Algebraic cycles & pure motives

Montpellier Jan '26

Goal (course): Explain why pure motives are useful for algebraic cycles

(despite their appearence, motives are sensible to the arithmetic of the ground field)

Missing: (1) Why mixed motives are even more useful to alg cycles

(2) Other applications of motives (periods, ...)

Outline (Lecture I):

§1. Chow rings

§2. Main statements of the course

§3. Motives

§1.

$X_{/k}$ = smooth, f.t., conn., of dim d_X

$$\begin{array}{l} CH_n(X) \\ \parallel \\ CH^{d_X-n}(X) \end{array} = \left\{ \underset{\substack{\text{formal}\\\text{finite}\\\text{sum}}}{\sum} m_i[Z_i],\ m_i \in \mathbb{Z},\ Z_i \subset X \text{ closed, integral (possibly singular) of dim } n\ (= \text{codim } d_X - n) \right\} \Big/ \text{rational equivalence}$$

Intuition: rational equivalence $\Leftrightarrow$ deforming over $\mathbb{P}^1$ (generalization of linear equiv on divisors) i.e., mod out by the group

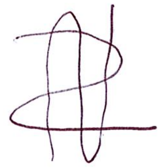
$$\langle \{\pi_*(\mathrm{div}(f))\} \mid f \in k(Y),\ Y \subset X \ \dim n+1,\ \pi: Y^{norm} \to Y \subset X \rangle$$

Facts: (1) Nice functorialities (f^*, f_*)

(2) Intersection product $CH^n(X) \times CH^m(X) \to CH^{n+m}(X)$

s.t. $[Z] ; [W] \mapsto [Z \cap W]$

If they meet transversally



(3) If $X/_k$ proper, there is a well defined degree map $\deg : CH_0(X) \to \mathbb{Z}$

$$\sum m_i [P_i] \mapsto \sum m_i \deg[k(P_i) : k]$$

Ex: (1) $X = \mathbb{P}^2$, $CH^0 = \mathbb{Q}[\mathbb{P}^2]$, $CH_0 = \mathbb{Q}[pt]$

$CH^1 = \mathbb{Q}[\text{line}]$, $[C \text{ of deg } d] = d \cdot [\text{line}]$

Inters prod $\Longrightarrow$ Bézout!

(2) $X = \mathbb{P}^1 \times \mathbb{P}^1$, $CH^0 = \mathbb{Q}[\mathbb{P}^1 \times \mathbb{P}^1]$, $CH_0 = \mathbb{Q}[pt]$

CH^1 is 2-dim^l gen by $V = |$ $H = —$

$V \cdot H = 1$, $V \cdot V = 0$, $H \cdot H = 0$

$\wr$
V'

| |

Inters prod $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(i.e. rat equiv forces negativity!)

(3) C = sm proj curve of genus g (with a k-pt) ③

$\Rightarrow \quad 0 \to Jac(C)(k) \to CH_0(C) \xrightarrow[\deg]{} \mathbb{Z} \to 0$

- If $k = \mathbb{C}$, $CH_0(C)$ is huge! ($Jac(C)(\mathbb{C}) \simeq \mathbb{C}^g/\text{lattice}$)
- If k = nb field, $CH_0(C)$ is of f.t. (Mordell-Weil)
- If $k = \mathbb{F}_q$, $\deg \otimes \mathbb{Q}$ is an iso

Convention: From now on $\mathbb{Q}$-coeff! ($CH = CH \otimes \mathbb{Q}$)

Def: (Numerical equivalence) Let X/k sm proper,

$$CH^n(X) \times CH^{d_X - n}(X) \to CH_0(X) \xrightarrow{\deg} \mathbb{Q}$$

num := Kernel of this pairing

§2. Main statements

X/k = sm proper, $\ell \neq char(k)$

$H_\ell(X) := H_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell)^{Gal_k}$

$\rho_n(X) = \dim_{\mathbb{Q}} CH(X)/_{num} \quad (< \infty)$

Our goals:

[Katz - Messing, 1974]: Let $f: X \to X$

$P^n_\ell(X) :=$ char pol of $f \curvearrowright H^n_\ell(X)$.

$\mathbb{Q}_\ell[T]$

Then $P^n_\ell(X) \in \mathbb{Q}[T]$ is ℓ-independent

[Kimura, 2005]: $S/_{\mathbb{C}}$ = sm proj surface,

$H^0(S,\Omega^1) = 0 = H^0(S,\Omega^2)$

(+ S is dominated by product of curves)

$\Rightarrow \deg: CH_0(S) \to \mathbb{Q}$ is an iso

(i.e. all pts are rationally equiv.)

