\} \setminus \Lambda_i).
\end{aligned}$$

Thus, the key lemma implies that for all T large enough

$$\left\|\mathbb{E}_{\lambda_T}\left(\frac{\sum_i A_i}{\sum_i B_i}\right) - \Xi_\varphi\right\| \leq \text{const}\, \varepsilon_0 \quad (3.4)$$

with the constant independent of T and ε_0.

But the quotient whose expectation we are calculating is nearly constant! For every $\omega \in \{h^t(\omega_0) : 0 < t < T\}$, $\sum_i B_i = T^* \pm (K + N \max r^*)$, and

$$\begin{aligned}
\sum_i A_i &= \xi(g^{T^*}\omega) \pm (K + N)\, n_0 \frac{\max \|f\|}{\min r^*_{n_0}} \\
&= \xi(g^{T^*}\omega_0) \pm \left[(K + N)\, n_0 \frac{\max \|f\|}{\min r^*_{n_0}} + \|\xi(g^{T^*}\omega) - \xi(g^{T^*}\omega_0)\|\right].
\end{aligned}$$

Now $\mathrm{dist}(g^{T^*}(\omega), g^{T^*}(\omega_0)) \leq e^{-T^*}T = T_0$, so $\|\xi(g^{T^*}\omega) - \xi(g^{T^*}\omega_0)\|$ is bounded by some constant independent of T^*. Thus we obtain $\sum_i A_i = \xi(g^{T^*}\omega_0) \pm \text{const}$, where the constant is independent of T^*.

We see that

$$\mathbb{E}_{\lambda_T}\left(\frac{\sum_i A_i}{\sum_i B_i}\right) = \frac{\xi(g^{T^*}\omega_0) \pm \text{const}}{T^* \pm \text{const}} = [1+o(1)]\frac{1}{T^*}\xi(g^{T^*}\omega_0) + o(1).$$

Comparing this with (3.4), we see that for T large, $\|\frac{1}{T^*}\xi(g^{T^*}\omega_0) - \Xi_\varphi\| < \text{const}\,\varepsilon_0$, with the constant independent of ε_0. Since ε_0 was arbitrary, ω_0 has asymptotic cycle Ξ_φ, and proposition 3.1 is proved (assuming lemma 3.2).

The remainder of the section contains the proof of lemma 3.2. The proof requires the construction of certain self maps of horocycles. These are constructed and studied in the next subsections.

Preparations I: Distortion Estimates. We find uniform bounds on the Radon-Nikodym derivative of certain maps between local symbolic stable manifolds, equipped with the hyperbolic length measure.

Define an equivalence relation on Σ via

$$x \sim y \Leftrightarrow \exists p, q \text{ s.t. } x_p^\infty = y_q^\infty$$

(the equivalence class of x can be viewed as the weak stable "manifold" of x for the action of the shift). If x, y are non-eventually periodic, then the following are independent of the choice of p, q:

$$\begin{aligned} R^+(x,y) &:= r_q(y) - r_p(x), \\ R(x,y) &:= r_q(y) - r_p(x) + h(y) - h(x) = \lim_{n\to\infty}[r^*_{q+n}(y) - r^*_{p+n}(x)], \\ F(x,y) &:= f_p(x) - f_q(y) = \lim_{n\to\infty}[f_{p+n}(x) - f_{q+n}(y)]. \end{aligned} \tag{3.5}$$

These are minus the functions appearing in lemma 2.1, part 5. Note that $R^+(x,y)$ and $F(x,y)$ only depend on x_0^∞, y_0^∞.

Fix some admissible sequence x_0^∞ and two words $\underline{a}, \underline{b}$ (not necessarily of the same length) such that $(\underline{a}, x_0), (\underline{b}, x_0)$ are admissible and such that $a_0 = b_0$. Define

$$\kappa^* : \{w \in \Sigma : w_{-|\underline{a}|}^\infty = \underline{a}x_0^\infty\} \to \{w \in \Sigma : w_{-|\underline{b}|}^\infty = \underline{b}x_0^\infty\}$$

by

$$\kappa^*(w) = \begin{cases} (w_{-\infty}^{-|\underline{a}|-1}, \underline{b}, \dot{x}_0^\infty) & w = (w_{-\infty}^{-|\underline{a}|-1}, \underline{a}, \dot{x}_0^\infty) \\ (w_{-\infty}^{-|\underline{b}|-1}, \underline{a}, \dot{x}_0^\infty) & w = (w_{-\infty}^{-|\underline{b}|-1}, \underline{b}, \dot{x}_0^\infty) \\ w & \text{otherwise,} \end{cases}$$

where the zero coordinate is at the beginning of the dotted word. This induces the following self–map of $W^{ss}(\pi(x, \xi, s + h(x))$:

$$\kappa[\pi(w, \xi, s + h(w))] = \pi(\kappa^*(w), \xi, s + h(\kappa^*(w))).$$

This map is well–defined and one–to–one on $W^{ss}_{\text{loc}}(\pi(x, \xi, s + h(x)))$ minus a countable set (Lemma 2.1, part 6).

Lemma 3.3. $\kappa : W^{ss}_{\text{loc}}(\pi(x, \xi, s+h(x))) \to W^{ss}_{\text{loc}}(\pi(x, \xi, s+h(x)))$ *is absolutely continuous with respect to the hyperbolic length measure* ℓ*, and there is some constant* $D' > 1$*, independent of* x_0^∞*,* $\underline{a}$*, and* $\underline{b}$ *such that*

$$(D')^{-1}\exp[-|R^+(\underline{a}x_0^\infty, \underline{b}x_0^\infty)|] \le \frac{d\ell \circ \kappa}{d\ell} \le D' \exp[|R^+(\underline{a}x_0^\infty, \underline{b}x_0^\infty)|].$$

Proof. Split the domain of κ into three parts: A, where it flips $\underline{a}$ to $\underline{b}$, B where it flips $\underline{b}$ to $\underline{a}$, and C, where $\kappa = id$.

We estimate the derivative on A. Fix n and a word $(x'_{-n}, \dots, x'_{-1})$ such that $(x'_{-|\underline{a}|}, \dots, x'_{-1}) = \underline{a}$. We study the distortion in the length of the horocycle piece $K_{x'_{-n},\dots,x'_{-1}} := \{\pi(w, \xi, s + h(w)) : w_{-n}^{\infty} = ((x')_{-n}^{-1}, x_0^{\infty})\}$:

$$\ell[K_{x'_{-n},\dots,x'_{-1}}] = \ell[\{\pi(w, \xi, s + h(w)) : w_{-n}^{\infty} = ((x')_{-n}^{-1}, x_0^{\infty})\}]$$

$$\begin{aligned} &= \ell[\{\pi(\sigma^{-n}w, \xi - f_n(\sigma^{-n}w), s + h(w) + r_n^*(\sigma^{-n}w)) : w_{-n}^{\infty} = ((x')_{-n}^{-1}, x_0^{\infty})\}] \\ &= \ell[\{\pi(w', \xi, s + h(\sigma^n w') + r_n^*(w')) : (w')_0^{\infty} = ((x')_{-n}^{-1}, x_0^{\infty})\}] \quad (\because \ell \circ D_{f_n} = \ell) \\ &= \ell[\{\pi(w', \xi, s + h(w') + r_n(w')) : (w')_0^{\infty} = ((x')_{-n}^{-1}, x_0^{\infty})\}] \\ &= e^{-r_n(x'_{-n},\dots x'_{-1}, x_0^{\infty})}\psi(x'_{-n}, \dots x'_{-1}, x_0^{\infty}) \quad (\because \ell \circ g^s = e^{-s}\ell). \end{aligned}$$

A similar calculation shows that

$$(\ell \circ \kappa)[K_{x'_{-n},\dots,x'_{-1}}] = e^{-r_{n+|\underline{b}|-|\underline{a}|}(x'_{-n},\dots x'_{-|\underline{a}|-1}, \underline{b}, x_0^{\infty})}\psi(x'_{-n}, \dots x'_{-|\underline{a}|-1}, \underline{b}, x_0^{\infty}).$$

Dividing, we see that

$$\begin{aligned} \frac{(\ell \circ \kappa)[K_{x'_{-n},\dots,x'_{-1}}]}{\ell[K_{x'_{-n},\dots,x'_{-1}}]} &= \frac{\exp[r_n(x'_{-n}, \dots x'_{-1}, x_0^{\infty})]}{\exp[r_{n+|\underline{b}|-|\underline{a}|}(x'_{-n}, \dots x'_{-|\underline{a}|-1}, \underline{b}, x_0^{\infty})]} \\ &\qquad \times \frac{\psi(x'_{-n}, \dots x'_{-|\underline{a}|-1}, \underline{b}, x_0^{\infty})}{\psi(x'_{-n}, \dots, x'_{-1} x_0^{\infty})} \\ &\le \left(\frac{\max \psi}{\min \psi}\right) e^{|R^+(\underline{a}x_0^{\infty}, \underline{b}x_0^{\infty})| + \mathrm{var}(r)}. \end{aligned}$$

Thus, for a global constant D' independent of $x, \underline{a}$, and $\underline{b}$,

$$\frac{(\ell \circ \kappa)[K_{x'_{-n},\dots,x'_{-1}}]}{\ell[K_{x'_{-n},\dots,x'_{-1}}]} \le D' \exp[|R(\underline{a}x_0^{\infty}, \underline{b}x_0^{\infty})|].$$

Since this estimate holds for all $n > |\underline{a}|$, and since $\{K_{x'_{-n},\dots,x'_{-1}}\}$ generate the Borel sigma–algebra of A, we get that $\ell \circ \kappa|_A \ll \ell$, and $d\ell \circ \kappa / d\ell \le D' \exp[|R(\underline{a}x_0^{\infty}, \underline{b}x_0^{\infty})|]$. The lower bound on the derivative is obtained in exactly the same manner, as do the estimates for the distortion of κ on B. □

We turn to discuss a different class of maps between local symbolic manifolds. Fix two admissible sequences x_0^{∞}, y_0^{∞} and define

$$\vartheta^* : \{w \in \Sigma : w_0^{\infty} = x_0^{\infty}\} \to \{w \in \Sigma : w_0^{\infty} = y_0^{\infty}\}$$

by $\vartheta^*(w) = (w_{-\infty}^{-1}, \underline{w}_{w_{-1}y_0}, \dot{y}_0^{\infty})$, where $\underline{w}_{w_{-1}y_0}$ is a bridge word out of the (fixed) collection $\mathfrak{B}$, and where the zero coordinate is the first symbol of the dotted word.

