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Density and equidistribution of half-horocycles on a geometrically finite hyperbolic surface

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Abstract

On geometrically finite negatively curved surfaces, we give necessary and sufficient conditions for a one-sided horocycle $(h^s u)_{s\geq 0}$ to be dense in the nonwandering set of the horocyclic flow. We prove that all dense one-sided orbits $(h^s u)_{s\geq 0}$ are equidistributed, extending results of [**Bu**] and [**Scha2**] where symmetric horocycles $(h^s u)_{-R\leq s\leq R}$ were considered.

1. *Introduction*

Hedlund [**H**] proved that on the unit tangent bundle of a finite volume hyperbolic surface, all nonperiodic positive orbits $(h^s v)_{s\geq 0}$ of the horocyclic flow (called in [**H**] "right-semihorocycles", and here positive half-horocycles) are dense.

On nonelementary *geometrically finite surfaces* of infinite volume, i.e. surfaces (of infinite volume) whose fundamental group is finitely generated and non virtually abelian, the wandering set of the horocyclic flow is nonempty. One restricts therefore the study of the dynamics to the nonwandering set $\mathcal{E}$ of the horocyclic flow. All nonwandering and non periodic full orbits $(h^s v)_{s\in\mathbb{R}}$ of the horocyclic flow are *dense* in the nonwandering set $\mathcal{E}$ of the horocyclic flow. More precisely, on such surfaces, the following trichotomy holds (see section 2 for definitions and details): 1) the horocyclic orbit is periodic iff it is tangent to the boundary at a parabolic point, 2) the horocyclic orbit is wandering iff its point of tangency to the boundary is outside the limit set Λ_Γ of the group, 3) the horocyclic orbit is dense in $\mathcal{E}$ iff it is tangent to the boundary at a horospherical point.

However, as soon as the surface has infinite volume, we can easily find horocycles $(h^s u)_{s\in\mathbb{R}}$ that are globally dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow, but with one side dense and the other not (see figure 2).

In this note, we characterize these horocycles with one side dense and the other not. If $u \in T^1S$, and $\tilde{u}$ is any of its lifts on the unit tangent bundle $T^1\mathbb{D}$ of the hyperbolic disc, we denote by $u^- \in S^1$ (resp. u^+) the negative (resp. positive) endpoint in the boundary $S^1 = \partial\mathbb{D}$ of the geodesic line defined by $\tilde{u}$. We prove:

Theorem 1.1. *Let S be a nonelementary geometrically finite hyperbolic surface. Let $u \in T^1S$ be a vector whose full unstable horocyclic orbit $(h^s u)_{s\in\mathbb{R}}$ is dense in $\mathcal{E}$. Then the positive half-horocycle $(h^s u)_{s\geq 0}$ is dense in $\mathcal{E}$ iff u^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, where the circle S^1 is oriented in the counterclockwise direction, and the limit set $\Lambda_\Gamma \subset S^1$ is the smallest non-empty Γ-invariant closed subset of S^1.*

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On geometrically infinite surfaces, this theorem remains valid for vectors $u \in T^1S$ that are periodic for the geodesic flow (see proposition 3.12).

To prove theorem 1.1, inspired by ideas of [**C**], we introduce the notion of *right horospherical vector*, and prove (proposition 3.9) that on general hyperbolic surfaces, a positive half-horocycle $(h^s u)_{s\geq 0}$ is dense in its nonwandering set $\mathcal{E}$ iff u is a right horospherical vector. We deduce theorem 1.1 from the fact that on geometrically finite surfaces, right horospherical vectors are easy to characterize.

Our initial motivation concerned the equidistribution properties of half-horocycles. Furstenberg's unique ergodicity result [**F**] for the horocyclic flow ensures that on the unit tangent bundle T^1S of a compact hyperbolic surface, all horocyclic orbits are equidistributed towards the unique (h^s)-invariant measure λ: for all $u \in T^1S$ and $f : T^1S \to \mathbb{R}$ continuous, $\frac{1}{T}\int_0^T f \circ h^s u\, ds \to \frac{1}{\lambda(T^1S)}\int_{T^1S} f\, d\lambda$, where λ is the Liouville measure. Of course, the same result holds for $(h^s u)_{s\leq 0}$.

This result was extended by Dani and Smillie [**DS**] to finite volume hyperbolic surfaces: all nonperiodic positive orbits $(h^s u)_{s\geq 0}$ are equidistributed towards λ.

On nonelementary geometrically finite hyperbolic surfaces, there is [**Bu**] [**Ro**], up to normalization, a unique (h^s)-invariant ergodic measure m, sometimes called the Burger-Roblin measure, that has full support in the nonwandering set of (h^s); and it is infinite. Therefore, one considers ratios $\frac{\int_{-T}^{T} f \circ h^s u\, ds}{\int_{-T}^{T} g \circ h^s u\, ds}$ and one can prove [**Bu**][**Scha2**] that they converge to $\frac{\int_{T^1S} f\, dm}{\int_{T^1S} g\, dm}$ for all continuous functions $f, g : T^1S \to \mathbb{R}$ with compact support, and all nonwandering and non periodic vectors $u \in T^1S$. We can now complete the trichotomy of the first page (see lemma 2.3 and remark 3.1): 1) if a horocycle is periodic, the normalized Lebesgue measure on its orbit defines a (h^s)-invariant ergodic probability measure; 2) if it is wandering, the Lebesgue measure on the orbit defines an infinite totally dissipative (h^s)-invariant ergodic measure, and 3) if the horocycle is dense in $\mathcal{E}$, it is equidistributed towards the unique (h^s)-invariant ergodic measure m supported on the full nonwandering set $\mathcal{E}$.

In both articles [**Bu**][**Scha2**], equidistribution is obtained for symmetric horocycles $(h^s u)_{-T\leq s\leq T}$ only, and not for positive horocycles $(h^s u)_{0\leq s\leq T}$. Omri Sarig asked whether similar results hold for one-sided averages. Indeed, symmetric averages are very natural from a geometric point of view, but not from the ergodic point of view, where a difference of behaviour between the negative and the positive orbit is an interesting phenomenon. Motivated by this question, we investigated density and equidistribution properties of half-horocycles.

In theorem 1.1, we characterized horocycles that are dense in $\mathcal{E}$, but have one side dense and the other not. For these horocycles, one cannot hope for equidistribution of both one-sided orbits. However, according to Hopf ergodic theorem (as the measure m is ergodic and conservative), almost all one-sided horocycles should be equidistributed towards m.

On nonelementary geometrically finite hyperbolic surfaces, the above phenomenon of dense horocycles with a non-dense side is the only obstruction to the equidistribution of one-sided horocycles. Indeed, methods of [**Scha1**] and [**Scha2**] apply here to give:

THEOREM 1.2. *Let S be a nonelementary geometrically finite surface, and $u \in T^1S$ such that $(h^s u)_{s\geq 0}$ is dense in the nonwandering set $\mathcal{E}$ of the unstable horocyclic flow. Then $(h^s u)_{s\geq 0}$ is equidistributed towards the unique (up to normalization) (h^s)-invariant ergodic measure m which has full support in the nonwandering set of $(h^s)_{s\in\mathbb{R}}$.*

In other words, for all continuous functions with compact support $f, g : T^1S \to \mathbb{R}$, *with* $\int_{T^1S} g\,dm > 0$, *we have*

$$\frac{\int_0^T f \circ h^s u\,ds}{\int_0^T g \circ h^s u\,ds} \to \frac{\int_{T^1S} f\,dm}{\int_{T^1S} g\,dm}\,, \quad \textit{when} \quad T \to +\infty\,.$$

Note that (h^s)-periodic orbits are obviously equidistributed towards the normalized Lebesgue measure on the orbit. Of course, theorem 1.2 also holds for negative orbits $(h^s u)_{s\le 0}$.

Remark 1.3. In particular, under the above assumptions, the three following properties are equivalent.
– the half-horocycle $(h^s u)_{s\ge 0}$ is equidistributed towards the measure m,
– the half-horocycle $(h^s u)_{s\ge 0}$ is dense in $\mathcal{E}$,
– for all $T \in \mathbb{R}$, the half-horocycle $(h^s u)_{s\ge T}$ is dense in $\mathcal{E}$.
Indeed, the first property implies clearly the two other (equivalent) properties, and the other implication is exactly the result of theorem 1.2.

Most results extend to surfaces of variable negative curvature. However, to avoid too many preliminaries, we postpone the discussion about such surfaces to the end of the paper.

Section 2 is devoted to preliminaries. Theorem 1.1 is proved in section 3, where we also discuss the case of geometrically infinite surfaces, and theorem 1.2 in section 4.

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2. *Preliminaries*

Hyperbolic geometry

The hyperbolic disc $\mathbb{D} = D(0,1)$ is endowed with the metric $\frac{4dx^2}{(1-|x|^2)^2}$. Let o be the origin of the disc. Denote by $\pi : T^1\mathbb{D} \to \mathbb{D}$ the canonical projection. The *boundary at infinity* is $S^1 = \partial\mathbb{D}$.

The map $z \in \mathbb{D} \mapsto \frac{i(1+z)}{1-z}$ is an isometry between $\mathbb{D}$ with the above metric and the upper half plane $\mathbb{H} = \mathbb{R} \times (0, +\infty)$ endowed with the hyperbolic metric $\frac{dx^2+dy^2}{y}$. Therefore, the group of isometries preserving orientation of $\mathbb{D}$ identifies with $PSL(2,\mathbb{R})$ acting by homographies on $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$. An isometry of $PSL(2,\mathbb{R})$ acts also on $T\mathbb{D}$ and $T^1\mathbb{D}$ via its differential. Moreover, the group $PSL(2,\mathbb{R})$ acts simply transitively on the unit tangent bundle $T^1\mathbb{D}$, so that we identify these two spaces through the map which sends the unit vector $(1,0)$ tangent to $\mathbb{D}$ at $o = (0,0)$ on the identity element of $PSL(2,\mathbb{R})$. Let d denote the hyperbolic distance on $\mathbb{D}$ and $\mathbb{H}$.

The *Busemann cocycle* is the continuous map defined on $S^1 \times \mathbb{D}^2$ by

$$\beta_\xi(x,y) := \lim_{z\to\xi} \left(d(x,z) - d(y,z)\right)\,.$$

Define the map $v \in T^1\mathbb{D} \mapsto (v^-, v^+, \beta_{v^-}(\pi(v), o)) \in (S^1 \times S^1) \setminus \mathrm{Diagonal} \times \mathbb{R}$, where $v^\pm$ are the endpoints in S^1 of the geodesic defined by v, and $\pi(v) \in \mathbb{D}$ is the basepoint in S of v. It defines a homeomorphism between $T^1\mathbb{D}$ and $\partial^2\mathbb{D} \times \mathbb{R} := (S^1 \times S^1) \setminus \mathrm{Diagonal} \times \mathbb{R}$, and we shall identify these two spaces in the sequel. An isometry $\gamma \in PSL(2,\mathbb{R})$ acts on $(S^1 \times S^1) \setminus$

Diagonal $\times\mathbb{R}$ by

$$\gamma.(v^-,v^+,t)=(\gamma.v^-,\gamma.v^+,t+\beta_{v^-}(o,\gamma^{-1}.o))\,.$$

Let Γ be a discrete subgroup of $PSL(2,\mathbb{R})$. Its *limit set* Λ_Γ is the set $\Lambda_\Gamma=\overline{\Gamma.o}\setminus\Gamma.o\subset S^1$. It is also the smallest closed Γ-invariant subset of S^1. The group Γ acts properly discontinuously on the *ordinary set* $S^1\setminus\Lambda_\Gamma$, which is a countable union of intervals. On the other hand, we will often use the fact that the action of Γ on Λ_Γ is minimal: for all $\xi\in\Lambda_\Gamma$, $\Gamma.\xi$ is dense in Λ_Γ.