[Kahn, 2003]: $k = \mathbb{F}_q$, $X = E_1 \times \dots \times E_r/_{\mathbb{F}_q}$, E_i = ell curve

$\Rightarrow$ rational and numerical equivalence coincide

[A., 2021]: A = abel fourfold $\Rightarrow$ the inters prod

$CH^2(A)/_{num} \times CH^2(A)/_{num} \to \mathbb{Q}$

has signature $(s_+, s_-) = (\rho_2 - \rho_1 + 1, \rho_1 - 1)$

§3. Motives

<u>Def</u>: An additive category is pseudo-abelian if projectors ($p^2 = p$) have Image.

<u>Fact</u>: An additive category $\mathcal{C}$ has a pseudo-abelianization $\mathcal{C}^{\natural}$:

$Ob(\mathcal{C}^{\natural})$: (M, p) with $M \in Ob(\mathcal{C})$, $p: M \to M$, $p^2 = p$

"$\mathrm{Im}\, p$"

$Mor(\mathcal{C}^{\natural})$: $\mathrm{Hom}((M,p),(N,q)) = \{ f: X \to Y, f = q \circ f \circ p \}$

[Voevodsky, 2000]: (1) $DM_{(gm)}(k)_{(\mathbb{Q})}$ is

Δ, $\mathbb{Q}$-lin, $\natural$, $\otimes$, $\vee$, Sym, $\mathbb{1}$, Tate twists, $End(\mathbb{1}) = \mathbb{Q}$

(2): $h : Var_k \to DM(k)^{op}$ s.t. $DM(k) = \langle \{h(X)\}_{Sm\ proj} \rangle_{\Delta}$

(3) $Hom_{DM}(\mathbb{1}(-n), h(X)) = CH^n(X)$, X smooth

(4) h satisfies Mayer Vietoris, localization, Poincaré, ...

Def (non classical) $CHM(k) \subset DM(k)$

$$CHM(k) := \langle \{h(X)\}_{X \in SmProj_k} \rangle_{\mathbb{Q}\text{-lin}, \natural, \otimes, \dots}$$

(NOT Δ !)

is called the category of Chow motives / pure mot.

[Ivorra, 2007]: $\exists$ ℓ-adic realization functor

$R_\ell : DM(k) \to D^b(Rep_{\mathbb{Q}_\ell}(Gal_k))$ Δ, $\otimes$, ...

s.t.

$$\begin{tikzcd} Var_k \arrow[r, "coho"] \arrow[dr, "h"'] & D^b(Rep_{\mathbb{Q}_\ell}(Gal_k))^{op} \\ & DM(k)^{op} \arrow[u] \end{tikzcd}$$

Intuition: $CHM(k)$ corresponds to complexes whose cohomology in degree n is pure of weight n

(i.e. Frob eigenvalues α satisfy the Weil relation $|\alpha| = q^{\frac{n}{2}}$)

Rmk: (1) Purity is not stable by shift

(2) General objects are iterated extensions of pure ones

$\rightsquigarrow$ weight filtration in cohomology

$\rightsquigarrow$ weight structure for ~~motives~~ DM [Bondarko, 2010]

Lecture II : Chow motives, hom mot, num mot

§1. Chow motives

§2. Homological and numerical motives

§3. Examples

§1

<u>Rem</u>: $\mathrm{Hom}_{DM}(h(X), h(Y))$ has no relation with classical invariants

<u>Prop</u>: Assume $X \in Sm$ Proper, $Y \in Sm$

$$\Rightarrow \mathrm{Hom}_{DM}(h(X), h(Y)) = CH^{d_X}(X \times Y)$$

<u>Pf</u>: $\mathrm{Hom}(h(X), h(Y)) = \mathrm{Hom}(\mathbb{1}, \underbrace{h(X)^\vee}_{\substack{\text{Poincaré} \\ \| \\ h_c(X)(d_X)}} \otimes h(Y))$

$$\underset{\text{Kunneth}}{=} \mathrm{Hom}(\mathbb{1}(-d_X), h(X \times Y))$$

$$\underset{\text{Voevodsky}}{=} CH^{d_X}(X \times Y) \qquad \square$$

<u>Cor</u>: For smooth proper varieties, motives can be constructed using <u>correspondances</u> ($= CH(X \times Y)$)

Composition:

$$\mathrm{Hom}(h(X),h(Y)) \times \mathrm{Hom}(h(Y),h(Z)) \to \mathrm{Hom}(h(X),h(Z))$$

$$CH(X\times Y) \times CH(Y\times Z) \longrightarrow CH(X\times Z)$$

$$\alpha \quad , \quad \beta \longmapsto pr_{XZ*}(pr_{XY}^*\alpha \cdot pr_{YZ}^*\beta)$$

$$\left(\begin{tikzcd} X\times Y\times Z \arrow[d, "pr_{XY}"'] \arrow[d] \arrow[dr] & \\ X\times Y & \end{tikzcd}\right)$$