This induces the map from $W_{\mathrm{loc}}^{ss}(\pi(x, \xi, s + h(x))$ into $W_{\mathrm{loc}}^{ss}(\pi(y, \xi, s + h(y))$

$$\vartheta[\pi(w, \xi, s + h(w))] = \pi(\vartheta^*(w), \xi, s + h(\vartheta^*(w))).$$

As before, ϑ is well–defined and one–to–one on $W_{\mathrm{loc}}^{ss}(\pi(x, \xi, s + h(x)))$ minus a countable set. But we can not claim it is surjective (because of the insertion of a fixed bridge word).

Lemma 3.4. *$\vartheta : W_{\mathrm{loc}}^{ss}(\pi(x, \xi, s+h(x))) \to W_{\mathrm{loc}}^{ss}(\pi(y, \xi, s+h(y)))$ is absolutely continuous with respect to the hyperbolic length measure ℓ, and there is some constant $D'' > 1$, independent of x_0^{∞} and y_0^{∞} such that ${D''}^{-1} \le d\ell \circ \vartheta / d\ell \le D''$.*

Proof. Fix n and a word $(w_{-n},\ldots,w_{-1})$ such that $[w_{-1},x_0]\neq\varnothing$, and set

$$K_{w_{-n},\ldots,w_{-1}}:=\{\pi(w',\xi,s+h(w')):(w')_{-n}^{\infty}=(w_{-n}^{-1},x_0^{\infty})\}.$$

The same calculation as in the previous lemma shows that

$$\begin{aligned}\frac{(\ell\circ\vartheta)[K_{w_{-n},\ldots,w_{-1}}]}{\ell[K_{w_{-n},\ldots,w_{-1}}]} &= \frac{\exp[r_n(w_{-n},\ldots w_{-1},x_0^{\infty})]}{\exp[r_{n+|\underline{w}_{w_{-1}y_0}|}(w_{-n},\ldots w_{-1},\underline{w}_{w_{-1}y_0},y_0^{\infty})]}\\ &\qquad\times\frac{\psi(w_{-n},\ldots w_{-1},\underline{w}_{w_{-1}y_0},y_0^{\infty})}{\psi(w_{-n},\ldots,w_{-1}x_0^{\infty})}\\ &\le\left(\frac{\max\psi}{\min\psi}\right)\times\exp[\mathrm{var}(r)]\times\max_{\underline{w}\in\mathfrak{B}}\left[\exp\max_{x\in[\underline{w}]}|r_{M_{br}}(x)|\right].\end{aligned}$$

As in the proof of the previous lemma, this implies that for some constant $D_1''>1$, independent of x and y, $d\ell\circ\vartheta/d\ell\le D_1''$. A uniform lower bound D_2'' can be obtained in exactly the same manner. □

Preparations II: Permuting Points on a Horocycle. We use the transformations discussed above to construct certain maps which 'permute' points on a given horocycle. These bijections are defined symbolically in terms of the coding $\pi(x,\xi,s)$. They take the form

$$\begin{aligned}\theta[\pi(x,\xi,s+h(x))] &= \pi\big(\theta^*(x),\xi+F(x,\theta^*(x)),s+h(x)+R(x,\theta^*(x))\big)\\ &= \pi\big(\theta^*(x),\xi+F(x,\theta^*(x)),s+h(\theta^*x)+R^+(x,\theta^*(x))\big)\end{aligned}$$

where θ^* will be defined below, and $F(\cdot,\cdot),R(\cdot,\cdot),R^+(\cdot,\cdot)$, given by (3.5), are designed to ensure that $\theta[\pi(x,\xi,s)]$ stays on the horocycle of $\pi(x,\xi,s)$. Definitions of this form make sense everywhere on the horocycle where π is injective, therefore everywhere except a countable set.

The maps $\theta^*(x)$ we need fall into two groups (precise definitions and explanation of notation follow):

(1) maps $\theta^*=\kappa_{n,N}^*$ which exchange the positions of two N–blocks in x,
(2) maps $\theta^*=\vartheta_{T^*,\omega_1,\omega_2}^*$ which exchange some suffix of x by another suffix.

Exchanging blocks: $\kappa_{n,N}$. Fix $n,N\in\mathbb{N}$. Define $\kappa_{n,N}^*$ on Σ by exchanging the places of the N–blocks $(x)_0^{N-1}$, $(x)_{nN}^{nN+N-1}$, and then inserting bridge words (see §2) at the right places to ensure admissibility:

$$\begin{aligned}\kappa_{n,N}^*(x):=(x_{-\infty}^{-1},w_{x_{-1}x_{nN}},&\boxed{\dot{x}_{nN}^{nN+N-1}},w_{x_{nN+N-1},x_N},\\ &x_N^{nN-1},w_{x_{nN-1},x_0},\boxed{x_0^{N-1}},w_{x_{N-1},x_{(n+1)N}},x_{(n+1)N}^{\infty}).\end{aligned}$$

The zero coordinate is at the first symbol of the dotted word, and the boxed blocks are those that were switched. This gives rise to the following self–map of the horocycle $\mathrm{Hor}(\pi(x,s,h(x))$:

$$\kappa_{n,N}\big(\pi(x,\underline{\xi},s+h(x))\big):=\pi\big(\kappa_{n,N}^*x,\xi+F(x,\kappa_{n,N}^*x),s+h(\kappa_{n,N}^*x)+R^+(x,\kappa_{n,N}^*x)\big).$$

Lemma 3.5. *The map $\kappa_{n,N}$ is absolutely continuous and injective on a subset of full ℓ–measure of $\mathrm{Hor}(\pi(x,\xi,s+h(x)))$, and there exist constants $C_r'\ge1$, E', and L' independent of n,N and x such that*

(1) $(C_r')^{-1}\le d\ell\circ\kappa_{n,N}/d\ell\le C_r'$,
(2) *the Σ–coordinate of $\kappa_{n,N}(\pi(x,\xi,t))$ is $\sigma^l(\kappa_{n,N}^*(x))$ for some $|l|\le L'$,*

(3) $\mathrm{dist}(\omega, \kappa_{n,N}(\omega)) \le E'$ ($\mathrm{dist} = \mathrm{dist}_{T^1M}$).

Proof. The map $\kappa^*_{n,N}$ can be thought of locally as the map which exchanges the word $\underline{a}$ by $\underline{b}$, where

$$\underline{a} = (x_0, \ldots, x_{(n+1)N-1})$$
$$\underline{b} = (w_{x_{-1}x_{Nn}}, x_{Nn}^{(n+1)N-1}, w_{x_{N(n+1)-1}, x_N}, x_N^{Nn-1}, w_{x_{Nn-1}, x_0}, x_0^{N-1}, w_{x_{N-1}, x_{(n+1)N}}).$$

Note that $|\underline{a}| = (n+1)N$, and $|\underline{b}| = (n+1)N + 4M_{br}$, where M_{br} is the length of the bridge words in $\mathfrak{B}$.

The distortion of such maps was found in Lemma 3.3. A direct calculation using the boundedness and Hölder continuity of r shows that there exist a constant E'_R, which only depends on $\max r$, M_{br} and $\mathrm{var}(r)$ (but not on N, n), such that $|R^+(\underline{a}x^\infty_{(n+1)N}, \underline{b}x^\infty_{(n+1)N})| < E'_R$. Part (1) follows from lemma 3.3.

Next we study the Σ–coordinate of $\kappa_{n,N}(\pi(x, \xi, s+h(x)))$. This must be $\sigma^l(\kappa^*_{n,N}x)$ with l such that either $l \ge 0$ and

$$0 \le s + h(\kappa^*_{n,N}x) + R^+(x, \kappa^*_{n,N}x) - r^*_l(\sigma^l\kappa^*_{n,N}x) < r^*(\sigma^l\kappa^*_{n,N}x).$$

or $l < 0$ and $0 \le s + h(\kappa^*_{n,N}x) + R^+(x, \kappa^*_{n,N}x) + r^*_{-l}(\sigma^l\kappa^*_{n,N}x) < r^*(\sigma^l\kappa^*_{n,N}x)$. We have seen that $|R^+(x, \kappa^*_{n,N}x)| \le E'_R$. Since h is bounded and $\inf r^*_{n_0} > 0$, l must be bounded by some constant $L' = L'(\max|h|, E'_r, \min r^*_{n_0})$. Part (2) follows.

Finally we study the $\mathbb{Z}^d$–coordinate of $\omega' := \kappa_{n,N}(\pi(x, \xi, s + h(x)))$. If l is as above, then the symbolic coordinates of ω' are (assuming for simplicity that $l > 0$)

$$(\sigma^l(\kappa^*_{n,N}(x)), \xi + F(\underline{a}x^\infty_{(n+1)N}, \underline{b}x^\infty_{(n+1)N}) + f_l(\kappa^*_{n,N}(x)), \text{something}).$$

It is routine to check that $\|F(\underline{a}x, \underline{b}x)\|$ is uniformly bounded by some constant E'_F which only depends on $\max\|f\|$ and M_{br}. This means that the difference in $\mathbb{Z}^d$–coordinates is no more than $\|F(\underline{a}x^\infty_{(n+1)N}, \underline{b}x^\infty_{(n+1)N}) + f_l(\kappa^*_{n,N}(x))\| \le E'_F + L'\max\|f\|$.