A point $\xi\in\Lambda_\Gamma$ is a *radial* limit point if it is the limit of a sequence $(\gamma_n.o)$ of points of $\Gamma.o$ that stay at bounded hyperbolic distance of the geodesic ray $[o\xi)$ joining o to ξ. Let Λ_{rad} denote the *radial limit set*. The set of points of Λ_Γ fixed by a hyperbolic isometry is a subset of Λ_{rad}.

A *horocycle* of $\mathbb{D}$ is a euclidean circle tangent to S^1. It can also be defined as a level set of a Busemann function. A *horoball* is the (euclidean) disc bounded by a horocycle. The point $\xi\in\Lambda_\Gamma$ is *horospherical* if any horoball centered at ξ contains infinitely many points of $\Gamma.o$. In particular, Λ_{rad} is included in the horospherical set Λ_{hor}.

An isometry of $PSL(2,\mathbb{R})$ is *hyperbolic* if it fixes exactly two points of S^1, it is *parabolic* if it fixes exactly one point of S^1, and *elliptic* in the other cases. Let $\Lambda_{\mathrm{p}}\subset\Lambda_\Gamma$ denote the set of *parabolic* limit points, that is the points of Λ_Γ fixed by a parabolic isometry of Γ.

Any hyperbolic surface is the quotient $S=\Gamma\backslash\mathbb{D}$ of $\mathbb{D}$ by a discrete subgroup Γ of $PSL(2,\mathbb{R})$ without elliptic elements, and its unit tangent bundle T^1S identifies with $\Gamma\backslash PSL(2,\mathbb{R})$.

In this note, we always assume that Γ is *nonelementary*, that is $\#\Lambda_\Gamma=+\infty$. Moreover, we are mainly interested in *geometrically finite surfaces* S, i.e. surfaces whose fundamental group Γ is finitely generated. In such cases, the limit set Λ_Γ is the disjoint union of Λ_{rad} and Λ_{p} [**Bow**]. Moreover, the surface is a disjoint union of a compact part C_0, finitely many cusps (isometric to $\{z\in\mathbb{H},\ Im\,z\geq cst\}/\{z\mapsto z+1\}$), and finitely many 'funnels' (isometric to $\{z\in\mathbb{H},\ Re(z)\geq 0,\,1\leq|z|\leq a\}/\{z\mapsto az\}=\{z\in\mathbb{H},\ Re(z)\geq 0\}/\{z\mapsto az\}$, for some $a>1$.

When S is compact, $\Lambda_\Gamma=\Lambda_{\mathrm{rad}}=S^1$; the surface S is called *convex-cocompact* when it is a geometrically finite surface without cusps. In this case, $\Lambda_\Gamma=\Lambda_{\mathrm{rad}}$ is strictly included in S^1 and Γ acts cocompactly on the set $(\Lambda_\Gamma\times\Lambda_\Gamma)\setminus$ Diagonal $\times\mathbb{R}\subset T^1\mathbb{D}$. When S has finite volume, there are no funnels and $\Lambda_\Gamma=\Lambda_{\mathrm{rad}}\sqcup\Lambda_{\mathrm{p}}=S^1$.

Geodesic and horocycle flows

A hyperbolic geodesic in $\mathbb{D}$ is a diameter or a half-circle orthogonal to S^1. A vector $v\in T^1\mathbb{D}$ is tangent to a unique geodesic of $\mathbb{D}$. Moreover, it is orthogonal to exactly two horocycles passing through its basepoint $\pi(v)$, and tangent to S^1 respectively at v^+ and v^-. The set of vectors $w\in T^1\mathbb{D}$ such that $w^-=v^-$ and based on the same horocycle tangent to S^1 at v^- is the *strong unstable horocycle* or strong unstable manifold $W^{su}(v)\subset T^1\mathbb{D}$ of v. It satisfies

$$W^{su}(v)=\{w\in T^1\mathbb{D},\,d(g^{-t}v,g^{-t}w)\to 0\text{ when }t\to+\infty\,\}\,.$$

The *strong stable manifold* $W^{ss}(v)$ is defined in the same way.

The *geodesic flow* $(g^t)_{t\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector v of a distance t along its geodesic. In the identification of $T^1\mathbb{D}$ with $PSL(2,\mathbb{R})$, this flow corresponds to the right action by the one-parameter subgroup

$$\left\{a_t:=\begin{pmatrix} e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix},\,t\in\mathbb{R}\right\}.$$

The *strong unstable horocyclic flow* $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector v of a distance $|s|$ along its strong unstable horocycle. There are two possible orientations for this flow, and we consider the choice corresponding to the right action by the one parameter subgroup

$$\left\{n_s:=\begin{pmatrix} 1 & 0\\ s & 1\end{pmatrix},\,s\in\mathbb{R}\right\}$$

on $PSL(2,\mathbb{R})$. This flow turns vectors along their strong unstable horocycle, so that $\{h^s v,\, s \in \mathbb{R}\} = W^{su}(v)$.

Moreover, for all $s \in \mathbb{R}$ and all $t \in \mathbb{R}$, geodesic and horocyclic flows satisfy the crucial relation :

$$g^t \circ h^s = h^{se^t} \circ g^t \,. \tag{2.1}$$

REMARK 2.1. With our choice of orientation of S^1, when $s \to +\infty$, if $u \in T^1\mathbb{D}$ and $u_s^+ \in S^1$ is the positive endpoint of the geodesic defined by $h^s u$, then u_s^+ converges to u^-, with $u_s^+ \geq u^-$ in the counterclockwise orientation of S^1.

These two right-actions are well defined on the quotient space $T^1 S \simeq \Gamma \backslash PSL(2,\mathbb{R})$.

DEFINITION 2.2. *Let $(\phi^t)_{t\in\mathbb{R}}$ be a flow acting continuously on a topological space X. The* nonwandering set *of ϕ is the set of points $x \in X$ such that for all neighbourhoods V of x, there exists a sequence $t_n \to +\infty$ such that $\phi^{t_n} V \cap V \neq \varnothing$.*

LEMMA 2.3. *Let $u \in T^1 S$ be a vector. The following trichotomy holds:*
(i) *If $u^- \in \Lambda_{\mathrm{p}}$, $(h^s u)_{s\in\mathbb{R}}$ is periodic.*
(ii) *If $u^- \in \Lambda_{\mathrm{hor}}$, $(h^s u)_{s\in\mathbb{R}}$ is dense in the set $\mathcal{E}$ of vectors $v \in T^1 S$, such that $v^- \in \Lambda_\Gamma$.*
(iii) *If $u^- \notin \Lambda_\Gamma$, $(h^s u)_{s\in\mathbb{R}}$ is embedded in $T^1 S$ and diverges to an end of the surface.*

In particular in case (3), the horocycle has no accumulation points, and the Lebesgue measure on $(h^s u)_{s\in\mathbb{R}}$ is an infinite (h^s)-invariant ergodic totaly dissipative locally finite measure.

Proof. This result follows easily from classical results of Hedlund [**H**], Eberlein [**E1**] [**E2**] and Dal'bo [**Da**] in the general case. □

LEMMA 2.4. *The nonwandering set of the geodesic flow acting on $T^1 S$ is*

$$\Omega := \Gamma \backslash \left((\Lambda_\Gamma \times \Lambda_\Gamma) \setminus \text{Diagonal} \times \mathbb{R} \right).$$

The nonwandering set of the horocyclic flow acting on $T^1 S$ is

$$\mathcal{E} := \Gamma \backslash \big((\Lambda_\Gamma \times S^1) \setminus \text{Diagonal} \times \mathbb{R} \big).$$

It satisfies

$$\mathcal{E} = \cup_{s\in\mathbb{R}} h^s \Omega.$$

Proof. For the geodesic flow, the result is classical (see [**E1**]). For the horocyclic flow, it follows from the above lemma. The relation between $\mathcal{E}$ and Ω is elementary. □

In the proof of theorem 1.1, we will need the following lemma.

LEMMA 2.5. *We have $\overline{\cup_{s\geq 0} h^s(\Omega)} = \mathcal{E}$.*

Proof. Let $v \in \mathcal{E} \setminus (\cup_{s\geq 0} h^s \Omega)$, and $\tilde{v} = (v^-, v^+, t)$ be a lift of v to $T^1\mathbb{D}$. It means that $v^- \in \Lambda_\Gamma$, $v^+ \notin \Lambda_\Gamma$, and there is no $\tilde{w} = (v^-, w^+, t) \in \widetilde{\Omega}$, such that $\tilde{v} = h^s(\tilde{w})$, for some $s \geq 0$. (The set $\widetilde{\Omega} = (\Lambda_\Gamma \times \Lambda_\Gamma) \setminus \text{Diagonal} \times \mathbb{R}$ is simply the lift of the nonwandering set Ω of the geodesic flow to $T^1\mathbb{D}$.) In other words, in the interval $[v^+, v^-] \subset S^1$ (where the circle is oriented in the counterclockwise direction), there is no point of Λ_Γ, except v^- of course.

Consider a sequence $\tilde{v}_n \in \mathcal{E}$ converging to $\tilde{v}$, of the form $\tilde{v}_n = (v_n^-, v^+, t)$, $v_n^- \to v^-$, $v_n^- \neq v^-$, $v_n^- \in \Lambda_\Gamma$. As $\Lambda_\Gamma \cap [v^+, v^-] = \{v^-\}$, necessarily, v_n^- is greater than v^-, with the counterclockwise orientation of the circle. Consider now the sequence of vectors $\tilde{w}_n = (v_n^-, v^-, t)$. These vectors are all in $\widetilde{\Omega}$, and satisfy $\tilde{v}_n = h^{s_n}\tilde{w}_n$, $s_n \geq 0$. In particular, $\tilde{v} = \lim_{n\to\infty} h^{s_n}\tilde{w}_n$, so that the lemma is proved. □

Nonarithmeticity of the length spectrum

In our situation (nonelementary hyperbolic surfaces) we know that the *length spectrum* of the fundamental group Γ of S is nonarithmetic, that is the set $\{l(\gamma)\}$ of lengths of closed geodesics generates a dense subgroup of $\mathbb{R}$. We will use this crucial fact in the sequel.

Local product structure of the geodesic flow

The geodesic flow on the unit tangent bundle of any hyperbolic surface (including $\mathbb{D}$) is a *hyperbolic flow*. In particular, it has a (uniform) *local product structure*: *for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *s.t. if* $d(u, v) \leq \delta$, *there is a vector* $w = [u, v]$ *in* $W^{ss}_\varepsilon(g^t u) \cap W^{su}_\varepsilon(v)$, *where* $W^{ss}_\varepsilon(v)$ *is the intersection of the strong stable horocycle of* v *with the ball centered at* v *of radius* ε *and* $|t| \leq \varepsilon$.

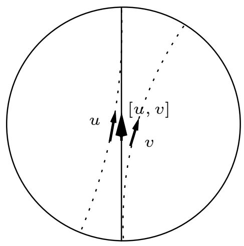

FIGURE 1. *Local product in the hyperbolic disc* $\mathbb{D}$

In $T^1\mathbb{D}$, the vector $[u, v]$ is the unique vector of endpoints v^- and u^+ on $W^{su}(v)$. If u and v are sufficiently close, $[u, v]$ belongs to $W^{ss}_\varepsilon(g^t u) \cap W^{su}_\varepsilon(v)$.