Ex (graphs) $f: Y \to X$, $g: Z \to Y$

$\alpha = {}^t\Gamma_f$ $\qquad \beta: {}^t\Gamma_g$

The intersection in $X\times Y\times Z$ is transverse: ~~and corresponds to~~

$(f\circ g(z), g(z), z)$

$\Rightarrow \alpha\circ\beta = {}^t\Gamma_{f\circ g}$

Classical def: ~~$CHM(k) =$~~ $Corr(k)$

$$\mathrm{Ob} = \{h(X)(n)\}_{\substack{X\in SmProj \\ n\in\mathbb{Z}}}, \text{ maps as above}$$

$$CHM(k) = Corr(k)^{\natural}$$

Classical realization:

$$\begin{tikzcd} R_\ell : DM(k) \arrow[r] & D^b(Rep(Gal_k)) \arrow[r] & GrRep(Gal_k) \\ CHM(k) \arrow[u, phantom, "\cup"] \arrow[urr, dashed, "\otimes, sym, 2"'] & & \end{tikzcd}$$

sign in the symmetry

<u>Rmk:</u> This realization is "just" the cycle class map:

$$\begin{tikzcd}
\mathrm{Hom}_{CHM}(h(X), h(Y)) \arrow[r, "R_\ell"] \arrow[d, equal, "\text{Prop above}"] & \mathrm{Hom}_{sGrRep}(H^*_\ell(X), H^*_\ell(Y)) \arrow[d, equal, "\text{Same argument in the prop}"] \\
CH^{d_X}(X \times Y) \arrow[dr, "cl^{d_X}_{X\times Y}"'] & \mathrm{Hom}_{sGrRep}(\mathbb{Q}_\ell, H^*_\ell(X \times Y)(d_X)) \arrow[d, equal, "\mathrm{Eval}_1"] \\
& H^{2d_X}(X \times Y)(d_X)
\end{tikzcd}$$

<u>Philosophy:</u> (1) Chow motives are just a reorganization of Chow rings & cycle class maps.

(2) Motives make a bridge between Chow rings and cohomologies.

(3) Originally Grothendieck thought we should use this bridge in order to prove cohomological statements using cycles. It is actually the other way around!

§2 Homological and numerical motives

<u>Classical</u> construction:

<u>Step 1</u>: Consider $CH/_{\hom} := CH/_{\ker cl_\ell}$ (or $CH/_{num}$)

<u>Step 2</u>: $\mathrm{Corr}(X,Y) := CH^{d_X}(X\times Y)/_{\hom \text{ or } num}$

<u>Step 3</u>: $\mathcal{M}ot(k) := (\mathrm{Corr}/_{\hom})^{\natural}$

$\mathrm{Num}(k) := (\mathrm{Corr}/_{num})^{\natural}$

<u>Rmk</u>: (1) Homological motives depend a priori on ℓ (conj it shouldn't)

(2) Hom is a priori finer than num

$$CH/_{\hom} \twoheadrightarrow CH/_{num}$$

(conj. iso)

<u>Picture</u>:

$$\begin{tikzcd} \mathrm{CHM}(k) \arrow[d, two heads] \arrow[dr, "R_\ell"] & \\ \mathcal{M}ot(k) \arrow[r, hook] \arrow[d, two heads] & \mathrm{sGrRep} \\ \mathrm{Num}(k) & \end{tikzcd}$$

full

faithful

<u>Conj</u>: (1) $\mathcal{M}ot(k) \twoheadrightarrow \mathrm{Num}(k)$ is an equivalence

(2) All maps are conservative (i.e. detect iso)

<u>Rem</u>: R_ℓ conservative is crazy!

<u>Def</u> Let $\mathcal{C} = (\mathbb{Q}\text{-})\mathrm{lin}, \otimes, \simeq, \mathbb{1}, \mathrm{sym}, \mathrm{End}(\mathbb{1}) = \mathbb{Q}$,

(1) (trace) $f : M \to M$ define $\mathrm{tr}(f) \in \mathbb{Q}$ as :

$$\left.\begin{array}{l} f \rightsquigarrow \mathbb{1} \to \check{M} \otimes M \\ \mathrm{id}_M \rightsquigarrow M \otimes \check{M} \to \mathbb{1} \\ \qquad \wr\!\wr \,\mathrm{sym} \\ \qquad \check{M} \otimes M \end{array}\right\} \text{compose : } \mathbb{1} \to \mathbb{1} \text{ is } \mathrm{tr}(f) \cdot \mathrm{Id}$$

(2) (Numerical $\otimes$-ideal) For all $M_1, M_2 \in \mathcal{C}$, define

$$\mathcal{N}(M_1, M_2) \subset \mathrm{Hom}(M_1, M_2)$$
$$\|$$
$$\{ f : M_1 \to M_2, \ \forall g : M_2 \to M_1, \ \mathrm{tr}(gf) = 0 \}$$