Thus ω, ω' lie in two copies of $\widetilde{M}_0 \subset T^1M$ which differ by a deck transformation whose $\mathbb{Z}^d$–index is bounded. Since the diameter of $\widetilde{M}_0$ is finite, this implies that $\mathrm{dist}(\omega, \omega')$ is uniformly bounded, whence part (3). □

Changing suffices: $\vartheta_{T^*,\omega_1,\omega_2}$. Fix $T^* > 0$ and two vectors ω_1,ω_2 on the same horocycle $\mathrm{Hor}(\omega)$. We define a map $\vartheta_{T^*,\omega_1,\omega_2} : g^{-T^*}[W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1)] \to g^{-T^*}[W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2)]$ as follows: write $\omega^*_i := g^{T^*}(\omega_i) = \pi(x^*_i, \xi^*_i, s^*_i + h(x^*_i))$ $(i = 1, 2)$, and define

(1) $\vartheta^*_{x^*_1,x^*_2} : \{z^* \in \Sigma : (z^*)^\infty_0 = (x^*_1)^\infty_0\} \to \{x^* \in \Sigma : (x^*)^\infty_0 = (x^*_2)^\infty_0\}$ by
$$\vartheta^*_{x^*_1,x^*_2}(z) = \big(z^{-1}_{-\infty}, w_{z_{-1},(x^*_2)_0}, (\dot{x}^*_2)^\infty_0\big);$$

(2) $\widetilde{\vartheta}_{\omega_1,\omega_2} : W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1) \to W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2)$ by
$$\widetilde{\vartheta}_{\omega_1,\omega_2}(\pi(z, \xi^*_1, s^*_1 + h(z))) = \pi(\vartheta^*_{x^*_1,x^*_2}z, \xi^*_2, s^*_2 + h(\vartheta^*_{x^*_1,x^*_2}z));$$

(3) The map $\vartheta_{T^*,\omega_1,\omega_2} : g^{-T^*}[W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1)] \to g^{-T^*}[W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2)]$ is
$$\vartheta_{T^*,\omega_1,\omega_2} := g^{-T^*} \circ \widetilde{\vartheta}_{\omega_1,\omega_2} \circ g^{T^*}.$$

Lemma 3.6. *Suppose* $\mathrm{dist}(g^{T^*}\omega_1, g^{T^*}\omega_2) \le T_0$. *The map* $\vartheta_{T^*,\omega_1,\omega_2}$ *is an absolutely continuous injective map from a subset of full* ℓ*–measure of* $g^{-T^*}[W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1)]$ *into* $g^{-T^*}[W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2)]$, *and there exist constants* $C''_r \ge 1$, E'' *and* L'' *which only depend on* T_0 *such that*

(1) $(C_r'')^{-1} \leq d\ell \circ \vartheta_{T^*,\omega_1,\omega_2}/d\ell \leq C_r''$,
(2) *the* Σ*–coordinate of* $\vartheta_{T^*,\omega_1,\omega_2}(\pi(x,\xi,T))$ *is* $\sigma^{-m}(\vartheta^*_{x_1^*,x_2^*}\sigma^n x)$ *with* $n \geq 0$ *such that* $|r_n^*(x) - T^*| \leq 5(\max r^* + \max|h|)$ *and* $|m-n| \leq L''$,
(3) $\mathrm{dist}(\omega, \vartheta_{T^*,\omega_1,\omega_2}\omega) \leq E''$ ($\mathrm{dist} = \mathrm{dist}_{T^1M}$).

Proof. Lemma 3.4 implies that $\widetilde{\vartheta}_{\omega_1,\omega_2}$ is absolutely continuous and that

$$\frac{1}{D''} \leq \frac{d\ell \circ \widetilde{\vartheta}_{\omega_1,\omega_2}}{d\ell} \leq D''$$

with D'' independent of ω_i. Since the geodesic flow contracts the length of horocycles uniformly, part (1) follows with $C_r'' := D''$.

To compare the symbolic coordinates of ω and $\vartheta_{T^*,\omega_1,\omega_2}(\omega)$, write $\omega = \pi(x,\xi,t)$ with $0 \leq t < r^*(x)$. Since $\omega \in g^{-T^*}W^{ss}_{\mathrm{loc}}(\omega_1^*)$, there is some n s.t.

$$g^{T^*}(\omega) = \pi(\sigma^n(x), \xi + f_n(x), t + T^* - r_n^*(x))$$

where $\sigma^n(x)_0^\infty = (x_1^*)_0^\infty$, $\xi + f_n(x) = \xi_1^*$, $t + T^* - r_n^*(x) = s_1^* + h(\sigma^n(x))$. Thus

$$\begin{aligned}\vartheta_{T^*,\omega_1,\omega_2}(\omega) &= g^{-T^*}\pi\Big(\vartheta^*_{x_1^*,x_2^*}(\sigma^n x), \xi_2^*, s_2^* + h\big(\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)\big)\Big)\\ &= \pi\left(\begin{array}{c}\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)\\ \xi_2^* - f_m(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x))\\ s_2^* + h(\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)) - T^* + r_m^*(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x))\end{array}\right)\end{aligned}$$

with m s.t.

$$0 \leq s_2^* + h(\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)) - T^* + r_m^*(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)) < r^*(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)).$$

The obvious bounds $|s_2^*| \leq \max r^* + \max|h|$, $|r_m^* - r_m| \leq 2\max|h|$, and the double inequality above imply that

$$|r_m(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x)) - T^*| \leq 5(\max r^* + \max|h|).$$

The identity $t + T^* - r_n^*(x) = s_1^* + h(\sigma^n(x))$ means that

$$|r_n^*(x) - T^*|, |r_n(x) - T^*| \leq 5(\max r^* + \max|h|).$$

We see that $|r_n(x) - r_m(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x))| \leq 10(\max r^* + \max|h|)$. By construction,

$$(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x))_0^\infty = (x_{n-m+M_{br}}, \ldots, x_{n-1}, w_{x_{n-1}(x_2^*)_0}, (x_2^*)_0^\infty) \tag{3.6}$$

This allows us to estimate $|m-n|$:

$$\begin{aligned}10(\max r^* + \max|h|) &\geq |r_n(x) - r_m(\sigma^{-m}\vartheta^*_{x_1^*,x_2^*}(\sigma^n x))|\\ &\geq \inf|r_{|n-m+M_{br}|}| - 2M_{br}\max|r| - 2\mathrm{var}(r)\\ &\geq \inf r^*_{|n-m+M_{br}|} - 2\max|h| - 2M_{br}\max|r| - 2\mathrm{var}(r)\\ &\geq \left\lfloor\frac{|n-m|}{n_0}\right\rfloor \inf r^*_{n_0} - 2\max|h| - 3M_{br}\max|r| - 2\mathrm{var}(r).\end{aligned}$$

Since $\inf r^*_{n_0} > 0$, we see that $|m-n| \leq L''$, with $L'' = L''(r, r^*, M_{br}, h, n_0)$ with L'' independent of ω_1, ω_2 and T^*.

We can now estimate the difference between the $\mathbb{Z}^d$ and $\mathbb{R}$ coordinates of $\omega = \pi(x,\xi,t)$ and $\omega' := \vartheta_{T^*,\omega_1,\omega_2}(\omega)$ which we write as $\omega' = \pi(x',\xi',t')$.

We saw above that $\xi = \xi_1^* - f_n(x)$. Comparing this to the symbolic coordinates of ω' found above, and recalling that $f(x) = f(x_0, x_1)$, we see that

$$\begin{aligned}\|\xi' - \xi\| &\leq \|\xi_1^* - \xi_2^*\| + \|f_m(x') - f_n(x)\| \\ &\leq \|\xi_1^* - \xi_2^*\| + (L'' + M_{br}) \max \|f\| \ (\because (3.6)).\end{aligned}$$

Since by assumption $\text{dist}(\omega_1^*, \omega_2^*) \leq T_0$, $\|\xi_1^* - \xi_2^*\|$ is bounded by a constant which only depends on T_0 and the geometry of $\widetilde{M}_0$. This implies that $\|\xi' - \xi\|$ is bounded above by some constant which only depends on T_0, L'' and $\max \|f\|$.

The upper bound on the difference between $\mathbb{Z}^d$–coordinates implies a uniform upper bound E'' on the distance between ω and ω' in T^1M, whence part (3). □

Proof of the Key Lemma. Define once and for all the following constants:

- $\ell_{\min} := \inf\limits_{\omega' \in T^1M} \ell(W^{ss}_{\text{loc}}(\omega')) = \min \psi$, $d_{\max} := \sup\limits_{\omega' \in T^1M} \text{diam}[W^{ss}_{\text{loc}}(\omega')]$ (with the diameter measured using the intrinsic horocycle metric);
- $T_0 := 100 d_{\max}$;
- $C_r := \max\{C'_r, C''_r\}$, where C'_r, C''_r are as in lemmas 3.5 and 3.6 (with T_0 as above);
- $L := \max\{L', L''\}$ where L', L'' are as in lemmas 3.5 and 3.6 (with T_0 as above);
- $E := \max\{E', E''\}$ where E', E'' are as in lemmas 3.5 and 3.6 (with T_0 as above);
- $K^* := 100L(\max r^* + \max |h| + \max |r| + \max \|f\| + \text{var}(r))$, where r, r^*, f are as in lemma 2.1, L as in lemma 3.3;
- Recall that $\widetilde{M}_0 \subset T^1M$ is a fundamental domain for the action of the covering group on T^1M. Fix $C = C(\widetilde{M}_0, E)$ so large that
$$\text{dist}(\omega, \widetilde{M}_0) < 10E \Longrightarrow \omega \in \bigcup_{\|\underline{a}\| < C} D_{\underline{a}}(\widetilde{M}_0) =: \widetilde{M}_0(C),$$
where $\{D_\xi : \xi \in \mathbb{Z}^d\}$ are the deck transformations of the cover.

Finally, fix some arbitrarily small $\varepsilon_0 > 0$.

Step 1. Application of Egoroff's theorem, and choice of $N(\varepsilon_0)$.

Recall that $\frac{1}{T}\xi(g^T\omega) \to \Xi_\varphi$ as $T \to \infty$ m_φ–almost everywhere. Symbolically, this means that

$$\frac{f_n(x)}{r_n^*(x)} \xrightarrow[n\to\infty]{} \Xi_\varphi \text{ for } m_\varphi\text{–a.e. } \omega = \pi(x, \xi, t).$$

The reason is that this fraction is asymptotic to $\frac{1}{T_n}[\xi(g^{T_n}\omega) - \xi(\omega)]$, where $T_n =$ is the n-th hitting time to the Poincaré section.