3. *Density of positive half-horocycles*

Recall that $W^{su}(v) = \{h^s v,\ s \in \mathbb{R}\}$ is compact iff $v^- \in \Lambda_{\mathrm{p}}$, and dense in $\mathcal{E}$ iff $v^- \in \Lambda_{\mathrm{hor}}$ (see lemma 2.3). Denote by $W^{su}_+(v) = \{h^s v,\ s \geq 0\}$ the positive half-horocycle.

We suppose in the sequel that S^1 is oriented in the counterclockwise direction.

Geometry of funnels

REMARK 3.1. If the surface $S = \mathbb{D}/\Gamma$ has a funnel isometric to $\{z \in \mathbb{H},\ Re(z) \geq 0\}/\{z \mapsto az\}$, with $a > 1$, the geodesic line $Re(z) = 0$ of $\mathbb{D}$ induces on the quotient the closed geodesic bounding the funnel.

Any geodesic line crossing this closed geodesic and entering into the funnel never returns back to the other side. In particular, if the lift of the funnel is exactly $\{z \in \mathbb{H}, \mathrm{Re}(z) \geq 0\}$, then the limit set Λ_Γ does not intersect the right half line $\mathbb{R}^*_+$.

Any horocycle centered in $\mathbb{R}^*_+$ stays in the funnel except during a finite interval of time. It is embedded in S. A horocycle centered in 0 or ∞ has one side which does not enter the funnel, and the other side enters the funnel and never returns back to the other side. This "half-horocycle" is embedded in S. The Lebesgue measure induced by the parametrization of the horocyclic flow on this embedded half-horocycle is an infinite locally finite totally dissipative measure.

From this elementary remark, we deduce the following key facts.

FACT 3.2. On a geometrically finite hyperbolic manifold, the points on the boundary of an interval of $S^1 \setminus \Lambda_\Gamma$ are hyperbolic. More precisely, both extremities of such an interval are the endpoints $p^\pm$ of the axis of a lift of the closed geodesic bounding the corresponding funnel.

FACT 3.3. Assume S is geometrically finite. If $v^- \in \Lambda_{\mathrm{hor}}$ is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, then $W^{su}_+(v) = \{h^s v, s \geq 0\}$ is not dense in $\mathcal{E}$, it is a closed embedded subset of T^1S, and $(g^{-t}v)_{t\geq 0}$ is asymptotic to the closed geodesic turning around a funnel.

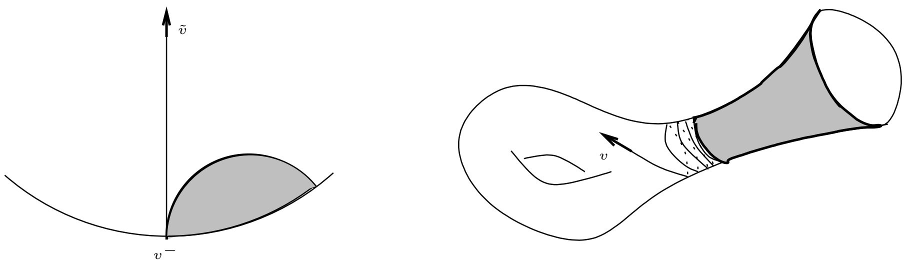

FIGURE 2. *A vector whose right horocycle is not dense in* Ω

Right horocyclic vectors and right horocyclic points

For a vector $v \in T^1S$, we denote by $\tilde{v}$ a lift to $T^1\mathbb{D}$, and by $v^\pm \in S^1$ the endpoints of this lift on the boundary.

If $v \in T^1\mathbb{D}$, we denote by $Hor(v) \subset \mathbb{D}$ the horoball centered at v^- and containing the base point of v in its boundary. We denote by $Hor^+(v) \subset Hor(v)$ the "right part" of the horoball, i.e. the set of basepoints of vectors of $\cup_{t\geq 0} W^{su}_+(g^{-t}v) = \cup_{t\geq 0} \cup_{s\geq 0} h^s g^{-t} v = \cup_{t\geq 0} \cup_{s\geq 0} g^{-t} h^s v$ (according to the relation (2.1)).

Fix a point $o \in \mathbb{D}$. If S is a geometrically finite surface, we assume that o belongs to a lift of the compact part of S.

In [**C**], a vector $v \in T^1S$ is called *horospherical* if there exists a (g^t)-nonwandering vector $z \in \Omega$, $t_i \to +\infty$ and $v_i \in W^{su}(v) \cap \Omega$ s.t. $g^{-t_i} v_i \to z \in \Omega$. It is equivalent to saying that $v^- \in \Lambda_{\mathrm{hor}}$, that is that all horoballs centered at v^- contain infinitely many points of the orbit $\Gamma.o$ (see lemma 3.7 below for a proof).

DEFINITION 3.4. *If* $v \in T^1\mathbb{D}$, *and* $\alpha > 0$, *we define the* cone *of width* α *around* v *as the set* $\mathcal{C}(v, \alpha)$ *of points at hyperbolic distance at most* α *from the geodesic ray* $(g^{-t}v)_{t\geq 0}$ *inside the horoball* $Hor(v)$.

Definition 3.5. *Let S be a nonelementary hyperbolic surface. A vector $v \in T^1S$ is a* right horocyclic vector *if for a lift $\tilde{v} \in T^1\tilde{S}$, for all $\alpha > 0$ and $D > 0$, the orbit $\Gamma.o$ intersects the right horoball $Hor^+(g^{-D}\tilde{v})$ minus the cone $\mathcal{C}(g^{-D}\tilde{v}, \alpha)$.*

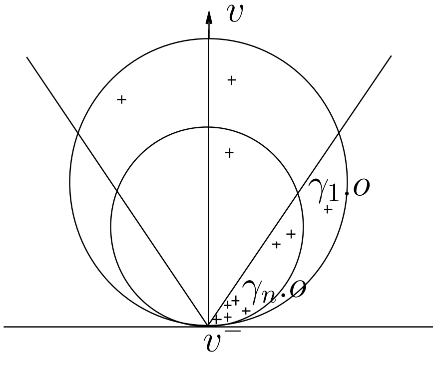

Figure 3. *Lift of a right-horocyclic vector*

Of course if v is a right horocyclic vector, then v^- is horospherical, and v is a horospherical vector in the sense of [**C**].

Remark 3.6. This definition depends only of v^-. Indeed, if w is another vector with $v^- = w^-$, any cone $\mathcal{C}(w, \alpha)$ around w is included in a cone $\mathcal{C}(g^\tau v, \beta)$ around v for some $\beta > 0$, $\tau \in \mathbb{R}$ depending on v, w, α. A point $\xi \in \Lambda_\Gamma$ which is the negative endpoint of a right horocyclic vector will therefore be called a *right horocyclic point.*

Lemma 3.7. *Let S be a hyperbolic surface. A vector $v \in T^1S$ is a right horocyclic vector if and only if there exists a (g^t)-nonwandering vector $z \in \Omega$ such that for all $\alpha > 0$, there exists a sequence $t_n \to +\infty$, $v_n \in W^{su}_+(v)$ s.t. $g^{-t_n}v_n$ converges to $z \in \Omega$, but $g^{-t_n}\tilde{v}_n \notin \mathcal{C}(\tilde{v}, \alpha)$, where $\tilde{v}$ and $\tilde{v_n}$ are lifts resp. of v and v_n on the same horocycle of $T^1\mathbb{D}$.*

The definition of right horocyclic vector is probably more natural, but the above equivalent property will be useful in the sequel.

Proof. Let us begin with the following elementary fact.

Fact 3.8. *There exists $R > 0$, such that for all $\xi \in \Lambda_\Gamma$, there exists $\eta \in \Lambda_\Gamma$, such that the geodesic $(\xi\eta)$ intersects the ball $B(o, R)$.*

Indeed, assuming it is false, we could find a sequence $R_n \to \infty$, $\xi_n \in \Lambda_\Gamma$, $\xi_n \to \xi \in \Lambda_\Gamma$, s.t. for all $\eta \in \Lambda_\Gamma$, the distance $d(o, (\xi_n\eta))$ is greater than R_n. Passing to the limit, for $\eta \neq \xi$, we obtain $d(o, (\xi\eta)) = +\infty$, which gives a contradiction.

Now, let v be a right horocyclic vector. Let $D_n \to +\infty$, $\alpha_n \to +\infty$, and $\tilde{v}$ be a lift of v to $T^1\mathbb{D}$. There exists a point $\gamma_n.o$ in $Hor^+(g^{-D_n}\tilde{v}) \setminus \mathcal{C}(\tilde{v}, \alpha_n)$. Using fact 3.8, we can find $\eta \in \Lambda_\Gamma$, $\eta \neq v^-$, s.t. the geodesic $(v^-\eta)$ intersects the ball $B(\gamma_n.o, R)$. Choose a vector

$\tilde{w}_n \in \widetilde{\Omega} \cap T^1B(\gamma_n.o, R)$ tangent to this geodesic, where $\widetilde{\Omega} = (\Lambda_\Gamma \times \Lambda_\Gamma) \setminus \mathrm{Diagonal} \times \mathbb{R}$ is the lift of the nonwandering set of the geodesic flow. It satisfies $w_n^- = v^-$, $w_n^+ = \eta$, $\tilde{w}_n = g^{-t_n}\tilde{v}_n$, $t_n \geq D_n - R$, $\tilde{v}_n \in W_+^{su}(\tilde{v})$, and $\tilde{w}_n$ does not belong to the cone $\mathcal{C}(\tilde{v}, \alpha_n - R)$. Passing to T^1S, we get a sequence of vectors w_n of the compact set $T^1B(o, R) \cap \Omega$. Up to a subsequence, it converges to some $z \in \Omega$. We proved that there exists $z \in \Omega$, s.t. for all $\alpha > 0$, there exists $t_n \to +\infty$, and $v_n \in W_+^{su}(v)$ s.t. $g^{-t_n}v_n \to z$, and $g^{-t_n}\tilde{v}_n \notin \mathcal{C}(\tilde{v}, \alpha)$.

Conversely, assume the existence of such a $z \in \Omega$. Fix $\alpha > 0$ and $D > 0$. Let $\rho = d(o, \pi(z))$, $\alpha > 0$, and $\beta = \alpha + \rho + 1$. There exists $t_n \to \infty$, $v_n \in W_+^{su}(v)$, $g^{-t_n}v_n \to z$, and $\tilde{v}_n \notin \mathcal{C}(\tilde{v}, \beta)$. For n large enough, $t_n \geq D + \rho + 1$, and $d(g^{-t_n}v_n, z) \leq 1$. There exists an element $\gamma_n.o \in B(\pi(g^{-t_n}\tilde{v}_n), \rho)$. By construction, this element is in $Hor^+(g^{-D}\tilde{v}) \setminus \mathcal{C}(g^{-D}\tilde{v}, \alpha)$. Thus, $\tilde{v}$ is a right horocyclic vector. □

Proof of theorem 1.1

We will prove

Proposition 3.9. *Let S be a hyperbolic surface. A vector $v \in T^1S$ is a right horocyclic vector if and only if $(h^s v)_{s \geq 0}$ is dense in $\mathcal{E}$.*

and

Lemma 3.10. *Let S be a geometrically finite surface. If $v^- \in \Lambda_{\mathrm{hor}}$, v^- is a right horocyclic point iff v^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.*

Theorem 1.1 is an immediate consequence of these two results. Let us now prove them. Proposition 3.9 will be proved thanks to proposition 3.12 below, which is interesting in itself. Lemma 3.10 is elementary and will be proved later.