<u>Exe:</u> $\mathcal{C} = \mathrm{CHM}$, $M = h(X)$, $f \leftrightarrow Z \hookrightarrow X \times X$

$$\mathrm{tr}(f) = \deg(Z \cdot \Delta_X)$$

$\mathcal{N}$ = numerically trivial correspondences

<u>Facts:</u> (1) $\otimes$-functors verify the factorization thm:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, two heads, "\text{full}"'] & \mathcal{C}' \quad \otimes\text{-func.} \\ \mathcal{C}/\mathrm{Ker} F \arrow[ur, hook, "\text{faithful}"'] & \end{tikzcd}$$

$\otimes$

(2) $\mathcal{N}$ is the biggest $\otimes$-ideal :

$$\mathcal{C} \twoheadrightarrow \mathcal{C}/\mathrm{Ker} F \twoheadrightarrow \mathcal{C}/\mathcal{N}$$

Exe: Apply this to $R_\ell : CHM(k) \to sGrRep$:

$$\begin{tikzcd}
CHM(k) \arrow[d] \arrow[dr, "R_\ell"] & \\
Mot(k) = \left(CHM(k)/_{\ker R_\ell}\right)^\natural \arrow[r] \arrow[d] & sGrRep \\
Num(k) = \left(CHM(k)/_{\mathcal{N}}\right)^\natural &
\end{tikzcd}$$

§ 3. Examples

[Liebermann, 1966]: A = abelian var $/k$.

Then, in $Mot(k)$:

$$h(A) = \bigoplus_n h^n(A), \quad R_\ell(h^n(A)) = H^n_\ell(A)$$

Pf: Consider $[2] : A \to A$,

alors $[2]^* \circlearrowright H^1_\ell(A)$ is $2 \cdot Id$

$\Updownarrow$

Tate module

hence, $[2]^* \circlearrowright H^n_\ell(A)$ is $2^n \cdot Id$

$\quad\quad\quad\quad\quad\quad\quad \shortparallel$

$\quad\quad\quad\quad\quad\quad \Lambda^n H^1_\ell(A)$

Define $\quad p_n \in \mathbb{Q}[[2]^*]$

$\quad\quad\quad\quad \cap \quad\quad \cap$

$\quad\quad\quad Corr(A, A)$

$$P_n := \prod_{i \neq n} \frac{[2]^* - 2^i [\mathrm{id}]}{2^n - 2^i}$$

~~P_n is a projec~~ $P_n \circlearrowright H^i_\ell$ as $\begin{matrix} 0 & i \neq n \\ \mathrm{id} & i = n \end{matrix}$

$\Rightarrow$ it is a projector $\Rightarrow$ defines $h^n(A)$ $\square$

[Katz-Messing, 1974]: $X/\mathbb{F}_q$ sm proper

$\Rightarrow$ In $Mot(\mathbb{F}_q)$, $h(X) = \bigoplus_n h^n(X)$

$R_\ell(h^n(X)) = H^n_\ell(X)$

<u>Pf:</u> We construct P_n (definying h^n)

$$\underset{\mathbb{Q}([\mathrm{Frob}]^*)}{\overset{P_n}{\cap}}$$

let $F_n[T]$ = char pol of $\mathrm{Frob}_X \circlearrowright H^n_\ell$

$$\underset{(\mathbb{Q}[T]}{\overset{F_n[T]}{\cap}}$$

$(\mathbb{Q}[T]$, ℓ independent $\underset{\text{Formula + Weil I}}{\overset{\text{trace}}{\Leftarrow}}$ Z function ℓ-ind)

By Weil I $\{F_n\}$ are coprime $\underset{\text{Bézout}}{\Rightarrow}$

$$1 = A \cdot F_n + B \cdot \prod_{i \neq n} F_i \quad \in \mathbb{Q}[T]$$

$\Rightarrow$ in $\mathrm{Corr}(X,X)$: $\mathrm{id} = A[\mathrm{Frob}] \cdot F_n[\mathrm{Frob}] + \underbrace{B[\mathrm{Frob}] \cdot \prod_{i \neq n} F_i[\mathrm{Frob}]}_{\shortparallel\ P_n}$

By Cayley-Hamilton $P_n \circlearrowright H^i = \begin{matrix} 0 & i \neq n \\ \mathrm{id}\ (\mathrm{mod}\ F_n) & i = n \end{matrix}$ $\square$

<u>Conj</u>: p_n are projectors also in $CHM(\mathbb{F}_q)$

Lecture III

§1. ℓ-independency

§2. Finite dim$^\ell$ motives (Kimura - O'Sullivan ~1995)

§3. Bloch's conj for some surfaces

[Katz - Messing, 1974]: $X/_k$ sm proper, $f: X \to X$

$p_\ell^n \in \mathbb{Q}_\ell[T]$ = the char pol of $f \circlearrowleft H^n_\ell(X)$

$\Rightarrow p_\ell^n \in \mathbb{Q}[T]$ is ℓ-independent.