We construct a large set Λ where the limit is nearly achieved in finite time. There exists an $N = N(\varepsilon_0)$ s.t. $m_\varphi\{\omega \in \widetilde{M}_0(C) : \omega = \pi(x, \xi, s)\ ,\ \big\|\frac{f_N(x)}{r_N^*(x)} - \Xi_\varphi\big\| \geq \varepsilon_0\} < \varepsilon_0$. Better yet, since $r_n^* \geq \lfloor n/n_0 \rfloor \inf r_{n_0}^*$ tends to infinity uniformly and f is bounded, we can take $N = N(\varepsilon_0, K^*)$ so large that the set

$$\Lambda(\varepsilon_0) := \left\{\omega = \pi(x, \xi, s) \in T^1M : \left\|\frac{f_N(x) + e_f}{r_N^*(x) + e_r} - \Xi_\varphi\right\| < \varepsilon_0 \text{ for all } |e_r|, \|e_f\| < K^*\right\}$$

satisfies $m_\varphi[\Lambda(\varepsilon_0) \cap \widetilde{M}_0(C)] > (1 - \varepsilon_0) m_\varphi[\widetilde{M}_0(C)]$ and $m_\varphi(\Lambda(\varepsilon_0) \cap \widetilde{M}_0) > 1 - \varepsilon_0$. This completes step 1.

We know that $\Lambda(\varepsilon_0)$ is large with respect to m_φ. The next step is to study the size of the intersection of $\Lambda(\varepsilon_0)$ with the horocycle of ω_0. Recall from (3.1) the definition of A_T and λ_T, and define λ_T^C to be the length measure on $A_T \cap \widetilde{M}_0(C)$, normalized s.t. $\lambda_T^C[\widetilde{M}_0] = 1$:

$$\lambda_T^C \quad := \quad \frac{1}{\int_0^T 1_{\widetilde{M}_0}(h^t(\omega_0))dt} \int_0^T 1_{\widetilde{M}_0(C)}(h^t\omega_0)\delta_{h^t(\omega_0)}dt.$$

Note that λ_T is a probability measure, but λ_T^C is not.

Step 2. If ω_0 is generic for m_φ, then there exists $\tau = \tau(\varepsilon_0)$ such that for all $T \geq \tau$, $\lambda_T(\widetilde{M}_0 \setminus \Lambda(\varepsilon_0)) \leq \lambda_T^C(\widetilde{M}_0(C) \setminus \Lambda(\varepsilon_0)) < 2\varepsilon_0$.

Proof. Set $\Lambda = \Lambda(\varepsilon_0)$, $N = N(\varepsilon_0)$. The discontinuities of the map $\omega \mapsto \frac{f_N(x(\omega))}{r_N^*(x(\omega))}$ are contained in the union of the boundaries of all sets of the form $\pi\{(x, \xi, t) : x \in [a_0, \ldots, a_{N-1}], \xi \in \mathbb{Z}^d, 0 \leq t < r^*(x)\}$. The Bowen–Series coding of [**Se1**, **Se2**] has the property that these boundaries are geodesics arcs. Therefore, m_φ assigns to this set measure zero.

Thus for every $\varepsilon > 0$, we can construct sets $F_\varepsilon \subset U_\varepsilon \subset \Lambda \cap \widetilde{M}_0(C)$ such that F_ε is compact, U_ε is open, and $m_\varphi[\Lambda \cap \widetilde{M}_0(C) \setminus F_\varepsilon] < \varepsilon$. Urysohn's lemma provides a continuous function $0 \leq \rho_\varepsilon(\cdot) \leq 1$ such that $\rho_\varepsilon = 1$ on F_ε and $\rho_\varepsilon = 0$ outside U_ε (whence outside Λ). This function has compact support, and

$$\int \rho_\varepsilon dm_\varphi > m_\varphi(\Lambda \cap \widetilde{M}_0(C)) - \varepsilon > 1 - (\varepsilon_0 + \varepsilon).$$

Next construct a continuous function with compact support $\theta_\varepsilon(\cdot)$ which approximates the indicator function of $\widetilde{M}_0$ from above in the sense that $0 \leq \theta_\varepsilon \leq 1$, $\theta_\varepsilon = 1$ on the closure of $\widetilde{M}_0$, and $1 < \int \theta_\varepsilon dm_\varphi < 1 + \varepsilon$.

By construction, and since ω_0 is assumed to be m_φ–generic,

$$\lambda_T^C(\Lambda) = \frac{\int_0^T 1_{\Lambda\cap\widetilde{M}_0(C)}(h^t\omega_0)dt}{\int_0^T 1_{\widetilde{M}_0}(h^t\omega_0)dt} \geq \frac{\int_0^T \rho_\varepsilon(h^t\omega_0)dt}{\int_0^T \theta_\varepsilon(h^t\omega_0)dt} \xrightarrow[T\to\infty]{} \frac{\int \rho_\varepsilon dm_\varphi}{\int \theta_\varepsilon dm_\varphi} > \frac{1-(\varepsilon_0+\varepsilon)}{1+\varepsilon}.$$

Thus there exists τ_ε such that for all $T > \tau_\varepsilon$, $\lambda_T^C(\Lambda) > [1-(\varepsilon_0+\varepsilon)]/(1+\varepsilon)$. Choosing ε sufficiently small, we get that $\lambda_T^C(\Lambda) > 1 - 2\varepsilon_0$ for all $T > \tau_\varepsilon$ sufficiently large. Equivalently, $\lambda_T^C(\widetilde{M}_0(C) \setminus \Lambda) < 2\varepsilon_0$ for all $T > \tau_\varepsilon$. Note that τ_ε is a function of ε_0. This finishes the proof of the second step.

The key lemma calls for the estimation of $\lambda_T(\widetilde{M}_0 \cap \{r_{nN}^* < \ln T^* - K\} \setminus \Lambda_n)$ for suitable choice of $K = K(\varepsilon_0)$, where here and throughout $N = N(\varepsilon_0), \Lambda = \Lambda(\varepsilon_0)$, $\Lambda_n = \Lambda_n(N(\varepsilon_0), \varepsilon_0)$, $T^* := \ln(T/T_0)$ (with T_0 as above), and

$$\{r_{nN}^* < T^* - K\} := \{\omega \in T^1M : r_{nN}^*(x(\omega)) < T^* - K\}.$$

Our choice for K is $K = K(\varepsilon_0) := 10(N + L)\max r^*$. We estimate the λ_T–measure of $\widetilde{M}_0 \cap \{r_{nN}^* < T^* - K\} \setminus \Lambda_n$ by partitioning $\widetilde{M}_0 \cap \{r_{nN}^* < T^* - K\} \setminus \Lambda_n(\varepsilon_0)$ into two pieces, which we then treat separately. Define for this purpose

$$\mathcal{W}(\mathrm{I}) \quad := \quad \bigcup W_{\mathrm{loc}}^{ss}(\omega'), \text{ where the union is over all } \omega' \text{ s.t. } W_{\mathrm{loc}}^{ss}(\omega') \subseteq g^{T^*}[A_T],$$

$$\mathcal{W}(\mathrm{II}) \quad := \quad \bigcup W_{\mathrm{loc}}^{ss}(\omega'), \text{ where the union is over all } \omega' \text{ s.t. } W_{\mathrm{loc}}^{ss}(\omega') \not\subseteq g^{T^*}[A_T], \text{ but } \mathcal{W}_{\mathrm{loc}}^{ss}(\omega') \cap g^{T^*}[A_T] \text{ has positive length.}$$

Note that $\mathcal{W}(\mathrm{I})$ and $\mathcal{W}(\mathrm{II})$ are unions of local stable manifolds. Let $\mathcal{N}(\mathrm{I}), \mathcal{N}(\mathrm{II})$ be the minimal number of local stable manifolds needed to cover these sets, up to sets of length zero. Next define

$$\begin{aligned} A_T(\mathrm{I}) &:= A_T \cap \widetilde{M}_0 \cap \{r^*_{nN} < T^* - K\} \cap g^{-T^*}[\mathcal{W}(\mathrm{I})], \\ A_T(\mathrm{II}) &:= A_T \cap \widetilde{M}_0 \cap \{r^*_{nN} < T^* - K\} \cap g^{-T^*}[\mathcal{W}(\mathrm{II})]. \end{aligned}$$

Clearly $A_T \setminus \Lambda_n = [A_T(\mathrm{I}) \cup A_T(\mathrm{II})] \setminus \Lambda_n$. The plan is to fix n, and estimate $\lambda_T[A_T(\mathrm{I}) \setminus \Lambda_n]$ and $\lambda_T[A_T(\mathrm{II}) \setminus \Lambda_n]$.

Step 3. $\sup_n \lambda_T[A_T(\mathrm{I}) \setminus \Lambda_n] \leq 2C_r\varepsilon_0$ *for all* $T > \tau(\varepsilon_0)$.

Proof. We use the holonomy $\kappa_{n,N}$ which exchanges the first and n-th blocks of N symbols in x, see lemma 3.5 above. The idea is to show that

$$\kappa_{n,N}(A_T(\mathrm{I}) \setminus \Lambda_n) \subset A_T \cap (\widetilde{M}_0(C) \setminus \Lambda), \tag{3.7}$$

which implies by Lemma 3.5 and the definition of C_r that

$$\begin{aligned} \lambda_T[A_T(\mathrm{I}) \setminus \Lambda_n] &= \frac{\ell[A_T(\mathrm{I}) \setminus \Lambda_n]}{\ell[A_T \cap \widetilde{M}_0]} \leq C_r \frac{\ell \circ \kappa_{n,N}[A_T(\mathrm{I}) \setminus \Lambda_n]}{\ell[A_T \cap \widetilde{M}_0]} \\ &\leq C_r \frac{\ell[A_T \cap \widetilde{M}_0(C) \setminus \Lambda]}{\ell[A_T \cap \widetilde{M}_0]} \qquad \because (3.7) \\ &= C_r \lambda_T^C(\widetilde{M}_0(C) \setminus \Lambda) < 2C_r\varepsilon_0 \quad \text{(step 2)}, \end{aligned}$$

which proves the step.

We prove (3.7). Suppose $\omega = \pi(x, \xi, s+h(x)) \in A_T(\mathrm{I}) \setminus \Lambda_n$, and let $\omega' := \kappa_{n,N}(\omega)$.

(1) $\omega' \in A_T$: The choice of K is such that $\omega \in \{r^*_{nN} < T^* - K\}$ forces

$$r^*_{(n+1)N+L}(\omega) \leq r^*_{nN}(\omega) + (N+L)\max r^* < T^*.$$

This implies that $x(g^{T^*}\omega)_0^\infty = x_p^\infty$ for $p > (n+1)N + L$.