Fact 3.11. *If $y \in \overline{W_+^{su}(x)} = \overline{\{h^s x, s \geq 0\}}$, then $W_+^{su}(y) = \{h^s y, s \geq 0\} \subset \overline{W_+^{su}(x)} = \overline{\{h^s x, s \geq 0\}}$.*

Proof. Evident with the parametrization of W^{su} by the horocyclic flow. □

Proposition 3.12. *Let S be a hyperbolic surface. If $p \in \Omega$ is a periodic vector for the geodesic flow, then its positive half-horocycle $(h^s(p))_{s \geq 0}$ is dense in the nonwandering set $\mathcal{E}$ of the horocyclic flow if and only if p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.*

This result is valid on any hyperbolic surface, without geometrical finiteness assumption. It is even true on elementary surfaces, but trivial (the limit set is finite, so that p^- is necessarily the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$).

Let us prove proposition 3.12.

Proof. Assume first that $p \in T^1S$ is a periodic vector for the geodesic flow, such that p^- is the first endpoint of an interval $]p^-\eta[$ of $S^1 \setminus \Lambda_\Gamma$. Denote by γ_p the isometry whose attractive (resp. repulsive) fixed point is p^+ (resp. p^-). In this case, η is necessarily equal to p^+ (if not, observe that the sequence $\gamma_p^{-n}\eta$ is a sequence of points of Λ_Γ in the interval $]p^-\eta[$, which gives a contradiction). On the quotient surface S, the area delimited by the geodesic (p^-p^+) and the interval $[p^-p^+]$ on the boundary (oriented counterclockwise) induces a funnel.

And we see easily that the positive half-horocycle $(h^s p)_{s\geq 0}$ is embedded in T^1S (see remark 3.1).

Assume now that p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, and prove the converse direction. We follow and adapt the strategy of [**C**].

$*$ *Step 1: Prove that* $\overline{g^{\mathbb{R}} W^{su}_+(p)} \supset \Omega$ (see also [**C**, Lemma1]).

Indeed, there exists a (g^t)-nonwandering vector $v \in \Omega$, s.t. $(g^t v)_{t\geq 0}$ is dense in Ω. Let $\tilde{v}$ (resp. $\tilde{p}$) be a lift of v (resp. p) to $T^1\mathbb{D}$, and v^+ its positive endpoint in S^1. The orbit $\Gamma.v^+$ is dense in Λ_Γ. As p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, we can find a sequence $v_n^+ \in \Gamma.v^+$ converging to p^-, with $v_n^+ \geq p^-$ (in the counterclockwise order).

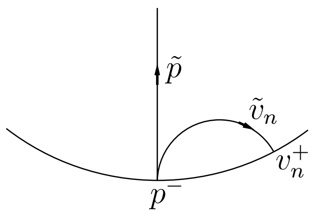

FIGURE 4. *Construction of a dense geodesic in the weak unstable manifold* $W^{wu}(p)$

If n is large enough, the unique vector $\tilde{v}_n$ of $W^{ss}(\tilde{p}) \cap (p^- v_n^+)$ belongs to the positive half-horocycle $W^{su}_+(\tilde{p})$, so that on T^1S, $v_n \in W^{su}_+(p)$ and $g^{\mathbb{R}} v_n$ is dense in Ω, because $(g^t v_n)_{t\geq 0}$ and $(g^t v)_{t\geq 0}$ are asymptotic by construction. Therefore, $g^{\mathbb{R}} W^{su}_+(p)$ is dense in Ω.

$*$*Step 2:* $\overline{W^{su}_+(p)} \supset g^{\mathbb{R}} W^{su}_+(p)$ (We still follow [**C**]).

• Recall first that on a nonelementary negatively curved surface, the *length spectrum* is non arithmetic (see [**Da**]), that is the set of lengths of periodic orbits $\{l(\gamma), <\gamma> \text{ periodic}\}$ generates a dense subgroup of $\mathbb{R}$.

Fix $\varepsilon > 0$, and a periodic vector $p_0 \in T^1S$, s.t. $\exists m, n \in \mathbb{Z}$, with $|ml(p) + nl(p_0)| < \varepsilon$. Without loss of generality, assume $n > 0$, $m > 0$ and $nl(p_0) - ml(p) = \varepsilon_0$, with $0 < |\varepsilon_0| < \varepsilon$.

Let $\delta = \delta(\varepsilon) > 0$ be the constant appearing in the local product structure property (see end of section 2).

• There exists $v \in W^{su}_+(p) \cap W^{ws}(p_0)$, such that $W^{su}_{2\varepsilon}(v) \subset W^{su}_+(p)$. Let $\sigma_1 > 0$ be such that $g^{\sigma_1} v \in W^{ss}_{\delta/2}(p_0)$.

Indeed, as p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, we can lift p_0 to $\tilde{p}_0$ in such a way that $p_0^+ \in [p^-p^+]$. Let $v \in W^{su}_+(p) \cap W^{ws}(p_0)$ be the vector obtained as the projection on T^1S of the unique $\tilde{v}$ of $W^{su}_+(\tilde{p}) \cap W^{ws}(\tilde{p}_0)$. If p_0^+ is well chosen (i.e. close enough to p^-), we have $W^{su}_{2\varepsilon}(v) \subset W^{su}_+(p)$. The geodesic orbit $(g^t v)_{t\in\mathbb{R}}$ is negatively asymptotic to the periodic orbit of p, positively asymptotic to the periodic orbit of p_0.

It is the only part of the proof where we use the assumption that p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.

• Using transitivity of the geodesic flow, we construct a vector $w \in W^{su}_{\delta/2}(p_0) \cap W^{ws}(p)$. As p_0 is periodic, we have of course $g^{-knl(p_0)} w \in W^{su}_{\delta/2}(p_0) \cap W^{ws}(p)$, where $k \in \mathbb{N}$ is any nonnegative integer, and $n > 0$ was given above by the nonarithmeticity property.

• For all $k \in \mathbb{N}$, there exists a vector $w_k \in W^{su}_\varepsilon(g^{\sigma_1} v) \cap W^{ss}_\varepsilon(g^{-knl(p_0)\pm\varepsilon} w)$. It is obtained by the local product structure, which allows to glue the past of $g^{\sigma_1} v$ with the future of $g^{-knl(p_0)} w$.

And when one pushes this vector backward by the geodesic flow, one gets a vector $v_k = g^{-\sigma_1} w_k \in W^{su}_{\varepsilon}(v) \cap W^{ws}(p)$. In particular, $v_k \in W^{su}_+(p)$.

• Note that for all $j \in \mathbb{Z}$, and all $k \in \mathbb{N}$, $g^{jl(p)} v_k \in W^{su}_+(p) \cap W^{ws}(p)$ (by periodicity of $(g^t p)_{t\in\mathbb{R}}$).

• For $k = 0$, let $j_0 \geq \sigma_1 + 2\varepsilon$ be the smallest integer such that $g^{j_0 l(p)} v_0$ is ε-close from the periodic orbit of p. This integer exists as $v_0 \in W^{ws}(p)$. Let $0 \leq \tau_0 \leq l(p)$ be such that $g^{j_0 l(p)} v_0$ belongs to $W^{ss}_{\varepsilon}(g^{\tau_0}(p))$.

• For all $k \geq 1$, let j_k be the smallest integer such that $j_k l(p) \geq j_0 l(p) + knl(p_0) + kml(p)$. By the above, $g^{j_0 l(p) + knl(p_0)} v_k$ is 2ε-close to $g^{\tau_0} p$.

Therefore,

$$g^{j_k l(p)} v_k = g^{(j_k - km - j_0) l(p)} g^{kml(p) - knl(p_0)} g^{j_0 l(p) + knl(p_0)} (v_k)$$

is 2ε-close from $g^{(j_k - km - j_0) l(p)} g^{kml(p) - knl(p_0)} g^{\tau_0}(p) = g^{k\varepsilon_0 + \tau_0}(p)$.

• We constructed a sequence of elements $(g^{j_k l(p)} v_k)_{k\in\mathbb{N}}$ of $W^{su}_+(p)$ that are 3ε-dense in the orbit of p. Let now $\varepsilon \to 0$ to get $\overline{W^{su}_+(p)} \supset \{g^t p, t \in \mathbb{R}\}$.

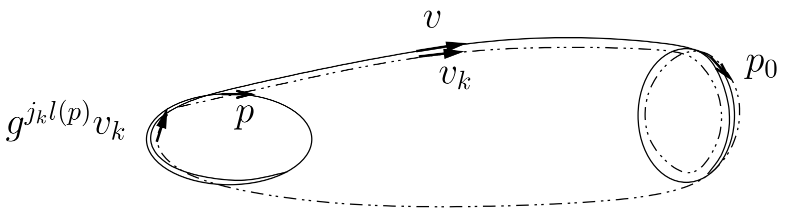

FIGURE 5. *Proof of proposition* 3.12

∗ *Step 3: Conclusion.*
The closure of the half-horocycle $\overline{W^{su}_+(p)}$ is invariant under h^s, for all $s \geq 0$, in the sense that $h^s \overline{W^{su}_+(p)} \subset \overline{W^{su}_+(p)}$ for all $s \geq 0$, and contains Ω. Lemma 2.5 implies that $\overline{W^{su}_+(p)}$ contains the full (h^s)-nonwandering set $\mathcal{E}$. This ends the proof. □

Proof of proposition 3.9 Assume first that $W^{su}_+(v) = (h^s v)_{s\geq 0}$ is dense in $\mathcal{E}$, and prove that v is a right horocyclic vector.

Let p be a vector on a periodic geodesic, $l(p)$ its length, and $d(p)$ the distance between o and its orbit. (If v is periodic, we assume that p and v have different orbits.) Fix $\alpha > 0$ and $D > 0$. Without loss of generality, we assume $D \geq l(p) + d(p) + 2$. Consider the cone $\mathcal{C} = \mathcal{C}(g^{-D}\tilde{v}, \alpha)$, where $\tilde{v}$ is a lift of v to $T^1\mathbb{D}$. The distance between (the basepoint of) $h^s(g^{-D}\tilde{v})$ and the cone $\mathcal{C}$ goes to infinity when $s \to +\infty$.

Fix $\varepsilon \in]0, 1[$. By density of $W^{su}_+(v)$ in $\mathcal{E}$, we can find an infinite sequence $v_k \in W^{su}_+(v)$, $v_k = h^{s_k} v$, $s_k \to \infty$, s.t. v_k is so close to p that $(g^{-t} v_k)_{0\leq t\leq 2D}$ and $(g^{-t} p)_{0\leq t\leq 2D}$ stay ε-close each other. We deduce that $g^{-2D} v_k$ is at distance ε from $g^{-2D} p$, hence from the orbit of p, and therefore at distance less than $1 + l(p) + d(p)$ from the projection $\pi(o)$ of o on S. Lift v to $\tilde{v} \in T^1\mathbb{D}$, and v_k to $\tilde{v_k} \in W^{su}_+(\tilde{v})$ As $v_k = h^{s_k} v$ goes to infinity on $W^{su}_+(v)$, the distance between $g^{-2D}\tilde{v_k}$ and $\mathcal{C} = \mathcal{C}(g^{-D}\tilde{v}, \alpha)$ goes to infinity. Therefore, we can assume that this distance is greater than $l(p) + d(p) + 2$. There exists a point of $\Gamma.o$ at distance at most $d(p) + l(p) + 1$ of $g^{-2D} v_k$. By construction, this point is inside $Hor^+(g^{-D}\tilde{v}) \setminus \mathcal{C}(\tilde{v}, \alpha)$. This construction works for all $\alpha > 0$ and $D > 0$ large enough, so that v is a right horospherical vector.