<u>Pf</u>: Reduce to $k = \mathbb{F}_q$:

finite nb of equation:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[loop, "f"] & \mathcal{X} \arrow[d, "\text{sm proper}"] \arrow[loop, "\tilde f"'] & \mathcal{X}_0 \arrow[l] \arrow[d] \arrow[loop, "\tilde f_0"] \\
k \arrow[r] & \eta \arrow[r] & S_{/\mathbb{Z}} \text{ sch of f.t.} & \mathbb{F}_q \arrow[l]
\end{tikzcd}$$

base ch: $H_\ell(X) \xrightarrow[sp]{\sim} H_\ell(\mathcal{X}_0)$

$\circlearrowleft f \qquad \circlearrowleft \tilde f_0$

Now $k = \mathbb{F}_q$: The statement is equiv. to $\operatorname{tr}(f^m \mid H^n_\ell(X)) \in \mathbb{Q}$ ($+ \ell$-ind.)

We have $\exists\, h^n(X) \in \mathrm{Mot}(k)$ s.t. $R_\ell(h^n(X)) = H^n_\ell(X)$

In Lecture II §2: $\operatorname{tr}(f^m \mid h^n(X))$

$$\operatorname{tr}(f^m \mid h^n(X)) \in \mathbb{Q}$$

$\|$ R_ℓ is a $\otimes$-sym funct.

$\operatorname{tr}(f^n \mid H^n_\ell(X)$ as sGrRep$)$

$\|$

$(-1)^n \operatorname{tr}(f^n \mid H^n_\ell)$ as Vect sp.

(Concretely: $\operatorname{tr} = \sum c_{ij} \deg(\Gamma_{f^i} \cdot \Gamma_{\mathrm{Frob}^j}) \in \mathbb{Q}$), $c_{ij} \in \mathbb{Q}$

ℓ-independency: the proj p_n def $h^n \in \mathrm{Mot}(\mathbb{F}_q)$ is ℓ-ind. $\square$

§2. Finite dim^l motives

(Kimura - O'Sullivan ~1995)

<u>Def</u>: $\mathcal{C} = \otimes$, $\mathbb{Q}$-lin, sym, ⊕, ⊕ cat

$M \in \mathcal{C}$ is finite dim^l if

$M = M_{even} \oplus M_{odd}$ s.t.

$$\Lambda^n M_{even} = 0 \qquad n \gg 0$$

$$\mathrm{Sym}^n M_{odd} = 0$$

Ex: $M \in s\mathrm{GrRep}$

$M = M_{even} \oplus M_{odd}$

(M_{even}: even degree; M_{odd}: odd degree)

(in $D(\mathrm{Rep})$ a complex concentrated in even degree is even - similarly odd)

Conj: All Chow motives are finite dim^l

Thms: (1) Motives of curves are finite dim^l

formal
(2) Being fin. dim. is stable by $\oplus, \otimes, \vee, \natural$

(3) $f: M \to M \in \mathcal{C}$, $F: \mathcal{C} \to \mathcal{D}$ $\otimes$-fct (e.g. $F = R_\ell$)

Then $F(f) = 0 \Rightarrow f^N = 0$

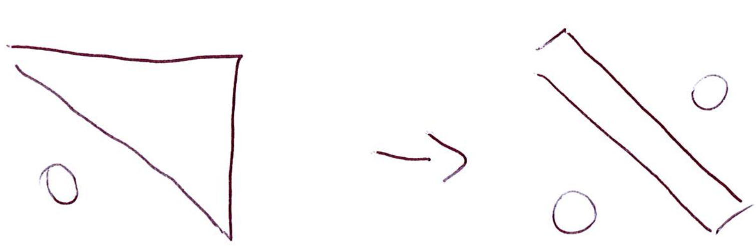
Cor: Are fin. dim^l: abel var, Fermat varieties, ...

(2) (3).

Rmk: $M \in CHM$. Conjecturally the map

$$\mathrm{End}(M) \xrightarrow{R_\ell} \mathrm{End}(R_\ell(M))$$

behaves as



upper triang matrices $\to$ diag. matrices

(each entry is a coho degree)

§3. Bloch's conj for some surfaces

[Kimura, 2005]: Let $S/_{\mathbb{C}}$ = sm proj st

(a) $H^0(S, \Omega^1) = 0 = H^2(S, \Omega^2)$

(b) S is dominated by a prod of curves

$\Rightarrow$ $\deg : CH_0(S) \to S$ is an iso

<u>Pf:</u> We will show:

(1) Under (a), $h(S) = \mathbb{1} \oplus \mathbb{1}(-1)^{\rho_1} \oplus \mathbb{1}(-2) \oplus P$

with P phantom (i.e. $R(P) = 0$)

(2) Under (b), $h(S)$ is finite dim^l (and so is P)

(3) Fin. dim^l phantom motives are zero.