There exists some $m > 0$ such that $(\kappa^*_{n,N}(x))_m^\infty = x^\infty_{(n+1)N}$. By lemma 3.5, $x(\omega') = \sigma^l(\kappa^*_{n,N}(x))$ with $|l| < L$, so there must exist some $k \geq 0$ such that $x(\omega')_k^\infty = x^\infty_{(n+1)N+L}$, whence $\exists q > 0$ such that $x(\omega')_q^\infty = x_p^\infty = x(g^{T^*}\omega)_0^\infty$. It is not difficult to see that this entails the existence of $T^\# > 0$ s.t.

$$g^{T^\#}(\omega') \in W^{ss}_{\mathrm{loc}}(g^{T^*}\omega) \subset \mathcal{W}(\mathrm{I}) \subset g^{T^*}[A_T].$$

Thus $\omega' \in g^{T^* - T^\#}(A_T)$. But image of $\kappa_{n,N}$ is always in $\mathrm{Hor}(\omega_0)$, so we must have $T^\# = T^*$, and $\omega' \in A_T$.

(2) $\omega' \in \widetilde{M}_0(C)$: See lemma 3.5 part 3, and the choice of E and C.

(3) $\omega' \notin \Lambda$: By construction, the first N-block of $x(\omega')$ differs from the n-th N–block of $x(\omega)$ by at most $2|l| < 2L$ symbols. This means that

$$\begin{aligned} \|f_N(x(\omega')) - f_N(\sigma^{Nn}x(\omega))\| &\leq 2L\max\|f\| < K^* \\ |r^*_N(x(\omega')) - r^*_N(\sigma^{Nn}x(\omega))| &\leq |r_N(x(\omega')) - r_N(x(\omega))| + 4\max|h| \\ &\leq 2L\max|r| + \mathrm{var}(r) + 4\max|h| < K^*. \end{aligned}$$

Had ω' been in Λ, then it would have followed from these estimates that $\|\frac{f_N(\sigma^{Nn}x(\omega))}{r^*_N(\sigma^{Nn}x(\omega))} - \Xi_\varphi\| < \varepsilon_0$, contrary to the assumption that $\omega \notin \Lambda_n$.

This finishes the proof of (3.7), and completes the step.

Step 4. $\sup_n \lambda_T[A_T(\mathrm{II}) \setminus \Lambda_n] \leq \frac{4C_r^2 d_{\max}}{\ell_{\min}}\varepsilon_0$ *for all* $T > \tau(\varepsilon_0)$.

Proof. Fix $W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1) \subset \mathcal{W}(\mathrm{II})$ and $W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2) \subset \mathcal{W}(\mathrm{I})$. We use $\vartheta_{T^*,\omega_1,\omega_2}$ to map $g^{-T^*}W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1)$ into $g^{-T^*}W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2) \in g^{-T^*}[\mathcal{W}(\mathrm{I})] \subset A_T$, and then apply $\kappa_{N,n}$ to claim that

$$\kappa_{n,N} \circ \vartheta_{T^*,\omega_1,\omega_2}\left(g^{-T^*}W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1) \cap \widetilde{M}_0 \cap \{r^*_{nN} < T^* - K\} \setminus \Lambda_n\right) \subset A_T \cap \widetilde{M}_0(C) \setminus \Lambda. \quad (3.8)$$

Suppose this were proved. Using the bounds on the Radon-Nikodym derivatives of $\kappa_{n,N}$ and $\vartheta_{T^*,\omega_1,\omega_2}$, we can then deduce that

$$\lambda_T\left(g^{-T^*}W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1) \cap \widetilde{M}_0 \cap \{r^*_{nN} < T^* - K\} \setminus \Lambda_n\right) \leq C_r^2 \lambda_T^C(\widetilde{M}_0(C) \setminus \Lambda) < 2C_r^2\varepsilon_0.$$

Fixing $W^{ss}_{\mathrm{loc}}(\omega_2)$, and summing over all $W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1)$ needed to cover $\mathcal{W}(\mathrm{II})$ up to sets of length zero, we get $\lambda_T\left(A_T(\mathrm{II}) \setminus \Lambda_n\right) \leq 2C_r^2\mathcal{N}(\mathrm{II})\varepsilon_0$.

We claim that $\mathcal{N}(\mathrm{II}) \leq 2d_{\max}/\ell_{\min}$. To see this note that every local stable manifold in $\mathcal{W}(\mathrm{II})$ has length at least $\min\psi$, and is located at most $d_{\max}$ units of distance away from the endpoints of $g^{T^*}(A_T)$ (otherwise it would be contained in $g^{T^*}(A_T)$, which it is not). Since local stable manifolds are either equal or disjoint up to sets of length zero, this means that $\mathcal{N}(\mathrm{II}) \leq 2d_{\max}/\ell_{\min}$.

The step follows from this. Thus it is enough to prove (3.8).

The argument is similar to the one we used in step 3. Fix $\omega \in g^{-T^*}W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_1) \cap \widetilde{M}_0 \cap \{r^*_{nN} < T^* - K\} \setminus \Lambda_n$ and set $\omega' := (\kappa_{n,N} \circ \vartheta_{T^*,\omega_1,\omega_2})(\omega)$.

(1) $\omega' \in A_T$: Set $\omega'' := \vartheta_{T^*,\omega_1,\omega_2}(\omega)$. Then $\omega'' \in g^{-T^*}W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2)$, and $\exists k > 0$, $l \in [-L, L]$ such that $|r^*_k(x(\omega)) - T^*| < 5(\max r^* + \max|h|)$, $x(\omega'')^{\infty}_{k+M_{br}+l} = x(\omega_2)^{\infty}_k$, and $x(\omega'')^{k+l}_l = x(\omega)^k_0$ (lemma 3.6). This k is larger than $(n+1)N + 2L$, because by choice of K,

$$\begin{aligned} r^*_{(n+1)N+2L}(x(\omega)) &\leq r^*_{nN}(x(\omega)) + (N + 2L)\max r^* \\ &< T^* - 5(\max r^* + \max|h|), \end{aligned}$$

and $r^*_k(x(\omega)) > T^* - 5(\max r^* + \max|h|)$. Thus the tail $x(\omega_2)^{\infty}_k$ survives $\kappa_{n,N}$, which means that $x(\omega_2)^{\infty}_k$ appears as some tail of $x(\omega')$. This means that there exists $T^{\#} > 0$ such that $g^{T^{\#}}(\omega') \in W^{ss}_{\mathrm{loc}}(g^{T^*}\omega_2) \subset g^{T^*}(A_T)$. As before $T^{\#}$ must be equal to T^*, whence $\omega' \in A_T$.

(2) $\omega' \in \widetilde{M}_0$: $\vartheta_{T^*,\omega_1,\omega_2}$ moves points at most E'' units of distance. $\kappa_{n,N}$ moves points at most E' units of distance. In total we move at most $2E$ units of distance from $\widetilde{M}_0$, which still leaves us well inside $\widetilde{M}_0(C)$.

(3) $\omega' \notin \Lambda$: By construction, the first N–block of $x(\omega')$ differs from the n-th N-block of x by at most $4L$ coordinates (the edge effects of two shifts by at most L units). Similar considerations to those used above show that had ω' belonged to Λ, then ω would have had to belong to Λ_n, a contradiction.

(3.8) is proved and step 4 is done.

Steps 3 and 4 imply the key lemma, with $C_0 := 2C_r + 4C_r^2 d_{\max}/\ell_{\min}$, and $K(\varepsilon_0), T_0, N(\varepsilon_0), \tau(\varepsilon_0)$ as above. □

4. Proof that a vector with asymptotic cycle in $int(\mathfrak{C})$ is generic

The purpose of this section is to prove

Proposition 4.1. *For every $\varepsilon > 0$ and all non-negative, non-identically zero continuous functions f, g, there is a neighborhood $K_\varphi \ni \Xi_\varphi$ in $\mathbb{R}^d$ s.t. if T is large enough then*

$$\frac{\xi_{\ln T}(\omega)}{\ln T} \in K_\varphi \Longrightarrow e^{-\varepsilon}\frac{\int f dm_\varphi}{\int g dm_\varphi} \leq \frac{\int_0^T f(h^t\omega)dt}{\int_0^T g(h^t\omega)dt} \leq e^{\varepsilon}\frac{\int f dm_\varphi}{\int g dm_\varphi}. \tag{4.1}$$

Thus every vector with asymptotic cycle Ξ_φ is generic for m_φ. Indeed we have the stronger statement that if there is a sequence $T_n \to \infty$ such that $\xi_{T_n}(\omega)/T_n \to \Xi_\varphi$, then $\int_0^{e^{T_n}} f(h^t\omega)dt / \int_0^{e^{T_n}} g(h^t\omega)dt \to \int f dm_\varphi / \int g dm_\varphi$ for all f, g as above.

Another way of looking at this is to say that if $\xi_T(\omega)/T$ has more than one accumulation point in $int(\mathfrak{C})$ then $\int_0^T f(h^t\omega)dt / \int_0^T g(h^t\omega)dt$ oscillates without converging, and therefore cannot be generic for any measure. This agrees with proposition 3.1, but it does not prove it, because of the lack of information on ω's such that $\xi_T(\omega)/T \to \partial\mathfrak{C}$.

In what follows, we fix $\varepsilon > 0$ and look for $K_\varphi = K_\varphi(f, g, \varepsilon)$ as above.

Modification of the Coding. Fix some small $\varepsilon^* = \varepsilon^*(\varepsilon) > 0$, to be determined later. Recall the symbolic coding of section 2, in particular the definitions of ψ and Ψ_φ. Let $d_{\max}$ denote the maximal diameter of a symbolic local stable manifold, measured in the intrinsic metric of the horocycle which contains it.