Let us establish now the other direction, adapting methods of [**C**]. Let v be a right horocyclic vector. We will prove that there exists a periodic vector $p \in \overline{W^{su}_+(v)}$, with $W^{su}_+(p) = (h^s p)_{s\geq 0}$

dense in $\mathcal{E}$. If v is periodic and p is on the orbit of v, the result follows. So we shall always assume that the periodic vector p is not on the orbit of v.

• Let p be any fixed periodic vector s.t. $W^{su}_+(p) = (h^s p)_{s\geq 0}$ is dense in $\mathcal{E}$, or equivalently s.t. p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, according to proposition 3.12.

• Fix $0 < \varepsilon < 1$, and let δ be the constant associated to ε by the local product structure property (see the end of section 2).

• As v is right-horocyclic, we can find $t_n \to \infty$, $v_n = h^{s_n} v \in W^{su}_+(v) \cap \Omega$, $s_n \to +\infty$ (i.e. $v_n \to \infty$ on the leaf), s.t. $g^{-t_n} v_n$ converges to some $z \in \Omega$, with $g^{-t_n}\tilde{v}_n$ staying outside a given cone $\mathcal{C}(v, 2)$.

• Using the transitivity and the product structure of the geodesic flow, we can construct a vector $w \in W^{su}_{\delta/2}(z) \cap W^{ws}(p)$ negatively asymptotic to z^-, and positively asymptotic to p^+. Let $s \geq 0$ be such that for all $t \geq s$, $g^t w$ is ε-close to the orbit of p.

• Now, let n be large enough so that $t_n \geq 2s$ and $d(g^{-t_n} v_n, z) \leq \delta/2$. In particular, the distance between $g^{-t_n} v_n$ and w is at most δ.

As $v_n \in W^{su}_+(v)$ and $g^{-t_n} v_n$ is not in the cone $\mathcal{C}(v, 2)$, the local strong stable manifold $W^{su}_{2\varepsilon}(g^{-t_n} v_n)$ is included in $W^{su}_+(g^{-t_n} v)$. Let $\tilde{v}$ be a lift of v, $\tilde{v}_n$ the lift of v_n on $W^{su}_+(\tilde{v})$, $\tilde{z}$ (resp. $\tilde{w}$) be the lift of z (resp w) $\delta/2$-close to $g^{-t_n}\tilde{v}_n$. And let respectively $v^\pm$, $v_n^\pm$, $z^\pm$, $w^\pm$, be their endpoints in the boundary. Consider the geodesic joining v^- to w^+. As $d(g^{-t_n} v_n, w) \leq \delta$, the local product structure of the geodesic flow allows to glue the negative orbit of $g^{-t_n} v_n$ and the positive orbit of w to get a vector of $W^{su}_\varepsilon(g^{-t_n} v_n) \cap W^{ss}_\varepsilon(g^{\pm\varepsilon} w)$. On the unit tangent bundle $T^1\mathbb{D}$ of the universal cover, the orbit of such a vector lifts precisely to the geodesic joining v^- to w^+. By the above, this geodesic crosses $W^{su}_{2\varepsilon}(g^{-t_n}\tilde{v}_n) \subset W^{su}_+(g^{-t_n}\tilde{v})$, and therefore also $W^{su}_+(\tilde{v})$.

• Let $\tilde{y}$ be the unique vector of $W^{su}_+(\tilde{v})$ on this geodesic. By construction $(g^t\tilde{y})_{t\leq 0}$ is asymptotic to v^-, $g^t\tilde{y}$ belongs to a 2ε-neighbourhood of $\tilde{w}$ for $t \simeq -t_n$, and then it becomes positively asymptotic to $(g^t\tilde{w})_{t\geq -t_n}$. In particular, on T^1S, as s is the "time" needed on the orbit of w to join the ε-neighbourhood of the orbit of p, for $t \geq s - t_n$, the orbit of y becomes 2ε-close to the orbit of p. We chose $t_n \geq 2s$ so that for $t = 0$, y is 2ε-close to the orbit of p.

• For $\varepsilon > 0$ fixed, we obtained a vector $y \in W^{su}_+(v)$ in the 2ε-neighbourhood of the periodic orbit of p. Let $\varepsilon_k \to 0$. We get by the above construction a sequence $y_k \in W^{su}_+(v)$ of vectors closer and closer from the periodic orbit $(g^t p)_{0\leq t\leq l(p)}$. Up to a subsequence, y_k converges to some $g^\tau p$, $0 \leq \tau \leq l(p)$.

It implies that $g^\tau p \in \overline{W^{su}_+(v)}$. Of course $g^\tau p$ is periodic and $W^{su}_+(g^\tau p)$ is dense in $\mathcal{E}$. Fact 3.11 implies now that $W^{su}_+(v)$ is dense in $\mathcal{E}$. □

Proof of lemma 3.10 Assume first that $v^- \in \Lambda_\Gamma$ is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$. As the property of being right horospherical depends only on v^-, we can assume that v is a periodic vector on the closed geodesic closing the funnel.

By the definition of a funnel, it becomes clear that if o was chosen in a lift of the compact part of S, the intersection of the open right horoball $Hor^+(v)$ with $\Gamma.o$ is empty. Thus, $\tilde{v}$ is not a right horospherical vector.

Suppose now that v is not a right horospherical vector. There exists a cone $\mathcal{C}(v, \alpha)$, a positive number $T \geq 0$, and a right horoball $Hor^+(g^{-T}v)$, s.t. $\Gamma.o$ does not intersect $Hor^+(g^{-T}v) \setminus \mathcal{C}(v, \alpha)$. Let us shrink the horoball from a distance d equal to the diameter of the compact part $C(S)$ of S. Thus, the set $Hor^+(g^{-T-d}v) \setminus \mathcal{C}(v, \alpha)$ does not intersect the Γ-orbit $\Gamma.\widetilde{C(s)}$ of the lift of the compact part. In other words, viewed on S, the projection of $Hor^+(g^{-T-d}v) \setminus \mathcal{C}(v, \alpha)$, which is a connected set, is necessarily included in a cusp or a funnel. It implies immediately that v^- is a parabolic point or is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$. By assumption, v^- is the fixed point of a hyperbolic isometry, so it cannot be parabolic. Thus it excludes the case of a cusp, and the result is proven. □

Geometrically infinite surfaces

On these surfaces, the situation is -not surprisingly - more complicated, and we only discuss here partial results on the behaviour of positive (resp. negative) half-horocycles.

Proposition 3.12 gives a complete answer for periodic vectors. Recall the

THEOREM 3.13 Hedlund, [**H**], thm 4.2 . *Let $S = \mathbb{D}/\Gamma$ be a hyperbolic surface of the first kind, i.e. such that $\Lambda_\Gamma = S^1$. Let $v \in T^1S$ be s.t. $(g^{-t}v)_{t\geq 0}$ returns infinitely often in a compact set, i.e. such that $v^- \in \Lambda_{\mathrm{rad}}$. Then the positive half-horocycle $(h^s v)_{s\geq 0}$ is dense in T^1S.*

In [**Sa-Scha**], in the case of an abelian cover of a compact surface (also a surface of the first kind), we proved the equidistribution, and therefore the density of all positive half-horocyclic orbits $(h^s v)_{s\geq 0}$ of vectors v whose asymptotic cycle is not maximal.

QUESTION 3.14. It would be interesting to understand completely the behaviour of half-horocycles. For example,

(i) On a surface of the first kind ($\Lambda_\Gamma = S^1$), are all horospherical vectors also right horocyclic vectors (generalization of Hedlund's theorem) ?

(ii) On a surface of the second kind, can we prove Hedlund's theorem for vectors v such that $v^- \in \Lambda_{\mathrm{rad}}$ is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$? And/or sufficient conditions to be right horocyclic ?

4. *Proof of Theorem* 1.2

In this section, S is a nonelementary geometrically finite surface.

4.1. *The Patterson-Sullivan construction*

Let δ_Γ be the critical exponent of Γ, defined by

$$\delta_\Gamma := \limsup_{T\to\infty} \frac{1}{T} \log \#\{\gamma \in \Gamma, d(o, \gamma.o) \leq T\}\,.$$

The well known Patterson construction provides a *conformal density* of exponent δ_Γ on S^1, that is a collection $(\nu_x)_{x\in\mathbb{D}}$ of measures, supported on $\Lambda_\Gamma \subset S^1$, s.t. $\nu_o(S^1) = 1$, $\gamma_*\nu_x = \nu_{\gamma.x}$ for all $\gamma \in \Gamma$, and $\frac{d\nu_x}{d\nu_y}(\xi) = \exp(-\delta_\Gamma \beta_\xi(x, y))$.

The Bowen-Margulis-Patterson-Sullivan measure m^{ps} on T^1S is defined locally as the product

$$dm^{ps}(v) = \exp\left(\delta_\Gamma \beta_{v^-}(o, \pi(v)) + \delta_\Gamma \beta_{v^+}(o, \pi(v))\right) d\nu_o(v^-) d\nu_o(v^+) dt$$

in the coordinates $\Omega \simeq \Gamma\backslash(\Lambda_\Gamma^2 \setminus \mathrm{Diagonal} \times \mathbb{R})$.

Under our assumptions on S, it is well known [**Su**] that the Bowen-Margulis-Patterson-Sullivan measure is (g^t)-invariant, finite and ergodic†, that there exists a unique conformal density of exponent δ_Γ, that all measures ν_x are nonatomic and give full measure to the radial limit set. Moreover, the Bowen-Margulis-Patterson-Sullivan measure is the measure of maximal entropy of the geodesic flow, and it is fully supported on the nonwandering set Ω of the geodesic flow. Note that in general, this measure is NOT invariant under the horocyclic flow, except on

†In fact, this measure is always invariant, bu not necessarily finite or ergodic in variable negative curvature, and the assumption $(*)$ added in section 5 ensures finiteness and ergodicity of this measure finite volume surfaces, where $\Lambda_\Gamma = S^1$, $\Omega = \mathcal{E} = T^1S$, and ν_0 is the Lebesgue visual measure on S^1 (see [**K**] for details).

4.2. *Foliations and measures*

The orbits of the horocyclic flow on T^1S form a one-dimensional foliation $\mathcal{W}^{su}$ of T^1S (and of $T^1\mathbb{D}$ also, of course). We denote by $W^{su}(u)$ a leaf of any of these two foliations. Given any chart $\varphi : B \to \mathbb{R}^2 \times \mathbb{R}$ of this foliation, the source B of the chart is called a *box*, a *plaque* of B is a set $P = \varphi^{-1}(\{x\} \times \mathbb{R})$, a *transversal* is a set $T = \varphi^{-1}(\mathbb{R}^2 \times \{t\})$. We write $B = T \times P$. On $T^1\mathbb{D}$, in view of the homeomorphism $T^1\mathbb{D} \simeq (S^1 \times S^1) \setminus \mathrm{Diagonal} \times \mathbb{R}$, natural transversals are the weak stable manifolds of the geodesic flow, defined by $W^{ws}(\tilde{u}) := \{v \in T^1\mathbb{D}, v^+ = u^+\}$. They induce the weak stable manifolds of the geodesic flow on T^1S, that give also locally transversals to the horocyclic flow.

A *holonomy* $\zeta : T \to T'$ is the homeomorphism following the plaques between two transversals T and T' of the same box $B = T \times P = T' \times P$. A *transverse invariant measure* is a collection $\nu = \{\nu_T\}$ of Radon measures on all transversals T, satisfying $\nu_{T'} = \zeta_*\nu_T$ for all holonomies $\zeta : T \to T'$. The support of such a measure is the closure of the union of the supports of the measures ν_T. By invariance by holonomies of $\{\nu_T\}$, this support is a saturated set, that is a union of leaves of the foliation.