Using (1), (2) (3) $\Rightarrow$ $h(S) = \mathbb{1} \oplus \mathbb{1}(-1)^{\rho_1} \oplus \mathbb{1}(-2)$

Hence $CH_0(S) = \underset{\underset{\mathbb{Q}}{\parallel}}{Hom(\mathbb{1}, \mathbb{1})} \oplus \underset{\underset{CH^{n>0}(\operatorname{Spec} \mathbb{C})}{\parallel\ 0}}{Hom(\mathbb{1}, \mathbb{1}(-1)^{\rho_1} \oplus \mathbb{1}(-2))}$ $\square$

<u>Pf(1):</u> Pick a point $x \in S(\mathbb{C})$. The constant map

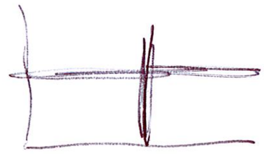
$S \to \operatorname{Spec} \mathbb{C} \xrightarrow{x} S$ induces

$h(S) \leftarrow \mathbb{1} \leftarrow h(S)$ which is a proj

defining $\mathbb{1}$ as a direct facto of $h(S)$

Dually: $h(S)^\vee \to \mathbb{1} \to h(S)^\vee$
$$\begin{array}{c} \| \\ h(S)(2) \to \mathbb{1} \to h(S)(2) \end{array}$$

$\rightsquigarrow \quad h(S) \to \mathbb{1}(-2) \to h(S)$

(N.B the correspondences are  $P_4 \subset S\times S$)

P_0

~~P_0 & P_4 are central~~

$$\Rightarrow \quad h(S) \underset{\uparrow}{=} \mathbb{1} \oplus \mathbb{1}(-2) \oplus Q$$

$P_0P_4 = 0\ P_4P_0 \Leftarrow$ no maps between $\mathbb{1}$ & $\mathbb{1}(-2)$

Now, under (a), Hodge theory gives $H^1(S) = H^3(S) = 0$
and by $(1,1)$-thf. $CH^1(S) \xrightarrow{\sim} H^2(S)$.

Pick $D_1, \dots, D_{\rho_1}$ orth. bases

$$\begin{tikzcd} \mathbb{1}(-1)^{\rho_1} \arrow[r, "{(D_1, \dots D_{\rho_1})}"] \arrow[rr, bend right, "\mathrm{id}"'] & h(S) \arrow[r, "{\left(\frac{D_1}{\deg(D_1\cdot D_1)}, \dots\right)}"] & \mathbb{1}(-1)^{\rho_1} \end{tikzcd}$$

$$\Rightarrow \quad h(S) = \mathbb{1} \oplus \mathbb{1}(-1)^{\rho_1} \oplus \mathbb{1}(-2) \oplus P$$

as above

but now $R(P) = 0$ $\square$

Pf (b) $C_1\times C_2 \dashrightarrow S \Rightarrow$

$\exists$ B = some blow-up

$$\begin{tikzcd} B \arrow[d] \arrow[dr, "\pi"] & \\ C_1\times C_2 \arrow[r, dashed] & S \end{tikzcd}$$

Now $h(C_1 \times C_2) = h(C_1) \otimes h(C_2) \Rightarrow$ finite dim.

$$h(B) = h(C_1 \times C_2) \oplus \underset{\substack{\uparrow \\ \text{Exc. div}}}{\mathbb{1}(-1)^n} \Rightarrow \text{——}$$

Blow-up form.

$$h(S) \underset{\substack{\uparrow \\ \text{sub factor by } \pi^*,\ \pi_*}}{\subset} h(B) \Rightarrow \text{——}$$

□

Pf(c): Let P be a f.d. Phantom:

$$R(Id_P) = 0 \underset{\text{Kimura(3)}}{\Longrightarrow} Id_P^n = 0$$

□

Lecture IV

§1 rat = num on $E_1 \times \dots \times E_r / \mathbb{F}_q$

§2. Arithmetic subtleties behind motives

§3. Signature on the inters. prod.

[Kahn, 2003]: If $X/\mathbb{F}_q$ is s.t.

(1) X satisfies Tate's conj (i.e. $cl^n_{X,\ell} : CH^n(X) \otimes \mathbb{Q}_\ell \to H^{2n}_\ell(X)^{Frob}$ is surj)

(2) $h(X)$ is fin-dim^l

(3) Frob $\circlearrowright H_\ell(X)$ is semisimple

$\Rightarrow$ rat = num on $CH(X)$

<u>Application</u>: $X = E_1 \times \dots \times E_r / \mathbb{F}_q$, indeed

(1) Tate's conj is known for ~~divisors~~ $n=1$ on abel var [Tate, 1966] and $\bigoplus_n H^{2n}(E_1 \times \dots \times E_r)(n)^{\mathrm{Frob}}$ is gen by $n=1$ [Spiess, 1999]

(2) Ok for abel var ($\forall k$)

(3) ——— (uses that Frob $\in \mathrm{End}(A)$ semisimple alg)

<u>Pf</u>: Pick $Z_1, \dots, Z_s \in CH^n(X)_{\mathbb{Q}}$ s.t.

$$\langle cl(Z_1), \dots, cl(Z_s) \rangle_{\mathbb{Q}_\ell} = H^{2n}_{\ell}(X)(n)^{\mathrm{Frob}}.$$

Similarly, $Y_1, \dots, Y_s \in CH^{d_X - n}(X)_{\mathbb{Q}}$ (Same s!)