We claim that we can change the coding to ensure that the new roof function and symbolic strong stable manifolds satisfy

$$\begin{aligned}
\max r^*_{\text{new}} &< \varepsilon^*, \\
\max |h_{\text{new}}| &< \varepsilon^* \\
(d_{\max})_{\text{new}} &< \varepsilon^*, \\
\max \psi_{\text{new}} &< \varepsilon^*,
\end{aligned}$$

$$\operatorname{diam}\bigl(\pi_{\text{new}}\{(x,\xi,s) : x_0 = a_0, \xi = \xi_0, 0 \leq s < r^*_{\text{new}}(x)\}\bigr) < \varepsilon^* \text{ for all } a_0, \xi_0.$$

Moreover, we claim that this modification can be done in such a way that

$$\frac{\max \psi_{\text{new}}}{\min \psi_{\text{new}}}, \frac{\max(\Psi_\varphi)_{\text{new}}}{\min(\Psi_\varphi)_{\text{new}}} < C_\varphi$$

where C_φ does not depend on ε^* or ε.

Here is how to do this.

The coding in lemma 2.1 is based on finding a Poincaré section for the *geodesic* flow, with a Markov section map T. Take a power $\alpha_L := \bigvee_{i=-L}^{L} T^{-i}\alpha$ of the Markov partition α of this section, with L so large that

$$V(L) := \sum_{k=L}^{\infty} \sup\{|r(x) - r(y)| : x_{-k}^k = y_{-k}^k\} < \tfrac{1}{2}\varepsilon^*.$$

Such L exist because r is Hölder continuous.

Increase the Poincaré section by adding to it the sets

$$g^{k\varepsilon^*/2}(A), \text{ for } A \in \alpha_L \text{ and all } k = 1, \ldots, \lfloor 2 \min_A r/\varepsilon^* \rfloor.$$

This is again a Poincaré section for the geodesic flow. It can be used to code the geodesic flow as a special flow over a (new) subshift of finite type with roof function r^*_{new} such that $\max r^*_{\text{new}} \leq \varepsilon^*/2 < \varepsilon^*$ and $\text{var}(r^*_{\text{new}}) \leq V(L) < \varepsilon^*$.

Recall that h_{new} is (any) Hölder continuous function such that $r_{\text{new}} := r^*_{\text{new}} + h_{\text{new}} \circ \sigma_{\text{new}} - h_{\text{new}}$ only depends on the non-negative coordinates (w.r.t Σ_{new}). There are explicit constructions of such functions which lead to functions h_{new} such that $|h_{\text{new}}| \leq \text{var}(r^*_{\text{new}})$, see for example the proof of lemma 1.6 in [**Bo2**]. Since $\text{var}(r^*) < \varepsilon^*$, this gives h_{new} with $|h_{\text{new}}| < \varepsilon^*$.

We claim that if L is large enough, then the diameter and length of all symbolic manifolds in the new coding are less than $\varepsilon^*/2$, whence $(d_{\max})_{\text{new}}, \max(\psi_{\text{new}}) < \varepsilon^*$.

To see this observe that for every x, the sets $\pi\{(y, \xi, s + h(y)) : y^L_{-L} = x^L_{-L}\}$ decrease to $\{\pi(x, \xi, s + h(x))\}$ as $L \to \infty$. Thus for every x there exists $L(x)$ such that the length and diameter of

$$\pi\left(\{(y, \xi, s) : y^{L(x)}_{-L(x)} = x^{L(x)}_{-L(x)}, \xi = \xi_0, 0 < s < \varepsilon^*/2\}\right)$$

are less than ε^*. (This $L(x)$ does not depend on ξ_0, because the deck transformations are isometries.) A compactness argument shows that $L(x)$ can be chosen to be uniform in x.

We check that the ratio of the maxima and minima of ψ_{new} and $(\Psi_\varphi)_{\text{new}}$ remains bounded by a constant independent of L, ε^* and ε.

First note that our modification of the coding does not change the value of $P(u_\varphi)$, because as mentioned above the diffeomorphism $P(\cdot)$ does not depend on the coding (it can be described by a 'section–free' formula, see [**BL2**]).

Unfortunately $(\Psi_\varphi)_{\text{new}} : \Sigma^+_{\text{new}} \to \mathbb{R}$ is affected by the coding, because it is defined to be the normalized eigenfunction of $L^{\text{new}}_{-P(u_\varphi),u_\varphi} : C(\Sigma^+_{\text{new}}) \to C(\Sigma^+_{\text{new}})$ and $r_{\text{new}}, f_{\text{new}}$ and Σ^+_{new} are different from r, f and Σ^+.

The first observation is that any change which is solely due to refining the Markov partition does not affect the minimum of maximum of Ψ_φ, because the subshift of finite type it generates is conjugate to the original subshift. It is therefore enough to focus on changes in the roof function r and the Poincaré section.

The modified Poincaré section can be coded using a subshift of finite type Σ^+_{new} with two types of states: the old set of states S of Σ^+, and a new set of states S' each of which has a unique predecessor (i.e. a state which can lead to it). It is easy to see that $\max r_{\text{new}} \leq \max r$, and $\max \|f_{\text{new}}\| \leq \max \|f\|$ (in fact $f_{\text{new}} = 0$ except perhaps on states which lead to S).

One can recover Σ^+ from Σ^+_{new} by inducing: If one induces the shift $\sigma_{\text{new}} : \Sigma^+_{\text{new}} \to \Sigma^+_{\text{new}}$ on the union of the states in S, then one gets a dynamical system isomorphic to $\sigma : \Sigma^+ \to \Sigma^+$. The same induction procedure can be used to recover $L_{-P(u_\varphi),u_\varphi}$ from $L^{\text{new}}_{-P(u_\varphi),u_\varphi}$: If τ is the first return time to the states in S, then $L_{-P(u_\varphi),u_\varphi}$ is conjugate to the operator $F \mapsto [(L^{\text{new}}_{-P(u_\varphi),u_\varphi})^{\tau(x)} F](x)$ on $C(\Sigma^+_{\text{new}})$. Standard results on inducing Ruelle operators imply that this conjugacy maps $(\Psi_\varphi)_{\text{new}}|_{\bigcup S}$ to Ψ_φ. It follows that

$$\max(\Psi_\varphi)_{\text{new}}|_S = \max \Psi_\varphi \text{ and } \min(\Psi_\varphi)_{\text{new}}|_{\bigcup S} = \min \Psi_\varphi.$$

Consider now points $x \in [a] \subset \bigcup S'$. Let k be the minimal natural number such that $x = \sigma^k(y)$ and $y \in \bigcup S$. Since every state in S' has exactly one predecessor, and since $\Psi_\varphi = L_{-P(u_\varphi),u_\varphi}\Psi_\varphi$,

$$\Psi_\varphi(x) = e^{-P(u_\varphi)(r_{\text{new}})_k(y)+\langle u_\varphi,(f_{\text{new}})_k(y)\rangle}\Psi_\varphi(y).$$

It is easy to verify that $|(r_{\text{new}})_k(y)| \leq |r^*(y)| + 2\max|h|$ and that $(f_{\text{new}})_k(y)$ is either equal to zero or to $f(y)$. We conclude that

$$\begin{aligned}\Psi_\varphi(x) &\leq e^{|P(u_\varphi)|(\max r^*+2\max|h|)+\|u_\varphi\|\max\|f\|}\max(\Psi_\varphi)_{\text{new}}|_{\bigcup S}\\ \Psi_\varphi(x) &\geq e^{-|P(u_\varphi)|(\max r^*+2\max|h|)-\|u_\varphi\|\max\|f\|}\min(\Psi_\varphi)_{\text{new}}|_{\bigcup S}.\end{aligned}$$

It follows that

$$\frac{\max(\Psi_\varphi)_{\text{new}}}{\min(\Psi_\varphi)_{\text{new}}} \leq \text{const}\times\frac{\max\Psi_\varphi}{\min\Psi_\varphi}$$

where the constant is independent of ε^*, ε. The constant can be taken to be a continuous function of φ. In particular, if φ ranges in a given compact set, then it is bounded.

The bound on $\max\psi_{\text{new}}/\min\psi_{\text{new}}$ can be obtained in exactly the same way. Indeed, $\psi_{\text{new}} = (\Psi_0)_{\text{new}}$ where 0 is the trivial homomorphism, see proposition 4.5 in [**BL2**].

This shows that the new coding is as required. Henceforth we work with this coding, and drop the decorations by 'new'.

Simple Sets. Working with $\varepsilon, \varepsilon^*$ and the symbolic coding of the previous section, we consider the following sets, which we call ε^**–simple sets*:

$$E = \pi\{(x,\xi,s+h(x)) : x \in [\underline{a}] := [\dot{a}_0,\ldots,a_{n-1}], \alpha \leq s < \beta\} \subset \Sigma\times\mathbb{Z}^d\times\mathbb{R}, \quad (4.2)$$

where $0 \leq \alpha \leq \beta \leq \inf_{[\underline{a}]} r^*$. We call $\beta-\alpha$ the *width* of E, and n its *depth.*

Note that $\text{diam}(E) < \varepsilon^*$, and that

$$m_\varphi(E) = \text{const}\, e^{\langle u_\varphi,\xi\rangle}\int_\alpha^\beta e^{-P(u_\varphi)s}ds\int_{[\underline{a}]}\psi d\nu_\varphi$$

with the constant independent of $\varepsilon^*, \varepsilon$.

Lemma 4.2. *If ε^* is sufficiently small, then for any two ε^*–simple sets E_1, E_2, there is a compact neighborhood $K_\varphi \ni \Xi_\varphi$ in $\mathbb{R}^d$ s.t. if T is large enough, then*

$$\frac{\xi_{\ln T}(\omega)}{\ln T} \in K_\varphi \Longrightarrow e^{-\varepsilon}\frac{m_\varphi(E_1)}{m_\varphi(E_2)} \leq \frac{\int_0^T 1_{E_1}(h^t\omega)dt}{\int_0^T 1_{E_2}(h^t\omega)dt} \leq e^{\varepsilon}\frac{m_\varphi(E_1)}{m_\varphi(E_2)}.$$

Proof. Lemma 4.2 is proved by finding the asymptotic behavior of $I_T(\omega,E) := \int_0^T 1_E(h^t\omega)dt$ as $T\to\infty$. Such asymptotics can be derived for *almost every* ω when $\varphi\equiv 0$ from the main lemma of [**LS1**]. Here we describe the necessary modifications to cover $\varphi\not\equiv 0$, and to replace the quantification 'm_φ–almost everywhere' by a condition on $\xi(g^{\ln T}\omega)/\ln T$.

In what follows we assume without loss of generality that $\omega\in\widetilde{M}_0$, and denote $A_T(\omega) := \{h^t(\omega) : 0\leq t\leq T\}$ and $T^* := \ln T$.