Transverse invariant measures for the horocyclic foliation are in one-to-one correspondance with (h^s)-invariant measures. It is a classical fact, which can easily be checked in restriction to any small box $B = T \times P$. A transverse invariant measure is *ergodic* iff the (h^s)-invariant measure associated to it is ergodic.

Remark that a periodic horocycle H^+ induces canonically a transverse invariant measure ν^{H^+} defined on a transversal T by $\nu_T := \sum_{t \in T \cap H^+} \delta_t$.

Definition 4.1. *A* Haar system *is a collection $\alpha = \{\alpha_{W^{su}}\}$ of measures on the leaves of the foliation, satisfying the following continuity condition: for all relatively compact boxes $B = T \times P$ and all continuous maps $\psi : T^1S \to \mathbb{R}$ with compact support included in B, the map $t \in T \mapsto \int_{P_t} \psi \, d\alpha_{W_t^{su}}$ is continuous, where P_t denotes the plaque $\{t\} \times P$ of B through t, and W_t^{su} the leaf of t.*

The Lebesgue measure $\lambda = \{\lambda_{W^{su}}\}$ on leaves induced by the parametrization of the flow defines a Haar system.

The conditional measures of the Patterson-Sullivan measure on leaves define a Haar system μ^{ps} defined on $T^1\mathbb{D}$ by $d\mu^{ps}_{W^{su}}(v) = \exp(\delta_\Gamma \beta_{v^+}(o, \pi(v)))d\nu_o(v^+)$. This family is Γ-invariant and gives rise to a Haar system on the horocyclic foliation of T^1S (see [**Scha2**] for details).

The Patterson-Sullivan measure induces also a transverse invariant measure, defined on a weak stable manifold $T = W^{ws}(w)$ of $T^1\mathbb{D}$ by $d\mu^{ps}_T(w) = \exp(-\delta_\Gamma t)d\nu_o(w^-)dt$, with $t = \beta_{w^-}(\pi(w), o))$. This transverse invariant measure is invariant under Γ, and induces therefore a transverse invariant measure on T^1S.

Given a transverse holonomy invariant measure ν and a Haar system α, we may define a measure $\nu \circ \alpha$ on T^1S, defined locally on boxes $B = T \times P$ by

$$\nu \circ \alpha(B) := \int_T \alpha(P_t) d\nu_T(t) \,.$$

This quantity does not depend on the choice of the transversal T in B.

The Patterson Sullivan measure m^{ps} is the “product” in this sense $m^{ps} = \mu^{ps}_T \circ \mu^{ps}_{W^{su}}$.

On geometrically finite surfaces, it follows from Roblin [**Ro**] (see also [**Bu**]) that *there is a unique transverse invariant measure $\{\mu^{ps}_T\}$ with support equal to $\mathcal{E}$, which gives zero measure to the set $\mathcal{E}_P$ of periodic horocycles.*

As a corollary, he obtains the classification of ergodic invariant measures for the horocyclic flow. Except the probability measures supported on periodic horocycles, and the infinite measures supported on wandering horocycles, *there is a unique (up to normalization) ergodic invariant measure fully supported in the nonwandering set* $\mathcal{E} \simeq \Gamma\backslash\big((\Lambda_\Gamma \times S^1)\setminus \text{Diagonal} \times \mathbb{R}\big)$ of $(h^s)_{s\in\mathbb{R}}$. It is an infinite locally finite measure, defined locally by

$$dm(v) = ds(v)\exp\left(\delta_\Gamma \beta_{v^-}(o,\pi(v))\right) d\nu_o(v^-)dt,$$

where $ds(v)$ denotes the natural Lebesgue measure on $(h^s v)_{s\in\mathbb{R}}$ associated with the parametrization by (h^s). In other words, it is the product $\mu_T^{ps}\circ\lambda_{w^{su}}$ of the transverse invariant measure induced by the Patterson-Sullivan measure by the Lebesgue measure on leaves.

Sketch of the proof

The strategy of the proof is exactly the same as in [**Scha1**] and [**Scha2**]. We consider 'one-sided versions' of all results of these articles. Due to the lengths of the proofs of technical results in [**Scha1**], all arguments are not presented with full details. However, following the recommandations of the referee, we recall all important arguments and ideas.

The main lines of the proof are as follows. We do not prove directly equidistribution of horocyclic orbits to the unique "interesting" ergodic invariant measure, because this measure is infinite. We consider auxiliary averages on horocycles. Using classical arguments (tightness in theorem 4.2 and classification of invariant measures due to Burger [**Bu**] and Roblin [**Ro**]), we prove equidistribution of these auxiliary averages towards the *finite* Bowen-Margulis-Patterson-Sullivan measure (theorem 4.3). We deduce then theorem 1.2 from the preceding.

Fix $u\in T^1S$, such that $u^-\in\Lambda_\Gamma$ is not the first endpoint of an interval of $S^1\setminus\Lambda_\Gamma$. Let $r>0$ be large enough so that $\mu^{ps}_{W^{su}}((h^s u)_{0\le s\le r})>0$. It is possible since $(h^s u)_{s\ge 0}$ is dense in $\mathcal{E}\supset\Omega$, by theorem 1.1, and $\mu^{ps}_{W^{su}}$ is an infinite measure of support equal to $W^{su}(u)\cap\Omega$.

Let $\psi : T^1S\to\mathbb{R}$ be a continuous compactly supported map. Consider the following averages:

$$M^+_{r,u}(\psi) = \frac{1}{\mu^{ps}_{W^{su}}((h^s u)_{0\le s\le r})}\int_{(h^s u)_{0\le s\le r}} \psi(v)\, d\mu^{ps}_{W^{su}}(v)\,.$$

These averages are supported on Ω. As mentioned above, we shall now prove first a tightness, or nondivergence result for these averages, then an equidistribution result for these averages, and last deduce theorem 1.2.

4.3. *A tightness result*

THEOREM 4.2. *Let S be a nonelementary geometrically finite hyperbolic surface, and $u\in\mathcal{E}\subset T^1S$. For all $\varepsilon>0$, there exist a compact set $K_{\varepsilon,u}\subset\Omega$ and $r_0>0$ such that for $r\ge r_0$, $M^+_{r,u}(K_{\varepsilon,u})\ge 1-\varepsilon$.*

This theorem is trivial when the surface is convex-cocompact, because the averages $M^+_{r,u}$ are probability measures on Ω, and saying that the surface is convex-cocompact means that the nonwandering set Ω is compact.

This result implies that all weak limit points of the sequence of probability measures $(M^+_{r,u})_{r>0}$ when $r\to+\infty$ are probability measures. We postpone the (long and technical) proof to paragraph 4.6.

4.4. *Equidistribution towards the Patterson-Sullivan measure*

Theorem 4.3. *Let S be a nonelementary geometrically finite hyperbolic surface, and $u \in \mathcal{E} \subset T^1S$. If the positive orbit $(h^s u)_{s\geq 0}$ is dense in Ω, then it is equidistributed: for all $\psi : T^1S \to \mathbb{R}$ continuous with compact support, we have*

$$M^+_{r,u}(\psi) \to \int_{T^1S} \psi \, dm^{ps} , \quad \text{when} \quad r \to \infty.$$

Recall that $\mathcal{E} = \{u \in T^1S, u^- \in \Lambda_\Gamma\}$. Introduce the notations $\mathcal{E}_P = \{u \in \mathcal{E},\ u^- \in \Lambda_P\}$, and $\mathcal{E}_{\rm rad} = \{u \in \mathcal{E},\ u^- \in \Lambda_{\rm rad}\}$. The proof of the theorem uses the two following lemmas.

Lemma 4.4. *If $u \in \mathcal{E}_{\rm rad}$, and u^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, every limit point of $(M^+_{r,u})_{r\geq 0}$ when $r \to \infty$ can be written as $\nu \circ \mu^{ps}$, where ν is a transverse measure invariant by holonomy and μ^{ps} is the Haar system associated with the Patterson-Sullivan measure m^{ps}.*

The proof of this lemma is postponed after the proof of theorem 4.3.

Lemma 4.5 [**Scha2**], lemma 3.6 or [**BeMa**]. *Under the assumptions of theorem 4.3, if $u \in \mathcal{E}_{\rm rad}$, any limit point of $(M^+_{r,u})_{r\geq 0}$ when $r \to +\infty$ gives zero measure to $\mathcal{E}_P$.*

The proof of this lemma is omitted, as the argument is exactly the same as in [**BeMa**] and [**Scha2**].

Proof of theorem 4.3 We follow [**Scha2**]. Thanks to theorem 4.2, all limit points of $(M^+_{r,u})_{r\geq 0}$ are probability measures. According to lemma 4.5, such a limit gives measure zero to the set of periodic horocycles.

Moreover, lemma 4.4 implies that a limit point of the family $(M^+_{r,u})_{r\geq 0}$ when $r \to \infty$ can be written as the product of a transverse invariant measure to the strong unstable foliation by the measure $\mu^{ps}_{W^{su}}$.

The uniqueness ([**Ro**]) of a transverse measure of full support in the nonwandering set $\mathcal{E}$ giving measure 0 to periodic horocycles allows to conclude the proof.

Proof of lemma 4.4 We follow closely [**Scha2**], lemma 3.5.

• As the support of $M^+_{r,u}$, for $r > 0$, is included in Ω, the support of any limit point of $(M^+_{r,u})_{r>0}$ when $r \to \infty$ is also included in Ω, so that it is enough to show that all $v \in \Omega$ have a relatively compact neighbourhood of the form $B = T \times P$ in restriction to which the result of the lemma is true. This neighbourhood can be assumed to be relatively compact.

• Define a transverse measure on all transversals T by

$$\nu^{+,r}_T = \frac{1}{\mu^{ps}_{W^{su}}((h^s u)^r_0)} \sum_{t \in T \cap \{h^s u, 0 \leq s \leq r\}} \delta_t \,.$$

• Observe that the average $M^+_{r,u}$, restricted to the box B, can be written as $M^+_{r,u}(B) = \nu^{+,r}_T \circ \mu^{ps}_{W^{su}}(B) + R(T,B,r)$, where the error term $R(T,B,r)$ is made of two possible contributions (see figure 6).

First, a term which can have been forgotten in the integral over the transversal T w.r.t. $\nu^{+,r}_T$, corresponding to the possible $t \in T \cap W^{su}_+(u)$, $t \notin \{h^s u, 0 \leq s \leq r\}$, which appear in $M^+_{r,u}$ because the plaque of t satisfies $\mu^{ps}_{W^{su}}(P_t \cap \{h^s u, 0 \leq s \leq r\}) > 0$. Second, a term

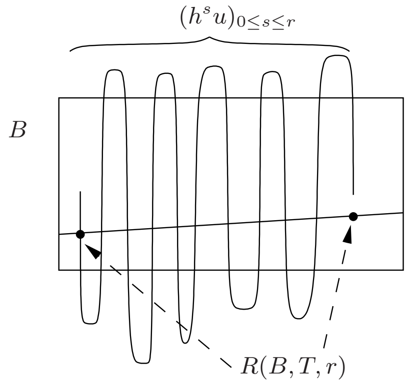

FIGURE 6. *Study of the error term*

corresponding to a possible $t \in T \cap \{h^s u, 0 \le s \le r\}$, whose contribution in $M^+_{r,u}$ is smaller than the contribution in $\nu_T^{+,r}$, in the case where the plaque P_t of t satisfies $\mu^{ps}_{W^{su}}(P_t \cap \{h^s u, 0 \le s \le r\}) < \mu^{ps}_{W^{su}}(P_t)$.