The inters. has coeff in $\mathbb{Q}$!

Can modify Y_i's s.t. $\deg(Z_i \cdot Y_j) = \delta_{ij}$

Hence $\mathbb{1}(-n)^s \xrightarrow{(Z_1, \dots, Z_s)} h(X) \xrightarrow{(Y_1, \dots, Y_s)^\vee} \mathbb{1}(-n)^s$, composite $= \mathrm{id}$

$\Rightarrow h(X) = \mathbb{1}(-n)^s \oplus M$

<u>Goal</u>: $\mathrm{Hom}_{CHM}(\mathbb{1}(-n), M) = 0$

<u>Tool</u>: $\forall M_i \in CHM(\mathbb{F}_q)\ \exists\ \mathrm{Frob}_{M_i} : M_i \to M_i$ s.t.

(a)
$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\mathrm{Frob}_{M_1}"] & M_2 \arrow[d, "\mathrm{Frob}_{M_2}"] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$
commutes

(b) $\mathrm{Frob}_{h(Y)} = h(\mathrm{Frob}_Y)$

[Soulé, 1984]

Let $f : \mathbb{1} \to M(n)$

$\Rightarrow \mathrm{Frob}_{M(n)} \circ f = f \circ \mathrm{Frob}_{\mathbb{1}} = f$

$\Rightarrow \underbrace{(\mathrm{Frob}_{M(n)} - \mathrm{Id})}_{\text{This is an iso}} \circ f = 0$

<u>Goal</u>

OTOH, Its realization is an iso (1 is not an eigenvalue in $R_\ell(M)(n)$)

Conclude with:

<u>Prop:</u> $F : N \to N$ with N finite dim^l

$R_\ell(F)$ iso $\Rightarrow F$ iso

<u>Pf:</u> Cayley-Hamilton: $\exists P(t)$, $P(F) = 0$ (but $P(0) \neq 0$)

$\overset{\text{Kimura}}{\Longrightarrow} P(F)^n = 0$ in CHM

$\Rightarrow \sum_{i>0} c_i F^i = -c_0 \mathrm{Id}$

$\;\;(\sum \,'' \,) \circ F \;\;//$

□

§2.

Question 1: Let $\alpha, \beta \neq 0$ be two alg classes

$$\in \mathrm{Im}\left(CH(X)_{\mathbb{Q}} \to H_\ell(X)\right)$$

Suppose $\alpha = \lambda \beta$ with $\lambda \in \mathbb{Q}_\ell^*$

Is $\lambda \in \mathbb{Q}$?

Answers: (1) Yes if char $k = 0$

$$H_\ell(X) = \underbrace{H_{sing}(X)}_{\substack{\ni \\ \alpha, \beta}} \otimes \mathbb{Q}_\ell$$

(2) Yes if hom $=$ num:

let γ s.t. $\deg(\alpha \cdot \gamma) \neq 0$ then

$$\underbrace{\deg(\alpha \cdot \gamma)}_{\in \mathbb{Q}} = \lambda \underbrace{\deg(\beta \cdot \gamma)}_{\in \mathbb{Q}}$$

Otherwise open! This is the missing point in:

[Clozel, 1999]: $A/\mathbb{F}_q$ ab var then

num $= \ell$-hom for infinitely many ℓ.

Rmk: (1) $H^n_\ell(X) \xrightarrow[\text{Hard-Lef}]{\sim} H^{2d_X-n}_\ell(X)(d_X-n)$

$\cong H^n_\ell(X)^\vee(-n)$

i.e. $H^n_\ell(X)$ is self dual (as Rep)

(2) In Hodge theory this self duality is a polarization:

$H^n_{sing}(X) \otimes H^n_{sing}(X) \to \mathbb{Q}(-n)$ is "almost a scalar product"

$\Rightarrow \forall M \underset{\text{sub H.S.}}{\subset} H^n$ The restriction is again a polarization $\Rightarrow$ M is also self dual

(3) False in ℓ-adic coho!

E/k nb field, ℓ s.t. $i \in \mathbb{Q}_\ell$

$I \circlearrowleft$, $I^2 = -id$

Then $I \curvearrowright H^1_\ell(E) = L_i \oplus L_{-i}$ eigen lines

$Gal_k \curvearrowright L_i$ is not self dual!

$(L_i^\vee \simeq L_{-i}(1))$

This is corrected by motives:

Question 2: $M \in Mot(k)_\mathbb{Q}$ of coho deg ~~weight~~ n ~~...~~

$\Rightarrow M \simeq \check{M}(-n)$?

§3. Signature of inters prod.