Step 1. For all ω and $T > 1$, $\exists N^+, N^- \in\mathbb{N}$ and $\exists\omega_i^* \in g^{T^*}(A_T(\omega)) = A_1(g^{T^*}\omega)$ $(i = 0,\ldots,N^+)$ s.t. if $J_{T^*}(\omega_i^*,E) := \ell[E\cap g^{-T^*}W^{ss}_{\text{loc}}(\omega_i^*)]$, then

$$\sum_{i=0}^{N^-} J_{T^*}(\omega_i^*,E) \leq I_T(\omega,E) \leq \sum_{i=0}^{N^+} J_{T^*}(\omega_i^*,E) \text{ and } 0 < \frac{N^+-N^-}{N^-} < \frac{4C_\varphi\varepsilon^*}{1-2\varepsilon^*}.$$

Proof. We view $I_T(\omega,E)$ as an integral on the horocyclic arc $A_T(\omega)$ with respect to its hyperbolic length measure $\ell=\ell_\omega$:

$$I_T(\omega,E)=\int_0^T 1_E(h^t\omega)dt=\ell_\omega[E\cap A_T(\omega)]=\ell_\omega[E\cap g^{-T^*}A_1(g^{T^*}\omega)],$$

where the last equality is because of the commutation relation between the geodesic and horocycle flows.

Let $W^{ss}_{\mathrm{loc}}(\omega_i^*)$ $(i=1,\dots,N^-)$ be the symbolic local stable manifolds contained in $A_1(g^{T^*}\omega)$. Add to the list the $W^{ss}_{\mathrm{loc}}(\omega_i^*)$ $(i=N^-+1,\dots,N^+)$ which intersect $A_1(g^{T^*}(\omega))$ with positive measure without being contained in it.

Lemma 2.1 part 6 allows us to say that for *every* ω, any two symbolic local stable manifolds are either equal or disjoint up to sets of length zero . Therefore, $I_T(\omega,E)$ can be sandwiched between $\sum_{i=0}^{N^\pm}J_{T^*}(\omega_i^*,E)$ as above.

The N^+-N^- symbolic local stable manifolds which intersect $A_1(g^T\omega)$ without being contained in it must be contained in a $d_{\max}$–neighborhood of the endpoints of $A_1(g^T\omega)$, thus $N^+-N^-<4d_{\max}/\min\psi$ and $N^->(1-2d_{\max})/\max\psi$. Since $\max\psi/\min\psi<C_\varphi$, $0<\frac{N^+-N^-}{N^-}<\frac{4C_\varphi d_{\max}}{1-2d_{\max}}<\frac{4C_\varphi\varepsilon^*}{1-2\varepsilon^*}$.

Step 2. Suppose ω,ω_i^* have symbolic coordinates $(x,0,t+h(x))$, $(x_i^*,\xi_i^*,t_i^*+h(x_i^*))$, and assume $T>e^{\varepsilon^*}$. Set $T_i^{\#}:=T^*-t_i^*$. Then for every ε^*–simple set E,

$$J_{T^*}(\omega_i^*,E)=e^{\pm|\beta-\alpha|}\sum_{k=0}^{\infty}\sum_{\sigma^k(y)=(x_i^*)_0^\infty}1_{[\alpha,\beta]}(r_k(y)-T_i^{\#})\delta_{\xi_i^*,\xi+f_k(y)}1_{[\underline{a}]}(y)\psi(y), \quad (4.3)$$

where the sum ranges over the one–*sided subshift of finite type Σ^+.*

Proof. See step 2 in lemma 1 of [**LS1**].

We note for future reference that $|T_i^{\#}-T^*|=|t_i^*|<\max r^*+\max|h|<2\varepsilon^*$, and that $|\beta-\alpha|\le\max r^*<\varepsilon^*$.

Step 3 [**BL2**]. *For every ε^*–simple set E, there exists a compact neighborhood K_φ^0 of Ξ_φ and $T_0>0$ such that for all $T\ge T_0$ and i,*

$$\frac{\xi_i^*}{T^*}\in K_\varphi^0\Rightarrow J_{T^*}(\omega_i^*,E)=\mathrm{const}\,e^{\pm C_\varphi\varepsilon^*}m_\varphi(E)e^{\pm\varepsilon}\frac{e^{T_i^{\#}H(\frac{\xi_i^*}{T_i^{\#}})}}{(\ln T)^{d/2}}\Psi_\varphi(x_i^*),$$

where $H(\cdot)$ is the minus the Legendre transform of $P(\cdot)$, and $H(\cdot)$, the constant in front of the expression, and C_φ do not depend on $\varepsilon,\varepsilon^*,E$ or ω.

Proof. This is done by estimating the sum (4.3) using an elaboration of Lalley's method [**L**], as in Babillot & Ledrappier [**BL1**] or [**BL2**].[2]

Step 4. There is a function $F_\varphi(\varepsilon,\varepsilon^*)\xrightarrow[\varepsilon,\varepsilon^*\to0^+]{}1$, s.t. for all ε^*–simple sets E_1, E_2, there is a compact neighborhood $K_\varphi\subset K_\varphi^0$ of Ξ_φ and $T_1>0$ such that for all $\omega\in T^1M$ and $T\ge T_1$ with $\frac{\xi_{T^*}(\omega)}{T^*}\in K_\varphi$,

$$\frac{1}{F_\varphi(\varepsilon,\varepsilon^*)}\frac{m_\varphi(E_1)}{m_\varphi(E_2)}\le\frac{I_T(\omega,E_1)}{I_T(\omega,E_2)}\le F_\varphi(\varepsilon,\varepsilon^*)\frac{m_\varphi(E_1)}{m_\varphi(E_2)}.$$

[2] For a detailed account of the calculation in the particular case $\varphi\equiv0$, see the appendix of [**LS1**]. The modifications needed to treat general φ's are routine.

Proof. Let K_φ^0 be an open neighborhood of Ξ_φ so small that the conclusion of step 3 holds for E_1 and E_2. Fix some smaller compact neighborhood $K_\varphi \subset K_\varphi^0$ of Ξ_φ (we shall see later how small), and assume $\frac{\xi_{T^*}(\omega)}{T^*} \in K_\varphi$.

By construction all the ω_i^* belong to a $d_{\max}$–neighborhood of $A_1(g^{T^*}\omega)$, a horocyclic arc of length 1. Their $\mathbb{Z}^d$–coordinates ξ_i^* must therefore be within a bounded distance D from each other and from $g^{T^*}(\omega)$.

As a result, if T is large enough, then $\xi_{T^*}(\omega)/T^* \in K_\varphi$ implies that $\xi_i^*/T^* \in K_\varphi^0$ for all i, and so by step 3

$$\frac{I_T(\omega,E_1)}{I_T(\omega,E_2)} \le e^{2C_\varphi\varepsilon^*}e^{2\varepsilon}\frac{m_\varphi(E_1)}{m_\varphi(E_2)}\left[1+\frac{\sum_{i=N^-+1}^{N^+} e^{T_i^\# H(\frac{\xi_i^*}{T_i^\#})}\Psi_\varphi(x_i^*)}{\sum_{i=1}^{N^-} e^{T_i^\# H(\frac{\xi_i^*}{T_i^\#})}\Psi_\varphi(x_i^*)}\right]. \tag{4.4}$$

To analyze the exponents, we compare $T_i^\# H(\frac{\xi_i^*}{T_i^\#})$ to $T^* H(\frac{\xi_{T^*}(\omega)}{T^*})$. Using the inequality $|T_i^\# - T^*| < 2\varepsilon^*$ and the estimate $\frac{\xi_i^*}{T_i^\#} = \frac{\xi_{T^*}(\omega)}{T^*} + o(1)$, we see that if K_φ is sufficiently small and T is sufficiently large, then

$$\begin{aligned}
&|T_i^\# H(\tfrac{\xi_i^*}{T_i^\#}) - T^* H(\tfrac{\xi_{T^*}(\omega)}{T^*})| \le \\
&\quad\le |T_i^\# - T^*|\cdot|H(\tfrac{\xi_i^*}{T_i^\#})| + T^*|H(\tfrac{\xi_i^*}{T_i^\#}) - H(\tfrac{\xi_{T^*}(\omega)}{T^*})| \\
&\quad\le 2\varepsilon^*[H(\Xi_\varphi)+\varepsilon^*] + (\|(\nabla H)(\Xi_\varphi)\| + \varepsilon^*)T^*\left\|\frac{\xi_i^*}{T_i^\#} - \frac{\xi_{T^*}(\omega)}{T^*}\right\|.
\end{aligned}$$

Now, for T large enough so that $\|\frac{\xi_i^*}{T_i^\#}\| < 2\operatorname{diam} K_\varphi + \|\Xi_\varphi\|$, we have

$$\begin{aligned}
T^*\left\|\frac{\xi_i^*}{T_i^\#} - \frac{\xi_{T^*}(\omega)}{T^*}\right\| &\le T^*\left\|\frac{\xi_i^*}{T_i^\#} - \frac{\xi_i^*}{T^*}\right\| + T^*\left\|\frac{\xi_i^*}{T^*} - \frac{\xi_{T^*}(\omega)}{T^*}\right\| \\
&\le T^*\left\|\frac{\xi_i^*}{T_i^\#}\right\|\left|1 - \frac{T_i^\#}{T^*}\right| + D \\
&\le (2\operatorname{diam} K_\varphi + \|\Xi_\varphi\|)\cdot 2\varepsilon^* + D.
\end{aligned}$$

Consequently, there is a constant k'_φ which only depends on φ such that for all T large enough, if $\frac{\xi(g^{T^*}\omega)}{T^*} \in K_\varphi$, then

$$\exp\left(|T_i^\# H(\tfrac{\xi_i^*}{T_i^\#}) - T^* H(\tfrac{\xi_{T^*}(\omega)}{T^*})|\right) \le \exp\left(k'_\varphi\varepsilon^* + D\|\nabla H(\Xi_\varphi)\|\right) =: k_\varphi(\varepsilon^*).$$

Thus the term in the brackets in (4.4) can be estimated by

$$1 + k_\varphi(\varepsilon^*)^2\frac{\max\Psi_\varphi}{\min\Psi_\varphi}\left(\frac{N^+ - N^-}{N^-}\right) \le 1 + \frac{4\varepsilon^* C_\varphi^2 k_\varphi(\varepsilon^*)^2}{1-2\varepsilon^*},$$

and so $I_T(\omega,E_1)/I_T(\omega,E_2) \le e^{2C_\varphi\varepsilon^*}e^{2\varepsilon}\frac{m_\varphi(E_1)}{m_\varphi(E_2)}F_\varphi(\varepsilon,\varepsilon^*)$ where

$$F_\varphi(\varepsilon,\varepsilon^*) := e^{2C_\varphi\varepsilon^*+2\varepsilon}\left(1 + \frac{4\varepsilon^* C_\varphi k_\varphi(\varepsilon^*)^2}{1-2\varepsilon^*}\right).$$

The lower bound is obtained in the same way, and the step is proved.