• As horocycles are one-dimensional, and the box $B = T \times P$ is relatively compact, the sum $R(T,B,r)$ of these two error terms can be bounded as follows:

$$|R(T,B,r)| \le \frac{1}{\mu^{ps}_{W^{su}}((h^s u)_{0\le s\le r})} 2 \sup_{t\in T} \mu^{ps}_{W^{su}}(P_t) < \infty\,.$$

FACT 4.6. *If $(h^s u)_{s\ge 0}$ is dense in $\mathcal{E}$, then $\mu^{ps}_{W^{su}}((h^s u)_{0\le s\le r}) \to +\infty$ when $r \to +\infty$.*

A way to see it is to consider a sequence $v_n \in \Omega$, $r_n \to +\infty$, such that $(h^s v_n)_{|s|\le r_n} \subset \{h^s u,\, s \ge 0\}$, and v_n converges to $v_\infty \in \Omega$. As $(h^s u)_{s\ge 0}$ is dense in $\mathcal{E} \supset \Omega$, such a sequence exists. As $\mu^{ps}_{W^{su}(v_\infty)}$ is an infinite measure, and the family $\{\mu^{ps}_{W^{su}}\}$ is a Haar system (i.e. varies continuously), the sequence $\mu^{ps}_{W^{su}}((h^s v_n)_{|s|\le r_n})$ goes to $+\infty$.

• Thanks to this fact, $R(T,B,r) \to 0$ when $r \to +\infty$, so that limit points of $(M^+_{r,u})_{t>0}$ are limit points of the sequence $\nu_T^{+,r} \circ \mu^{ps}_{W^{su}}$. They can be written as the product of a transverse measure with the Haar system $\mu^{ps}_{W^{su}}$. It is easy to show that any limit point of $\{\nu_T^{+,r}\}$ is a holonomy invariant transverse measure. We refer to [**Scha2**] for details.

4.5. *Proof of theorem* 1.2

We refer to [**Scha2**] for a proof with full details in the case of symmetric averages.

• Let m be the unique (h^s)-invariant ergodic measure with full support in $\mathcal{E}$. Fix a large compact set K of T^1S, such that $m(K) > 0$. Assume that K is proper, that is equal to the closure of its interior. Consider a continuous function φ with compact support in $\mathcal{E}$. Without loss of generality, we assume that it is nonnegative, that $\int_{T^1S} \varphi\, dm > 0$, and that its support is included in a box $B = T \times P$ included in K.

• Introduce the averages

$$M^{+,K}_{r,u}(\varphi) := \frac{\int_0^r \varphi \circ h^s u\, ds}{\int_0^r \mathbf{1}_K(h^s u)\, ds}\,,$$

and the transverse measure defined on all transversals T by

$$\nu_T^{K,r} := \frac{1}{\int_0^r \mathbf{1}_K(h^s u)\, ds} \sum_{t \in T \cap \{h^s u, 0\le s\le r\}} \delta_t\,.$$

• As in the proof of theorem 4.3, and more precisely lemma 4.4 we see that up to an error term, $M^{+,K}_{r,u}(\varphi) \simeq \nu_T^{K,r} \circ \lambda_{W^{su}}$, where $\lambda = \{\lambda_{W^{su}}\}$ is the Haar system induced by the parametrization of the horocyclic flow. The error term is bounded by a constant depending on φ divided by $\int_0^r \mathbf{1}_K(h^s u)\, ds$. This denominator goes to 0 as soon as K is not too small (i.e. intersects largely $\mathcal{E}$, because $(h^s u)_{s\geq 0}$ is dense in $\mathcal{E}$, and therefore comes back infinitely often in K).

Recall that up to an error term going to 0, we have

$$M^+_{r,u}(\varphi) = \nu_T^{+,r} \circ \mu^{ps}_{W^{su}} \,.$$

Moreover, we have

$$\nu_T^{K,r} = \frac{\mu^{ps}_{W^{su}}(\{h^s u, 0 \leq s \leq r\})}{\int_0^r \mathbf{1}_K \circ h^s u\, ds}\, \nu_T^{+,r} \,.$$

• The fact that $M^+_{r,u}$ converges to m^{ps} implies that $\nu_T^{+,r}$ converges to μ_T^{ps}. The averages $M^K_{r,u}$ define probability measures on K, so that the ratio

$$\frac{\mu^{ps}_{W^{su}}(\{h^s u, 0 \leq s \leq r\})}{\int_0^r \mathbf{1}_K \circ h^s u\, ds}$$

must have positive and finite limit points when $r \to +\infty$ (see [**Scha2**] page 162 for details). We deduce that accumulation points of the two sequences $(\nu_T^{K,r})$ and $(\nu_T^{+,r})$ when $r \to +\infty$ differ from a finite positive constant. It implies that any limit measure of the sequence $(M^K_{r,u})$ can be written as a (positive finite) constant times the measure $\mu_T^{ps} \circ \lambda$, that is the unique (h^s)-invariant ergodic measure with support $\mathcal{E}$. This concludes the proof.

4.6. *Proof of theorem* 4.2

Cusps and decomposition of the surface For simplicity, assume that S has exactly one cusp. If it has no cusp, Ω is compact, so theorem 4.2 is obvious. Denote by $(\xi_i)_{i\in\mathbb{N}} = \Gamma.\xi_1$ the Γ-orbit of parabolic limit points of Λ_Γ. As S is geometrically finite, there is a Γ-invariant family of disjoint horoballs $\mathcal{H}_i$ of $\mathbb{D}$, based at ξ_i, such that $\sqcup_{i\in\mathbb{N}}\mathcal{H}_i = \Gamma.\mathcal{H}_1$, Γ acts cocompactly on $(\Lambda^2_\Gamma \setminus \mathrm{Diagonal} \times \mathbb{R}) \setminus \cup_i T^1\mathcal{H}_i$. Assume that $\mathcal{H}_1$ is the closest horoball to the origin o, that the distance from o to $\partial\mathcal{H}_1$ is bounded by the diameter of the compact part C_0 of S, and that the geodesic ray $[o\xi_1)$ does not intersect other horoballs $\mathcal{H}_i$, $i \neq 1$. We will call $\mathcal{H}$ the image of the horoballs $\mathcal{H}_i$ on S; it is exactly the cusp of the surface. We know that $1 > \delta_\Gamma > 1/2$ for a nonlattice not convex-compact geometrically finite group. (†)

If $\mathcal{H}_i$ is thee horoball centered at the parabolic point ξ_i, and $(h^s u)_{s\geq 0}$ intersects $T^1\mathcal{H}_i$, let v_i be the unique vector of $(h^s u)_{s\geq 0}$ such that $v_i^+ = \xi_i$.

An elementary computation gives the following lemma (see also [**Scha1**] page 979).

LEMMA 4.7. *If $(h^s u)_{s\in\mathbb{R}}$ intersects $T^1\mathcal{H}_i$, then $(h^s u)_{s\in\mathbb{R}} \cap T^1\mathcal{H} = (h^s v_i)_{|s|\leq\sqrt{e^{h_i}-1}}$, where h_i is the distance between the base point of v_i and the boundary of $\mathcal{H}_i$.*

In the sequel, to simplify notations, we replace $\sqrt{e^h - 1}$ by $e^{h/2}$ (see [**Scha1**] for a rigorous way to do that).

† Indeed, [**Pe**] as the surface has infinite volume, it contains a funnel. Let $\xi \notin \Lambda_\Gamma$, and $p \notin \Gamma$ a parabolic isometry fixing ξ. Using the divergence of Γ and [**DOP**, Prop.2], we obtain $1 \geq \delta_{<p>*\Gamma} > \delta_\Gamma$. As all parabolic subgroups of Γ are divergent, the same proposition [**DOP**, Prop.2] gives $\delta_\Gamma > \delta_\Pi$ for all Π parabolic subgroups of Γ. And $\delta_\Pi = 1/2$ by an elementary computation on hyperbolic surfaces.

Strategy of the proof If $\mathcal{H}_i$ is a horoball of $\mathbb{D}$ associated to the cusp, denote by $\mathcal{H}_i^N \subset \mathcal{H}_i$ the shrunk horoball centered at the same point, and such that $d(\partial\mathcal{H}_i, \partial\mathcal{H}_i^N) = N$.

Fix $\varepsilon > 0$. Denote by $\mathcal{H}$ (resp. $\mathcal{H}^N$) the image of any of the horoballs $\mathcal{H}_i$ (resp. $\mathcal{H}_i^N$) on S. We will find a $N_0 \geq 0$ large enough, so that

$$\frac{\mu^{ps}_{W^{su}}((h^s u)_{0\leq s\leq r} \cap T^1\mathcal{H}^N)}{\mu^{ps}_{W^{su}}((h^s u)_{0\leq s\leq r})} \leq \varepsilon\,, \tag{4.1}$$

for $N \geq N_0$, independently of $r > 0$ (sufficiently positive so that the above quantity is well defined). The compact set $K_{\varepsilon,u}$ of the statement of the theorem will be chosen as

$$K_{\varepsilon,u} = \Omega \setminus T^1\mathcal{H}^N\,,$$

for $N \geq N_0$ large enough.

Lift the situation to $T^1\mathbb{D}$, where the cusp $\mathcal{H}$ lifts into $\Gamma.\widetilde{\mathcal{H}} = \sqcup_{i\in\mathbb{N}}\mathcal{H}_i$. Denote by $\xi_i \in \Lambda_P$ the limit point associated to $\mathcal{H}_i$, and $\tilde{u}$ a lift of u to $T^1\mathbb{D}$.

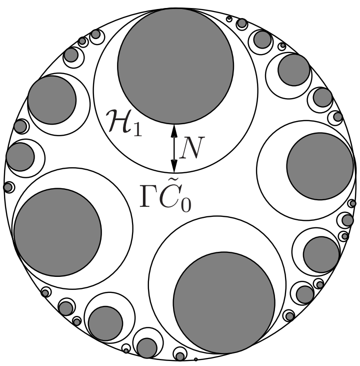

FIGURE 7. *Shrunk horospheres on $T^1\mathbb{D}$*

Define $I_{r,N} := \{i \in \mathbb{N}, (h^s\tilde{u})_{0\leq s\leq r} \cap T^1\mathcal{H}_i \neq \varnothing\}$. It can happen that $(h^s\tilde{u})_{0\leq s\leq r}$ intersects $T^1\mathcal{H}_i^N$ and $T^1\mathcal{H}_i$, but not completely, i.e. $(h^s\tilde{u})_{0\leq s\leq r}$ intersects $T^1\mathcal{H}_i^N$, and $(h^s\tilde{u})_{0\leq s\leq r} \cap T^1\mathcal{H}_i$ is strictly included in $(h^s\tilde{u})_{s\geq 0} \cap T^1\mathcal{H}_i$ (see figure 7 above). As the $\mathcal{H}_i$ are pairwise disjoint, and $(h^s\tilde{u})_{s\in\mathbb{R}}$ is one dimensional, there is at most one such index, that we denote by $i_{\max}(r,N) \in I_{r,N}$, when it is defined.