K = p-adic field

$M \in CHM(K)$ with good reduction

(e.g. M, $R_\ell(M) \subset H^{2n}(X)(n)$, X good red)

$\overline{M} \in CHM(\mathbb{F}_q)$ its reduction

$$q : \mathrm{Sym}^2 M \to \mathbb{1} \rightsquigarrow \left\{\begin{array}{l} q_B := R_{\mathrm{sing}}(q) \quad (K \hookrightarrow \mathbb{C}) \\ q_Z := \overline{q}|_{\underset{CHM(\mathbb{F}_q)}{\mathrm{Hom}(\mathbb{1}, \overline{M})}} \end{array}\right.$$

two $\mathbb{Q}$-quad forms

"alg cycles mod p"

Conj: Assume

(1) $\overline{M} \simeq \mathbb{1}^m$ (supersing reduction)

(2) q_B is a polarization

Then q_Z is positive definite

[A. 2021]: Conj holds for M, $\dim R_\ell(M) = 2$.

Cor: Signature of the inter prod Z^2 (A = abel fourfold)

Pf of Cor: Even though $H^4(A)$ is $\binom{8}{4} = 70$ dim^l

one can decompose it into smaller motives (CM theory)

Pf $\dim M = 2$, $p = \mathrm{char}(\mathbb{F}_q)$

(1) $\ell \neq p$: $q_B \otimes \mathbb{Q}_\ell \simeq q_Z \otimes \mathbb{Q}_\ell$

$$\begin{array}{ccc} q_B \otimes \mathbb{Q}_\ell & & q_Z \otimes \mathbb{Q}_\ell \\ \| \text{ Artin comp} & & \| \text{ supersing hyp} \\ \ell\text{-adic coho in char } 0 & \underset{\text{base chg}}{=} & \ell\text{-adic coho in char } p \end{array}$$

(2) $\operatorname{disc}(q_B) = \operatorname{disc}(q_Z) \in \mathbb{Q}^* / (\mathbb{Q}^*)^2$

as it is true mod ℓ $\tilde{\forall} \ell$.

(3) $\operatorname{disc}(q_B) > 0$ (polarization) $\Rightarrow \operatorname{disc} q_Z > 0 \Rightarrow q_Z$ pos def or neg def

dim ?!

Def: $\varepsilon_\nu(q \text{ of rk } 2) = \begin{cases} +1 & \text{if } q(x,y) = 1 \text{ has a sol } x, y \in \mathbb{Q}_\nu \\ -1 & \text{otherwise} \end{cases}$

$\nu = \ell, p, \infty$

__Goal__: $\varepsilon_\infty(q_Z) = 1$

(4) Product formula: $\prod_\nu \varepsilon_\nu(q_B) = 1 = \prod_\nu \varepsilon_\nu(q_Z)$

By (1): $\varepsilon_p(q_B) \cdot \varepsilon_\infty(q_B) = \varepsilon_p(q_Z) \cdot \varepsilon_\infty(q_Z)$

$\Rightarrow \varepsilon_\infty(q_Z) = \underbrace{\varepsilon_\infty(q_B)}_{\text{// Hodge theory}} \cdot \dfrac{\varepsilon_p(q_B)}{\varepsilon_p(q_Z)}$

p-adic Hodge theory

~~depe~~ depends on the type of HS

(5) $q_B \otimes \mathbb{Q}_p$ = p-adic coho in char 0 (as in (1))

$q_Z \otimes \mathbb{Q}_p$ = φ-Inv part of crystalline coho

$\underset{\text{Fontaine}}{\Longrightarrow}$ $(q_B \otimes \mathbb{Q}_p) \otimes B_{cris} \simeq (q_Z \otimes \mathbb{Q}_p) \otimes B_{cris}$

One can compute the base change matrix $\in M_{2\times 2}(B_{cris})$ and from there deduce $\varepsilon_p(q_B) / \varepsilon_p(q_Z)$

Ex: $\left(\begin{array}{c}\mathbb{Q}_p \rightsquigarrow \mathbb{R} \\ B_{crys} \rightsquigarrow \mathbb{C}\end{array}\right)$

$q_1 = a x^2 + b y^2 \,/\mathbb{R}$
$q_2 = c x^2 + d y^2 \,/\mathbb{R}$

$q_1 \otimes \mathbb{C} \xrightarrow{\sim} q_2 \otimes \mathbb{C}$

$\begin{pmatrix} 1 & \\ & i \end{pmatrix}$

$\Rightarrow q_1 \not\cong q_2$

[A., Marmora, 2025]: M of higher rank as in the conj $\underset{\text{some methods}}{\Rightarrow}$ $\mathrm{disc}(q_Z) > 0$, $\varepsilon_\infty(q_Z) = +1$

$\Rightarrow (s_+, s_-)$ signature of q_Z, then $4 \mid s_-$

[Agugliaro, 2025]: Find condition on M so that $(s_+, s_-) =$
- $s_- = 0$ ̈◡
- $s_+ = s_-$
- $s_+ = 0$

(tannakian methods)

Combining the two, get $s_- = 0$ in several cases.