We can now prove the lemma. Fix $\widetilde{\varepsilon}$. Choose $\varepsilon, \varepsilon^*$ so small that $e^{-\widetilde{\varepsilon}} < F(\varepsilon, \varepsilon^*) < e^{\widetilde{\varepsilon}}$. Then the lemma follows with $\widetilde{\varepsilon}$ instead of ε. $\square$

Proof of Proposition 4.1. Take ε and ε^* as in lemma 4.2. Call a function an ε^**–step function* if it is a finite linear combination of indicators of ε^*–simple sets.

It is easy to see that lemma 4.2 implies the proposition for every pair of ε^*–step functions f, g with non-zero integrals.

The proposition also holds for all non-negative non-identically zero continuous functions f, g with compact supports which do not intersect the section: Such functions can be sandwiched between ε^*–step functions with almost the same integrals, and ε^* and the width of the simple sets can be chosen arbitrarily small.

We claim that there is an open neighborhood V of the section such that for all non-negative, non-identically zero, continuous functions f, g supported inside V, if T is large enough and $\|\xi_{\ln T}(\omega)/\ln T - \Xi_\varphi\|$ is sufficiently small, then

$$e^{-3\varepsilon}\frac{\int f dm_\varphi}{\int g dm_\varphi} \le \frac{\int_0^T f(h^t\omega)dt}{\int_0^T g(h^t\omega)dt} \le e^{3\varepsilon}\frac{\int f dm_\varphi}{\int g dm_\varphi}.$$

Every point $\widetilde{\omega}$ in the section has a precompact open neighborhood $V_{\widetilde{\omega}}$ and a constant $0 < s(\widetilde{\omega}) < \min\{\varepsilon^*, \varepsilon\}$ such that the closure of $g^{s(\widetilde{\omega})}[V_{\widetilde{\omega}}]$ does not intersect the section. Suppose f_1, f_2 are non-negative, non-identically zero, continuous functions with compact support in $V_{\widetilde{\omega}}$; then $\widetilde{f_i} := f_i \circ g^{-s(\widetilde{\omega})}$ are uniformly continuous functions with compact supports which do not intersect the section. Thus they satisfy (4.1) for all T large enough with some compact neighborhood K_φ of Ξ_φ.

The commutation relation between the geodesic flow and the horocycle flow implies $\int_0^T f_i(h^t\omega)dt = e^{s(\widetilde{\omega})}\int_0^{Te^{-s(\widetilde{\omega})}} \widetilde{f_i}(h^\tau g^{-s(\widetilde{\omega})}\omega)d\tau$. If $\frac{\xi_{\ln T}(\omega)}{\ln T} \in int(K_\varphi)$, then for T large enough $\frac{\xi_{\ln T}(g^{s(\widetilde{\omega})}\omega)}{\ln T} \in int(K_\varphi)$. For such T, $\int_0^T f_1(h^t\omega)dt/\int_0^T f_2(h^t\omega)dt = \int_0^{Te^{-s(\widetilde{\omega})}} \widetilde{f_1}(h^\tau g^{-s(\widetilde{\omega})}\omega)d\tau/\int_0^{Te^{-s(\widetilde{\omega})}} \widetilde{f_2}(h^\tau g^{-s(\widetilde{\omega})}\omega)d\tau = e^{\pm\varepsilon}\int \widetilde{f_1}/\int \widetilde{f_2}$. The integral ratio is equal to $\int f_1/\int f_2$, because $m_\varphi \circ g^\tau = e^{-P(u_\varphi)\tau}m_\varphi$.

Thus (4.1) holds for all f_1, f_2 supported inside $V_{\widetilde{\omega}}$.

Now suppose $\operatorname{supp} f_1 \subset V_{\widetilde{\omega}_1}$ and $\operatorname{supp} f_2 \subset V_{\widetilde{\omega}_2}$ where $\widetilde{\omega}_1 \neq \widetilde{\omega}_2$. Choose non-negative non-identically zero continuous g_i with compact support in $V_{\widetilde{\omega}_i}\setminus$section. Writing

$$\frac{\int_0^T f_1 \circ h^t dt}{\int_0^T f_2 \circ h^t dt} = \frac{\int_0^T f_1 \circ h^t dt}{\int_0^T g_1 \circ h^t dt} \cdot \frac{\int_0^T g_1 \circ h^t dt}{\int_0^T g_2 \circ h^t dt} \cdot \frac{\int_0^T g_2 \circ h^t dt}{\int_0^T f_2 \circ h^t dt},$$

we see that for all T large enough, if $\|\xi_{\ln T}(\omega)/\ln T - \Xi_\varphi\|$ is sufficiently small, then

$$e^{-3\varepsilon}\frac{\int f_1 dm_\varphi}{\int f_2 dm_\varphi} \le \frac{\int_0^T f_1(h^t\omega)dt}{\int_0^T f_2(h^t\omega)dt} \le e^{3\varepsilon}\frac{\int f_1 dm_\varphi}{\int f_2 dm_\varphi}.$$

Let V be the union of $V_{\widetilde{\omega}}$ with $\widetilde{\omega}$ in the section. Every non-negative continuous function with compact support in V is the sum of finitely many non-negative continuous functions supported inside some $V_{\widetilde{\omega}}$. Thus proposition 4.1 holds for all f, g non-negative, non-identically zero, continuous functions supported inside V.

Now set $\mathcal{U}$ to be the family of all non-negative non-identically zero continuous functions with compact support, whose support does not intersect the section. Define $\mathcal{V}$ to be the family of all non-negative non-identically zero continuous functions supported inside V.

We saw that proposition 4.1 holds for all pairs of functions in $\mathcal{U}$, and for all pairs of functions in $\mathcal{V}$. Since $\mathcal{U}\cap\mathcal{V}\neq\varnothing$, proposition 4.1 holds for all pairs of functions in $\mathcal{U}\cup\mathcal{V}$. Since any non-negative continuous function of compact support is the sum of a function from $\mathcal{U}$ and a function from $\mathcal{V}$, proposition 4.1 holds for all non-negative continuous functions of compact support. □

5. A Question

The obvious question is what happens for other hyperbolic surfaces of infinite genus. Theorem 1.1 does not make sense for such surfaces, because the notion of asymptotic cycle is specific to $\mathbb{Z}^d$–covers.

It is desirable to find another criterion which does make sense in general.

The following observation is perhaps a step in this direction. Let Δ denote the Laplace–Beltrami operator on the hyperbolic surface M. A positive eigenfunction $F : M \to \mathbb{R}$ is called *minimal*, if it defines an extremal ray in the cone of positive eigenfunctions with the same eigenvalue. The minimal positive eigenfunctions are known for $\mathbb{Z}^d$–covers [**CG**], [**LP**]: they form a list $\{cF_\varphi : c > 0, \varphi : \mathbb{Z}^d \to \mathbb{R}$ a homomorphism$\}$, where F_φ satisfies $F_\varphi \circ D_\xi = e^{\varphi(\xi)}F_\varphi$ for all deck transformations D_ξ, $\xi \in \mathbb{Z}^d$. The similarity with the collection of Babillot–Ledrappier measures is not a coincidence, see [**LS2**].

Extend $F_\varphi : M \to \mathbb{R}$ to $F_\varphi : T^1M \to \mathbb{R}$ by setting $F_\varphi(\omega) = F_\varphi(\text{base point of }\omega)$. It is easy to see using the precompactness of $\widetilde{M}_0$ and the continuity of F_φ that for all homomorphisms φ,

$$\varphi(\Xi(\omega)) = \lim_{T\to\infty} \frac{1}{T} \ln F_\varphi(g^T\omega).$$

Thus $\Xi(\omega)$ is completely determined by the logarithmic growth of the minimal positive eigenfunctions of Δ along the forward geodesic ray of ω.

In particular, we get the following corollary of theorem 1.1: *Suppose M is a $\mathbb{Z}^d$–cover of a compact hyperbolic surface, then ω is generic for the horocycle flow with respect to the volume measure iff*

$$\lim_{T\to\infty} \frac{1}{T} \ln F(g^T\omega) = 0$$

for every positive minimal eigenfunction F of the Laplace–Beltrami operator of M.

Question: Does the above extend to other, perhaps all, hyperbolic surfaces with the Liouville property?

We remind the reader that a hyperbolic surface is called *Liouville* if all its bounded harmonic functions are constant. We need to assume the Liouville property, because of Kaimanovich's theorem [**K**], which says that the volume measure is ergodic for the horocycle flow on a hyperbolic surface, iff this surface is Liouville.

Note added in proof: F. Ledrappier and M. Pollicott have suggested to us ways to extend the results of this paper to the case when M_0 has variable negative curvature (the "horocycle flow" being replaced by the Margulis parametrization [**Mrg**] of the strong stable foliation, see [**Mrc**]). To do this, replace the Bowen–Series coding of the geodesic flow we used for lemma 2.1 by either Ratner's symbolic coding of the geodesic flow [**Ra1**], or by the coding obtained by making a time change to conjugate the geodesic flow to the geodesic flow of a conformally equivalent metric of constant curvature, as in [**G**] or [**Ka**], section 3. See [**PS2**].

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O. Sarig, Mathematics Department, The Pennsylvania State University, University Park, PA 16802 USA

E-mail address: sarig@math.psu.edu

B. Schapira, L.A.M.F.A. UMR 6140, Faculté de Mathématiques et Informatique, Université Picardie Jules Verne, 33 rue St Leu 80000 Amiens, France

E-mail address: Barbara.Schapira@u-picardie.fr