Remark that for $i \in I_{r,N}$, if v_i is the vector of $(h^s\tilde{u})_{s\geq 0}$ such that $v_i^+ = \xi_i$, and h_i is the distance between its base point and $\partial\mathcal{H}_i$, we have $(h^s\tilde{u})_{s\geq 0} \cap T^1\mathcal{H}_i = (h^s v_i)_{|s|\leq e^{h_i/2}}$, whereas $(h^s\tilde{u})_{s\geq 0} \cap T^1\mathcal{H}_i^N = (h^s v_i)_{|s|\leq e^{(h_i-N)/2}}$.

We can bound by above the numerator of 4.1 by

$$\sum_{i\in I_{r,N}} \mu^{ps}_{W^{su}}\left((h^s v_i)_{|s|\leq e^{(h_i-N)/2}}\right)\,. \tag{4.2}$$

Due to the boundary problem (the existence of the index $i_{\max}(r,N)$), it is more difficult to bound by below the denominator of 4.1. This denominater is greater than

$$\sum_{\substack{i\,\in\, I_{r,N},\\ i\,\neq\, i_{\max}(r,N)}} \mu^{ps}_{W^{su}}\left((h^s v_i)_{|s|\leq e^{h_i/2}}\right) \quad + \quad \mu^{ps}_{W^{su}}\left((h^s u)_{0\leq s\leq r} \cap T^1\mathcal{H}_{i_{\max}(r,N)}\right)\,. \tag{4.3}$$

Case of indices $i \neq i_{\max}(R,N)$

Lemma 4.8. *For $N \geq 0$, $r > 0$ and $i \in \mathbb{N}$, we have*

$$\frac{\mu^{ps}_{W^{su}}((h^s v_i)_{|s|\leq e^{(h_i-N)/2}})}{\mu^{ps}_{W^{su}}(h^s v_i)_{|s|\leq e^{h_i/2}})} \leq Ce^{(1-2\delta_\Gamma)N/2}\,.$$

In particular, as $\delta_\Gamma > 1/2$, for all $\varepsilon > 0$, there exists $N_0 > 0$, such that for $N \geq N_0$, $i \in I_{r,N}$,

$$\frac{\mu^{ps}_{W^{su}}((h^s v_i)_{|s|\leq e^{(h_i-N)/2}})}{\mu^{ps}_{W^{su}}(h^s v_i)_{|s|\leq e^{h_i/2}})} \leq \varepsilon\,.$$

This lemma says that the 'time' (measured with the measure $\mu^{ps}_{W^{su}}$) spent by a horocycle in a horoball $T^1\mathcal{H}_i^N$ is uniformly small compared to the 'time' spent in $T^1\mathcal{H}_i$, independently of the choice of the horocycle.

Before the proof, let us introduce some additional notations. Let $o \in \mathbb{D}$ be fixed outside all horoballs $\mathcal{H}_i$, $\xi \in S^1$ and $t \geq 0$. If $\eta \in S^1$ and $C \subset \mathbb{D}$ is a closed convex set, the *projection* of η on C is the point p of C which minimizes the function $x \in \mathbb{D} \mapsto \beta_\eta(x, o)$. Let $\xi(t)$ be the point of the geodesic ray $[o\xi)$ at distance t of o, and define the set $V(o, \xi, t)$ as the set of points $\eta \in S^1$ whose projection on $[o\xi)$ is at distance at least t of o. By abuse of notation, we call such sets *shadows*, because they are comparable to Sullivan's shadows [**Su**] (see for example [**Scha1**], section 2.4).

Let us prove the lemma.

Proof. • As $u^- \in \Lambda_{\mathrm{rad}}$, for all $i \in \mathbb{N}$, there exists $T_i \geq h_i/2$ s.t. $g^{-T_i}v_i \in \Gamma\tilde{C}_0$, where $\widetilde{C}_0$ is a lift of the compact part C_0 of the surface S in a fundamental domain of the action of Γ on $\mathbb{D}$. By definition of $(\mu^{ps}_{W^{su}})$, this family of measures is quasi-invariant under the geodesic flow in the sense that $g^t_*\mu^{ps}_{W^{su}} = e^{\delta_\Gamma t}\mu^{ps}_{W^{su}}$ for $t \in \mathbb{R}$, so that we get

$$\frac{\mu^{ps}_{W^{su}}((h^s v_i)_{|s|\leq e^{(h_i-N)/2}})}{\mu^{ps}_{W^{su}}((h^s v_i)_{|s|\leq e^{(h_i-N)}})} = \frac{\mu^{ps}_{W^{su}}((h^s g^{-T_i}v_i)_{|s|\leq e^{(h_i-N)/2-T_i}})}{\mu^{ps}_{W^{su}}((h^s g^{-T_i}v_i)_{|s|\leq e^{(h_i-N)/2-T_i}})}\,.$$

• Recall that $d\mu^{ps}_{W^{su}(u)}(v) = e^{\beta_{v^+}(x,\pi(v))}\,d\nu_x(v^+)$ for any choice of $x \in \mathbb{D}$, where ν_x is the Patterson measure on $\Lambda_\Gamma \subset S^1$ viewed from the point x. By continuity of the Busemann cocycle, for all $C > 1$, we can find $R > 0$, such that if the distance $d(\pi(u), x)$ between the basepoint of u and x is less than R, then in restriction to $(h^s u)_{|s|\leq R}$, the measures $\mu^{ps}_{W^{su}}$ and $(p^{-1}_{W^{su}})_*\nu_x$ differ at most from the multiplicative constant $C^{\pm 1}$. Here, if $W^{su}(u)$ is the horocycle $(h^s u)_{s\in\mathbb{R}}$, $p^{-1}_{W^{su}(u)}$ is the natural projection from $S^1 \setminus \{u^-\}$ to $W^{su}(u)$ which associates to ξ the unique vector of $W^{su}(u)$ pointing towards ξ.

Note that we can exchange quantifiers. For all $0 \leq R \leq 1$, there exists $C > 1$, such that the above property is true.

• As the distance from the base point $\pi(g^{-T_i}v_i)$ of $g^{-T_i}v_i$ to $\Gamma.o$ is less than the diameter of the compact part C_0, up to $C^{\pm 2}$, where C is given by the above when $R = \mathrm{diam}(C_0)$, the above ratio is uniformly close to

$$\frac{\nu_{\gamma.o}(V(\gamma.o, \xi_i, T_i - h_i/2 + N/2))}{\nu_{\gamma.o}\,(V(\gamma.o, \xi_i, T_i - h_i/2))}\,.$$

for some $\gamma \in \Gamma$.

• The Γ-invariance property of the family $(\nu_x)_{x\in\mathbb{H}}$ shows that the above ratio is equal to

$$\frac{\nu_o(V(o, \gamma^{-1}\xi_i, T_i - h_i/2 + N/2))}{\nu_o\,(V(o, \gamma^{-1}\xi_i, T_i - h_i/2))}\,.$$

• The so-called Shadow Lemma (see [**Scha1**] proposition 3.4, but also [**Str-Ve**]) says that as $\gamma^{-1}\xi_i$ is a parabolic point, we have

$$\nu_o(V(o,\gamma^{-1}\xi_i,t)) \asymp e^{-\delta_\Gamma t+(1-\delta_\Gamma)d((\gamma^{-1}\xi_i)(t),\Gamma.o)}\,, \tag{4.4}$$

where $\gamma^{-1}\xi_i(t)$ is the point of the geodesic ray $[o,\gamma^{-1}\xi_i)$ at distance t of o.

• Comparing the cases where $t = T_i - h_i/2 + N/2$ and $t = T_i - h_i/2$, the two corresponding distances from $(\gamma^{-1}\xi_i)(t)$ to $\Gamma.o$ differ exactly from a quantity $N/2$. We deduce that the ratio we are interested in is bounded (up to uniform constants) by $e^{(1-2\delta_\Gamma)N}$, with $\delta_\Gamma > 1/2$. The result of the lemma follows.

□

Case where $i = i_{\max}(r,N)$ We denote by $V^+(o,\xi,t)$ (resp. $V^-(o,\xi,t)$) the *positive* (resp. negative) *half-shadow*, that is the subset of points of $V(o,\xi,t)$ that are greater (resp. less) than ξ in the counterclockwise order.

To get a bound on the boundary term corresponding to $i_{\max}(r,N)$ in (4.1), (4.2) and (4.3), we want to prove the following analogue of lemma 4.8.

LEMMA 4.9. *For all $\varepsilon > 0$, there exists $N_0 > 0$, such that for $N \geq N_0$, $i = i_{\max}(r,N)$,*

$$\frac{\mu^{ps}_{W^{su}}((h^s v_i)_{|s|\leq e^{(h_i-N)/2}})}{\mu^{ps}_{W^{su}}(h^s v_i)_{-e^{h_i/2}\leq s\leq -e^{(h_i-N)/2}})} \leq \varepsilon\,.$$

Proof. By exactly the same steps as above, we see that it is enough to bound the ratio

$$\frac{\nu_o(V(o,\gamma^{-1}\xi_i,T_i-h_i/2+N/2))}{\nu_o(V^+(o,\gamma^{-1}\xi_i,T_i-h_i/2)\setminus V^+(o,\gamma^{-1}\xi_i,T_i-h_i/2+N/2))}$$

The careful reader can check that all results of [**Scha1**], and in particular proposition 3.4, can be adapted verbatim to get the following estimate of parabolic positive half-shadows

$$\nu_o(V^+(o,\gamma^{-1}\xi_i,t)) \asymp e^{-\delta_\Gamma t+(1-\delta_\Gamma)d((\gamma^{-1}\xi_i)(t),\Gamma.o)}\,, \tag{4.5}$$

the constants hidden in the notation $\asymp$ being of course different than in the case of full shadows $V(o,\xi,t)$.

Now, if N_0 is large enough, we deduce that uniformly in $i \in \mathbb{N}$, and in particular for $i = i_{\max}(r,N)$, we have

$$\frac{\nu_o(V(o,\gamma^{-1}\xi_i,T_i-h_i/2+N/2))}{\nu_o(V^+(o,\gamma^{-1}\xi_i,T_i-h_i/2)\setminus V^+(o,\gamma^{-1}\xi_i,T_i-h_i/2+N/2))} \leq Cst.e^{(1-2\delta_\Gamma)N/2}\,,$$

and the conclusion of the lemma follows. □

Conclusion of the proof of theorem 4.2 Lemmas 4.8 and 4.9 show that for $N \geq N_0$, the quantity (4.1) is bounded by ε. So we are done.

5. *Surfaces with variable negative curvature*

Most results proved here extend to surfaces S of variable negative curvature. More precisely, we assume that all sectional curvatures are pinched between two negative constants. Some definitions of the notions used here have to be adapted, and we refer to the preliminary sections of [**Scha1**] or [**Scha2**] for details. In particular, horocycles on T^1S and $T^1\widetilde{S}$ are defined as the strong unstable manifolds of the geodesic flow respectively on the unit tangent bundles T^1S of S and $T^1\widetilde{S}$ of the universal cover $\widetilde{S}$ of S. The main difference is that there is no canonical parametrization of horocycles by a nice horocyclic flow on $T^1\widetilde{S}$, even if it is possible to define such a flow (see [**Mrc**] for example).

The motivated reader can check that the proof of theorem 1.1 and all results of section 3 extend verbatim to the situation of pinched negatively curved surfaces.

Concerning the equidistribution, we need to be more careful. We add an assumption, denoted by $(*)$ in [**Scha1**] and [**Scha2**], which allows to control the geometry of the cusps, ensures in particular that the Bowen-Margulis measure is finite, and allows to prove a Shadow Lemma as in [**Scha1**], and to obtain theorem 4.2 in the variable negative curvature case. All other proofs do not depend on any (negative) curvature assumption. With this restriction, theorem 1.2 is valid on pinched negatively curved geometrically finite surfaces.

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